Asymmetric norms, partially ordered normed spaces and injectivity

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Outline

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Asymmetric norms and injectivity

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Some definitions

Let X be a *real* vector space. A function $p : X \to [0, \infty)$ is called **sublinear** (or an **asymmetric seminorm**) if for all $x, y \in X$, $\lambda \geq 0$,

(a) $p(\lambda x) = \lambda p(x)$;

(b) $p(x + y) \leq p(x) + p(y)$.

If in addition $p(x) = 0 = p(-x)$ iff $x = 0$, we call p an **asymmetric norm.**

If X is a real vector space and p an asymmetric norm on X, then the pair (X, p) will be called an **asymmetrically normed space**

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Then $p^s : X \to [0, \infty)$ defined by

$$p^s(x) = \max\{p(x), p^t(x)\} = \max\{p(x), p(-x)\}$$

is a norm on X.

A simple but important special case:

$X = \mathbb{R}, \quad p_1(x) = x^+ = x \vee 0.$

$p^t(x) = x^- = (-x) \vee 0, \quad p^s(x) = |x|.$

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Banach's theorem (1931)

Theorem (Banach)

Let X be a real a real vector space and p be a sublinear function from X to $\mathbb{R}$. Let X_0 be a vector subspace of X and let f_0 be a linear function from X_0 to $\mathbb{R}$ such that

$$f_0(x) \leq p(x) \text{ for all } x \in X_0.$$

Then there exists a linear function f from X to $\mathbb{R}$ that extends f_0 and for which

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The topology of an asymmetrically normed space

If p is an asymmetric norm on X,

$$d_p(x,y) = p(y-x)$$

defines a quasi-metric d_p on X which induces a T_0-topology on X.

The topology is T_0, but need not be T_1.

The basic neighbourhoods of x are the open balls $B_r^p(x) = \{y \in X : p(y-x) < r\}, \quad r > 0.$

Addition is jointly continuous, but scalar multiplication is only continuous for multiplication by non-negative scalars.

Later we will also need the closed balls $B_r^p[x] = \{y \in X : p(y-x) \leq r\}, \quad r > 0.$

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Linear maps between asymmetrically normed spaces

Let (X, p) and (Y, q) be asymmetrically normed spaces and $T : X \to Y$ be a linear map.

T is continuous with respect to the topologies induced by p and q ((p, q)-continuous for short) iff T is bounded, i.e. there is a $C > 0$ such that

$$q(Tx) \leq Cp(x) \text{ for all } x \in X.$$

If this is the case, the infimum of all such constants C will be denoted by $\|T|$:

$$\begin{aligned} \|T| &= \inf\{C > 0 : q(Tx) \leq Cp(x) \quad \forall x \in X\} \\ &= \sup\{q(Tx) : p(x) \leq 1\}. \end{aligned}$$

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Hahn-Banach for asymmetrically normed spaces

Theorem

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$$||f| = ||f_0|.$$

A canonical asymmetric norm on a normed Riesz space

If X is a normed Riesz space (vector lattice), then

$$p(x) = ||x^+|| = ||x \vee 0||, \quad x \in X,$$

defines an asymmetric norm on X,

A real linear functional f on X is (p, p_1)-continuous iff it is norm bounded and positive (i.e. if $x \geq 0 \Rightarrow f(x) \geq 0$).

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Let X be a normed Riesz space, X_0 a vector subspace of X and f_0 a bounded positive linear functional on X_0. Then there is a bounded positive extension f of f_0 such that $||f_0|| = ||f||$.

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Partially ordered normed spaces

If C is a cone in a real vector space X, then it induces a partial order $\leq_C$ on X defined by

$$x \leq_C y \iff y - x \in C, \qquad x, y \in X.$$

With this partial order, X becomes a **partially ordered vector space.**

A **partially ordered normed space** is a normed space equipped with a partial order induced by a cone.

The cone C is **normal** if the norm is **monotone**, i.e. $0 \leq_C x \leq_C y \Rightarrow ||x|| \leq ||y||$.

