Asymmetric norms, partially ordered normed spaces and injectivity

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2. Injective normed spaces
3. Injective asymmetrically normed spaces
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Some definitions

Let $X$ be a real vector space. A function $p : X \to [0, \infty)$ is called **sublinear** (or an **asymmetric seminorm**) if for all $x, y \in X$, $\lambda \geq 0$,

(a) $p(\lambda x) = \lambda p(x)$;
(b) $p(x + y) \leq p(x) + p(y)$.

If in addition $p(x) = 0 = p(-x)$ iff $x = 0$, we call $p$ an **asymmetric norm**.

If $X$ is a real vector space and $p$ an asymmetric norm on $X$, then the pair $(X, p)$ will be called an **asymmetrically normed space**.
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More definitions and an example

The function $p^t : X \rightarrow [0, \infty)$ defined by $p^t(x) = p(-x)$ is also an asymmetric norm on $X$.

Then $p^s : X \rightarrow [0, \infty)$ defined by

$$p^s(x) = \max\{p(x), p^t(x)\} = \max\{p(x), p(-x)\}$$

is a norm on $X$.

A simple but important special case:

$X = \mathbb{R}, \quad p_1(x) = x^+ = x \vee 0.$

$p^t(x) = x^- = (-x) \vee 0, \quad p^s(x) = |x|.$
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The injective hull of an asymmetrically normed space
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Links

The injective hull of an asymmetrically normed space
Banach’s theorem (1931)

**Theorem (Banach)**

Let $X$ be a real vector space and $p$ be a sublinear function from $X$ to $\mathbb{R}$. Let $X_0$ be a vector subspace of $X$ and let $f_0$ be a linear function from $X_0$ to $\mathbb{R}$ such that

$$f_0(x) \leq p(x) \text{ for all } x \in X_0.$$ 

Then there exists a linear function $f$ from $X$ to $\mathbb{R}$ that extends $f_0$ and for which

$$f(x) \leq p(x) \text{ for all } x \in X.$$
Hahn’s theorem (1927)

Theorem (Hahn)

Let $X$ be a real normed space. Let $X_0$ be a vector subspace of $X$ and let $f_0$ be a bounded linear function from $X_0$ to $\mathbb{R}$. Then there exists a bounded linear function $f$ from $X$ to $\mathbb{R}$ that extends $f_0$ and for which

$$\|f\| = \|f_0\|.$$  

This follows from Banach’s theorem by taking $p(x) = \|x\|$. 
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The topology of an asymmetrically normed space

If $p$ is an asymmetric norm on $X$,

$$d_p(x, y) = p(y - x)$$

defines a quasi-metric $d_p$ on $X$ which induces a $T_0$-topology on $X$.

The topology is $T_0$, but need not be $T_1$.

The basic neighbourhoods of $x$ are the open balls $B^p_r(x) = \{ y \in X : p(y - x) < r \}, \quad r > 0$.

Addition is jointly continuous, but scalar multiplication is only continuous for multiplication by non-negative scalars.

Later we will also need the closed balls $B^p_r[x] = \{ y \in X : p(y - x) \leq r \}, \quad r > 0$. 
The topology of an asymmetrically normed space

If $\rho$ is an asymmetric norm on $X$,

$$d_\rho(x, y) = \rho(y - x)$$

defines a quasi-metric $d_\rho$ on $X$ which induces a $T_0$-topology on $X$.

The topology is $T_0$, but need not be $T_1$.

The basic neighbourhoods of $x$ are the open balls $B_\rho^p(x) = \{y \in X : \rho(y - x) < r\}, \quad r > 0$.

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The basic neighbourhoods of $x$ are the open balls $B^O_p(x) = \{ y \in X : p(y - x) < r \}$, $r > 0$.

Addition is jointly continuous, but scalar multiplication is only continuous for multiplication by non-negative scalars.

Later we will also need the closed balls $B^O_I[x] = \{ y \in X : p(y - x) \leq r \}$, $r > 0$. 
Linear maps between asymmetrically normed spaces

Let \((X, p)\) and \((Y, q)\) be asymmetrically normed spaces and \(T : X \rightarrow Y\) be a linear map.