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A positive bounded linear functional on a linear subspace of a partially ordered normed space need not have an extension to a bounded positive linear functional on the whole space.

Theorem

Let X be a real normed space with closed unit ball B_X and ordered by a cone C_X, and let X_0 be a linear subspace of X. A bounded positive linear functional f_0 on X_0 has a bounded positive extension to X iff f_0 is bounded above on $X_0 \cap (B_X - C_X)$

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$$p_X(x) = \inf\{||x + x'||_X : x' \in C_X\}.$$

Then p_X is an asymmetric norm on X, and $C_X = \{x \in X : p_X(-x) = 0\}$

The set $A = B_X - C_X$ is a convex absorbent set such that $\cap\{\lambda A : \lambda \neq 0\} = \{0\}$, and for $x \in X$,

$$p_X(x) = \inf\{\lambda > 0 : x \in \lambda A\} = p_A(x).$$

Furthermore, $\{x \in X : p_X(x) < 1\} \subseteq B_X - C_X \subseteq \{x \in X : p_X(x) \leq 1\}$.

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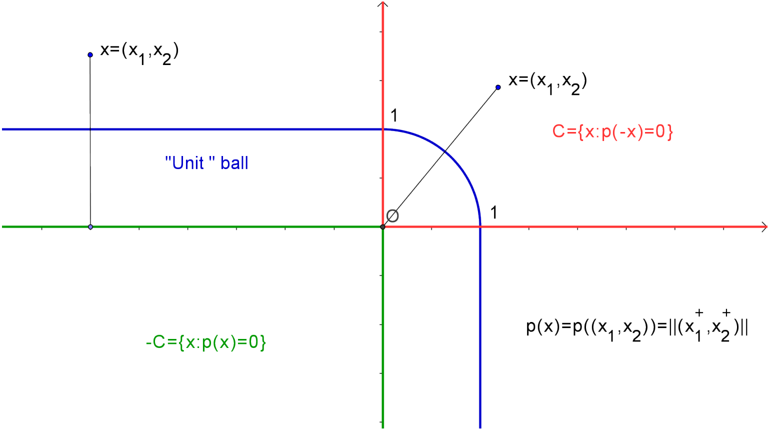
x=(x1,x2)
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1
C={x:p(-x)=0}
"Unit " ball
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Hahn-Banach for partially ordered normed spaces (take 2)

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If the partial order induced by C_X is a lattice order, then $p_X(x) = ||x^+||_X$ for $x \in X$.

In particular, if $X = \mathbb{R}$ with its usual norm and order, then $p_X = p_{\mathbb{R}} = p_1$.

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Injectivity

A normed space Y is 1-injective if the Hahn-Banach theorem for normed spaces remains true when $\mathbb{R}$ is replaced by Y. More precisely:

Definition (1-injectivity)

A normed space Y is 1-injective if for every normed space X, every linear subspace X_0 of X and every continuous linear map $T_0 : X_0 \to Y$ there is a continuous linear extension $T : X \to Y$ of T_0 such that $||T|| = ||T_0||$.

Every 1-injective normed space is a Banach space.

A Banach space Y is 1-injective iff for every Banach space X containing Y as a subspace there is a linear projection P from X onto Y such that $||P|| \leq 1$.

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Hyperconvexity

Definition

*A normed space X has the **binary intersection property** if every collection of closed balls in X, each pair of which has nonempty intersection, has nonempty intersection.*

Definition

*A normed $X)$ is called **hyperconvex** if for each family $(x_i)_{i \in I}$ of points in X and each family of positive real numbers $(r_i)_{i \in I}$, the conditions $d(x_i, x_j) \leq r_i + r_j$ whenever $i, j \in I$ imply that $\bigcap\{B_{r_i}[x_i] : i \in I\} \neq \emptyset$.*

(Here $B_{r_i}[x_i] = \{x \in X : ||x - x_i|| \leq r_i\}$.)

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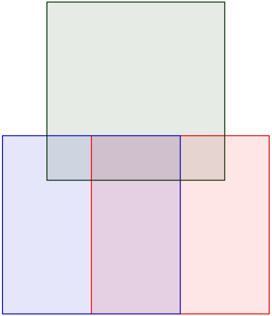
A normed space is hyperconvex iff it has the binary intersection property.

Binary intersection property: Example 1

$X = \mathbb{R}^2, ||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}$ has the binary intersection property.



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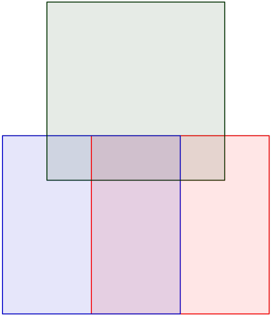

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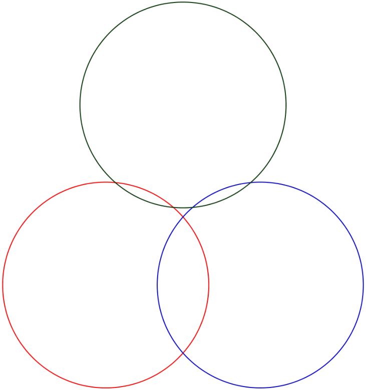

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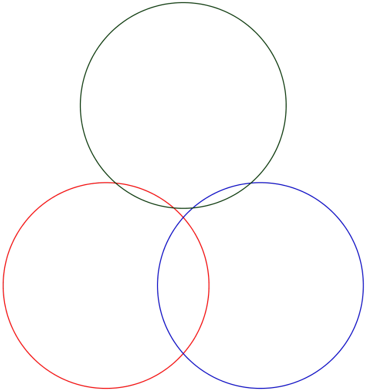

Characterizing injective Banach spaces

Theorem ((Nachbin, 1950))

For a Banach space X the following are equivalent:

(a) *X is 1-injective.*

(b) *X has the binary intersection property.*

(c) *X is a Dedekind-complete vector lattice with an order unit.*

(d) *X is isometrically isomorphic to the space $C(K)$ of continuous real-valued functions on an extremally disconnected compact Hausdorff space K, equipped with the supremum norm.*

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Characterizing injective Banach spaces

Theorem ((Nachbin, 1950))

For a Banach space X the following are equivalent:

(a) *X is 1-injective.*

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Injective asymmetrically normed spaces: Definitions

Definition

*An asymmetrically normed space (Y, q) is called **1-injective** if for every asymmetrically normed space (X, p) and every linear subspace X_0 of X, every continuous linear map $T_0 : (X_0, p) \to (Y, q)$ has a continuous extension $T : X \to Y$ such that $||T| \leq ||T_0|$.*

Definition

*An asymmetrically normed space (X, p) is **Isbell-convex** if for each family $(x_i)_{i \in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ it follows from $p(x_j - x_i) \leq r_i + s_j$ whenever $i, j \in I$, that*

$$\bigcap_{i \in I} B^p_{r_i}[x_i] \cap B^{p^t}_{s_i}[x_i] \neq \emptyset.$$

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More definitions

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An asymmetrically normed space (X, p) *is* ***Isbell-complete*** *if for each family* $(x_i)_{i \in I}$ *of points in* X *and families of nonnegative real numbers* $(r_i)_{i \in I}$ *and* $(s_i)_{i \in I}$ *such that if* $B^p_{r_i}[x_i] \cap B^{p^t}_{s_j}[x_j] \neq \emptyset$ *whenever* $i, j \in I$ *, then*

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Theorem

If an asymmetrically normed space is Isbell-complete (equivalently, Isbell-convex), it is 1-injective.

Theorem

If X is a Dedekind-complete Riesz space with order unit e and the asymmetric norm p on X is defined by

$$p(x) = \inf\{\lambda \geq 0 : x \leq \lambda e\}, \quad x \in X,$$

then (X, p) is Isbell-convex.

Corollary

If K is an extremally disconnected compact Hausdorff space and for $f \in C(K)$ we put $p(f) = \sup\{f(t) : t \in K\}$, then $(C(K), p)$ is Isbell-convex and therefore 1-injective.