\(T\) is continuous with respect to the topologies induced by \(p\) and \(q\) \(((p, q)\text{-continuous for short})\) iff \(T\) is bounded, i.e. there is a \(C > 0\) such that

\[ q(Tx) \leq Cp(x) \quad \text{for all } x \in X. \]

If this is the case, the infimum of all such constants \(C\) will be denoted by \(||T||:\)

\[ ||T|| = \inf\{C > 0 : q(Tx) \leq Cp(x) \quad \forall x \in X\} = \sup\{q(Tx) : p(x) \leq 1\}. \]
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$$\|T\| = \inf\{C > 0 : q(Tx) \leq Cp(x) \quad \forall x \in X\}$$
$$= \sup\{q(Tx) : p(x) \leq 1\}.$$
Hahn-Banach for asymmetrically normed spaces

Theorem

Let \((X, p)\) be an asymmetrically normed space. Let \(X_0\) be a vector subspace of \(X\) and let \(f_0\) be a bounded linear function from \((X_0, p)\) to \((\mathbb{R}, p_1)\). Then there exists a bounded linear function \(f\) from \(X\) to \(\mathbb{R}\) that extends \(f_0\) and for which

\[
\|f\| = \|f_0\|.
\]
A canonical asymmetric norm on a normed Riesz space

If $X$ is a normed Riesz space (vector lattice), then

$$p(x) = ||x^+|| = ||x \vee 0||, \quad x \in X,$$

defines an asymmetric norm on $X$,

A real linear functional $f$ on $X$ is $(p, p_1)$-continuous iff it is norm bounded and positive (i.e. if $x \geq 0 \Rightarrow f(x) \geq 0$).

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Theorem

Let $X$ be a normed Riesz space, $X_0$ a vector subspace of $X$ and $f_0$ a bounded positive linear functional on $X_0$. Then there is a bounded positive extension $f$ of $f_0$ such that $\|f_0\| = \|f\|$.

The result follows from Banach’s theorem, using the asymmetric norm $p$ defined above as sublinear functional.
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The result follows from Banach’s theorem, using the asymmetric norm $\rho$ defined above as sublinear functional.
Partially ordered normed spaces

If $C$ is a cone in a real vector space $X$, then it induces a partial order $\leq_C$ on $X$ defined by

$$x \leq_C y \iff y - x \in C, \quad x, y \in X.$$  

With this partial order, $X$ becomes a partially ordered vector space.

A partially ordered normed space is a normed space equipped with a partial order induced by a cone.

The cone $C$ is normal if the norm is monotone, i.e.

$$0 \leq_C x \leq_C y \Rightarrow \|x\| \leq \|y\|. $$
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Hahn-Banach for partially ordered normed spaces

A positive bounded linear functional on a linear subspace of a partially ordered normed space need not have an extension to a bounded positive linear functional on the whole space.

**Theorem**

Let $X$ be a real normed space with closed unit ball $B_X$ and ordered by a cone $C_X$, and let $X_0$ be a linear subspace of $X$. A bounded positive linear functional $f_0$ on $X_0$ has a bounded positive extension to $X$ iff $f_0$ is bounded above on $X_0 \cap (B_X - C_X)$. 
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A canonical asymmetric norm on a partially ordered normed space

Let \((X, \| \cdot \|_X)\) be a normed space with closed unit ball \(B_X\), partially ordered by the closed normal cone \(C_X\). For \(x \in X\), put

\[ p_X(x) = \inf \{ \|x + x'\|_X : x' \in C_X \}. \]

Then \(p_X\) is an asymmetric norm on \(X\), and

\[ C_X = \{ x \in X : p_X(-x) = 0 \} \]

The set \(A = B_X - C_X\) is a convex absorbent set such that

\[ \cap \{ \lambda A : \lambda \neq 0 \} = \{0\}, \]

and for \(x \in X\),

\[ p_X(x) = \inf \{ \lambda > 0 : x \in \lambda A \} = p_A(x). \]

Furthermore,

\[ \{ x \in X : p_X(x) < 1 \} \subseteq B_X - C_X \subseteq \{ x \in X : p_X(x) \leq 1 \}. \]
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Asymmetric norms and injectivity

Jurie Conradie

Hahn-Banach theorems

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Injective partially ordered normed spaces

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If the partial order induced by $C_X$ is a lattice order, then $p_X(x) = \| x^+ \|_X$ for $x \in X$.

In particular, if $X = \mathbb{R}$ with its usual norm and order, then $p_X = p_\mathbb{R} = p_1$. 
Hahn-Banach for partially ordered normed spaces (take 2)

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Injectivity

A normed space \( Y \) is 1-injective if the Hahn-Banach theorem for normed spaces remains true when \( \mathbb{R} \) is replaced by \( Y \). More precisely:

**Definition (1-injectivity)**

A normed space \( Y \) is 1-injective if for every normed space \( X \), every linear subspace \( X_0 \) of \( X \) and every continuous linear map \( T_0 : X_0 \rightarrow Y \) there is a continuous linear extension \( T : X \rightarrow Y \) of \( T_0 \) such that \( \| T \| = \| T_0 \| \).

Every 1-injective normed space is a Banach space.

A Banach space \( Y \) is 1-injective iff for every Banach space \( X \) containing \( Y \) as a subspace there is a linear projection \( P \) from \( X \) onto \( Y \) such that \( \| P \| \leq 1 \).
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Hyperconvexity

Definition

A normed space $X$ has the **binary intersection property** if every collection of closed balls in $X$, each pair of which has nonempty intersection, has nonempty intersection.

Definition

A normed $X$ is called **hyperconvex** if for each family $(x_i)_{i \in I}$ of points in $X$ and each family of positive real numbers $(r_i)_{i \in I}$, the conditions $d(x_i, x_j) \leq r_i + r_j$ whenever $i, j \in I$ imply that $\bigcap \{B_{r_i}[x_i] : i \in I\} \neq \emptyset$.

(Here $B_{r_i}[x_i] = \{x \in X : \|x - x_i\| \leq r_i\}$.)

A normed space is hyperconvex iff it has the binary intersection property.
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A normed $X$ is called **hyperconvex** if for each family $(x_i)_{i \in I}$ of points in $X$ and each family of positive real numbers $(r_i)_{i \in I}$, the conditions $d(x_i, x_j) \leq r_i + r_j$ whenever $i, j \in I$ imply that $\bigcap\{B_{r_i}[x_i] : i \in I\} \neq \emptyset$.

(Here $B_{r_i}[x_i] = \{x \in X : \|x - x_i\| \leq r_i\}$.)

A normed space is hyperconvex iff it has the binary intersection property.
Binary intersection property: Example 1

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Theorem ((Nachbin, 1950))

For a Banach space $X$ the following are equivalent:

(a) $X$ is 1-injective.

(b) $X$ has the binary intersection property.

(c) $X$ is a Dedekind-complete vector lattice with an order unit.

(d) $X$ is isometrically isomorphic to the space $C(K)$ of continuous real-valued functions on an extremally disconnected compact Hausdorff space $K$, equipped with the supremum norm.
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Injective asymmetrically normed spaces: Definitions

**Definition**

An asymmetrically normed space \((Y, q)\) is called **1-injective** if for every asymmetrically normed space \((X, p)\) and every linear subspace \(X_0\) of \(X\), every continuous linear map \(T_0 : (X_0, p) \to (Y, q)\) has a continuous extension \(T : X \to Y\) such that \(||T|| \leq ||T_0||\).

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An asymmetrically normed space \((X, p)\) is **Isbell-convex** if for each family \((x_i)_{i \in I}\) of points in \(X\) and families of nonnegative real numbers \((r_i)_{i \in I}\) and \((s_i)_{i \in I}\) it follows from \(p(x_j - x_i) \leq r_i + s_j\) whenever \(i, j \in I\), that

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*If an asymmetrically normed space is Isbell-complete (equivalently, Isbell-convex), it is 1-injective.*

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**Corollary**

*If K is an extremally disconnected compact Hausdorff space and for f ∈ C(K) we put p(f) = sup\{f(t) : t ∈ K\}, then (C(K), p) is Isbell-convex and therefore 1-injective.*
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Injectivity for partially ordered normed spaces

We could require that it must be possible to extend linear maps that are both continuous and positive to maps of the same kind, with preservation of norms.