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1. Can every 1-injective asymmetrically normed space be represented as $(C(K), p)$, as above?
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Injectivity for partially ordered normed spaces

We could require that it must be possible to extend linear maps that are both continuous and positive to maps of the same kind, with preservation of norms.

But with such a definition, $\mathbb{R}$ would not be injective.

Definition (Riedl (1964))

A partially ordered normed space Y with closed unit ball B_Y, ordered by the closed cone C_Y has **Property** P_1 *if for every partially ordered normed space X with closed unit ball B_X, ordered by the closed cone C_X, every linear subspace X_0 of X and every bounded linear map $T_0 : X_0 \to Y$ such that*

$$T_0(X_0 \cap (B_X - C_X)) \subseteq ||T_0||(B_Y - C_Y),$$

there is a bounded positive extension $T : X \to Y$ of T_0 such that $||T|| = ||T_0||$.

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Property P_1 and continuity

The inclusion

$$T_0(X_0 \cap (B_X - C_X)) \subseteq ||T_0||(B_Y - C_Y),$$

is equivalent to the $(p_X|_{X_0}, p_Y)$-continuity of T_0, with $||T_0| = ||T_0||$.

Linking asymmetrically to partially ordered normed spaces

Let (X, p) be an asymmetrically normed space.

(a) $C_p = \{y \in Y : p(-y) = 0\}$ is a cone in X; ordered by C_p, X is a partially ordered vector space. The order $\leq_p$ is defined by $x \leq_p y \Leftrightarrow p(x - y) = 0$.

(b) $p^s(y) = \max\{p(y), p(-y)\}$ defines a norm on X which is monotone with respect to the order induced by C_p.

(c) C_p is a p^s-closed normal cone in X.

Going back:

(d) On X we can define another asymmetric norm p_m by $p_m(y) = \{p^s(y + y') : y' \in C_p\}$.

(e) If q is an asymmetric norm on X such that $q(x) \leq p^s(x)$ and $C_p \subseteq C_q$, then $q(x) \leq q_m(x)$.

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(c) However, in general we do not have $p^s_X(x) = ||x||_X$.

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Continuity

Proposition

Let (X, p) and (Y, q) be asymmetrically normed spaces and $T : X \to Y$ be a linear map. Then T is continuous with respect to the topologies induced by p_m and q_m if and only if $T(C_p) \subseteq C_q$ and T is continuous with respect to the topologies induced by p^s and q^s. In this case

$$||T|| = ||T|.$$

A partial answer to the first question

Theorem

If the asymmetrically normed space (Y, q) is 1-injective and q is maximal, then there is an extremally disconnected compact Hausdorff space K such that (Y, q) is asymmetrically isomorphic to $(C(K), p)$, where

$$p(f) = \sup\{f(t) : t \in K\}, \text{ for } f \in C(K).$$

Two further questions:

3. If q is a maximal asymmetric norm on Y, is (Y, q) 1-injective?
4. If (Y, q) is 1-injective, is q maximal?

Taking $Y = \mathbb{R}$ and $p(x) = |x|$ shows that the answer to the third question is "no".

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Injective hulls

For any asymmetrically normed space (X, p), is there a 'smallest' 1-injective asymmetrically normed space (Y, q) 'containing' X?

More precisely: Is there a 1-injective asymmetrically normed space (Y, q) and an isometric isomorphism Ψ from X into Y such that there is no proper 1-injective subspace of Y containing $\Psi(X)$?

If such a pair (Y, Ψ) exists, we call Y (somewhat loosely) an **injective hull** of X.

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Some historical remarks about injective hulls

Existence for metric spaces: Isbell (1964)

Existence for Banach spaces: Cohen (1964)

Isbell also showed (non-constructively) that the metric injective hull has the structure of a Banach space.

Cianciaruso and De Pascale (1996) gave an explicit definition of algebraic operations on the injective hull of a Banach space.

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The construction: function pairs

(X, p) is an asymmetrically normed space.
A function pair $f = (f_1, f_2)$, where $f_i : X \to [0, \infty)$ for $i = 1, 2$, is called **ample** if $p(y - x) \leq f_2(x) + f_1(y)$.

f is **minimal** whenever $g = (g_1, g_2)$ is an ample pair such that if $g_1 \leq f_1, g_2 \leq f_2$, then $g_1 = f_1, g_2 = f_2$.

The set of all minimal function pairs on X will be denoted by $\mathcal{E}(X, p)$.

For every $z \in X$, we define the minimal function pair $f_z = (f_{z,1}, f_{z,2})$ by $f_{z,1}(x) = p(x - z), \quad f_{z,2}(x) = p(z - x)$.

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Scalar multiplication in $\mathcal{E}(X, p)$

Scalar multiplication:
For $\lambda \in \mathbb{R}$ and $f \in \mathcal{E}(X, p)$, we define the function pair $f^\lambda = (f_1^\lambda, f_2^\lambda)$ by

$$f_1^\lambda(x) = \begin{cases} \lambda f_1(\lambda^{-1}x) & \text{if } \lambda > 0, \\ p(x) & \text{if } \lambda = 0, \\ |\lambda| f_2(\lambda^{-1}x) & \text{if } \lambda < 0 \end{cases} \quad \text{and}$$

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Addition in $\mathcal{E}(X, p)$

If $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{E}(X, p)$, $x \in X$ we put $f \oplus g = ((f \oplus g)_1, (f \oplus g)_2)$, where

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The map $x \mapsto f_x$ preserves addition.

The only candidate for the additive identity is $f^0 = (f_1^0, f_2^0)$, with $f_1^0(x) = p(x), f_2^0(x) = p(-x))$.

The only candidate for the additive inverse of $f = (f_1, f_2)$ is $(-1)f = (f_1^{(-1)}, f_2^{(-1)})$.

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$\mathcal{E}(X,p)$ is an Isbell-convex hull of X

Theorem

If scalar multiplication on $\mathcal{E}(X,p)$ and addition $\oplus$ are defined as above, then $\mathcal{E}(X,p)$ is a vector space and the map $x \mapsto f_x$ is a linear isomorphism of X into $\mathcal{E}(X,p)$.

Proposition

The function $\tilde{p} : \mathcal{E}(X,p) \to [0,\infty)$ defined by $\tilde{p}(f) = \tilde{p}((f_1,f_2)) = f_2(0)$ is an asymmetric norm on $\mathcal{E}(X,p)$ and the map $x \mapsto f_x$ is an isometry.

Proposition

$(\mathcal{E}(X,p),\tilde{p})$ is an Isbell-convex asymmetrically normed space containing an isometrically isomorphic copy of X..

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The second question answered

Theorem

An 1-injective asymmetrically normed space (X, p) is Isbell-convex.

The order structure of the injective hull

Proposition

If (X, p) is an asymmetrically normed space and $\mathcal{E}(X, p)$ its injective hull, equipped with the asymmetric norm $\tilde{p}$, then the order $\leq_{\tilde{p}}$ on $\mathcal{E}(X, p)$ is given by

$$\begin{aligned} f \leq_{\tilde{p}} g \quad &\Longleftrightarrow \quad f_1(x) \geq g_1(x) \text{ for every } x \in X \\ &\Longleftrightarrow \quad f_2(x) \leq g_2(x) \text{ for every } x \in X, \end{aligned}$$

where $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{E}(X, p)$.

Theorem

If (X, p) is an asymmetrically normed space, then with the above order, $\mathcal{E}(X, p)$ is a Dedekind complete vector lattice, and the asymmetric norm $\tilde{p}$ on $\mathcal{E}(X, p)$ is maximal.

The last two questions answered

It follows from the previous theorem that if the asymmetrically normed space (X, p) is 1-injective, p must be maximal.

This answers Question 4 in the affirmative.

Combined with the previous partial answer to Question 1, this gives

Theorem

If the asymmetrically normed space (Y, q) is 1-injective, there is an extremally disconnected compact Hausdorff space K such that (Y, q) is asymmetrically isomorphic to $(C(K), p)$, where

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