But with such a definition, $\mathbb{R}$ would not be injective.

**Definition (Riedl (1964))**

A partially ordered normed space $Y$ with closed unit ball $B_Y$, ordered by the closed cone $C_Y$ has **Property P$_1$** if for every partially ordered normed space $X$ with closed unit ball $B_X$, ordered by the closed cone $C_X$, every linear subspace $X_0$ of $X$ and every bounded linear map $T_0 : X_0 \to Y$ such that

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Property $P_1$ and continuity

The inclusion

$$T_0(X_0 \cap (B_X - C_X)) \subseteq \| T_0 \|(B_Y - C_Y),$$

is equivalent to the $(p_X|_{X_0}, p_Y)$-continuity of $T_0$, with $\| T_0 \| = \| T_0 \|$.
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Let \((X, \rho)\) be an asymmetrically normed space.

(a) \(C_\rho = \{ y \in Y : \rho(-y) = 0 \}\) is a cone in \(X\); ordered by \(C_\rho\), \(X\) is a partially ordered vector space. The order \(\leq_\rho\) is defined by \(x \leq_\rho y \iff \rho(x - y) = 0\).

(b) \(\rho^s(y) = \max\{\rho(y), \rho(-y)\}\) defines a norm on \(X\) which is monotone with respect to the order induced by \(C_\rho\).

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Going back:

(d) On \(X\) we can define another asymmetric norm \(\rho_m\) by \(\rho_m(y) = \{\rho^s(y + y') : y' \in C_\rho\}\).

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Continuity

**Proposition**

Let \((X, p)\) and \((Y, q)\) be asymmetrically normed spaces and \(T : X \to Y\) be a linear map. Then \(T\) is continuous with respect to the topologies induced by \(p_m\) and \(q_m\) if and only if \(T(C_p) \subseteq C_q\) and \(T\) is continuous with respect to the topologies induced by \(p^s\) and \(q^s\). In this case

\[ \|T\| = \|T\|. \]
A partial answer to the first question

Theorem

If the asymmetrically normed space \((Y, q)\) is 1-injective and \(q\) is maximal, then there is an extremally disconnected compact Hausdorff space \(K\) such that \((Y, q)\) is asymmetrically isomorphic to \((C(K), p)\), where

\[ p(f) = \sup\{f(t) : t \in K\}, \text{ for } f \in C(K). \]

Two further questions:

3. If \(q\) is a maximal asymmetric norm on \(Y\), is \((Y, q)\) 1-injective?

4. If \((Y, q)\) is 1-injective, is \(q\) maximal?

Taking \(Y = \mathbb{R}\) and \(p(x) = |x|\) shows that the answer to the third question is “no”. 
A partial answer to the first question

**Theorem**

If the asymmetrically normed space $(Y, q)$ is 1-injective and $q$ is maximal, then there is an extremally disconnected compact Hausdorff space $K$ such that $(Y, q)$ is asymmetrically isomorphic to $(C(K), p)$, where 

$$p(f) = \sup\{f(t) : t \in K\}, \text{ for } f \in C(K).$$

Two further questions:

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Links

- The injective hull of an asymmetrically normed space
A partial answer to the first question

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Links
- The injective hull of an asymmetrically normed space
- Injective partially ordered normed spaces
- Injectivity
- Asymmetric norms and injectivity
- Hahn-Banach theorems
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**Theorem**

*If the asymmetrically normed space \((Y, q)\) is 1-injective and \(q\) is maximal, then there is an extremally disconnected compact Hausdorff space \(K\) such that \((Y, q)\) is asymmetrically isomorphic to \((C(K), p)\), where \(p(f) = \sup\{f(t) : t \in K\}\), for \(f \in C(K)\).*

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For any asymmetrically normed space \((X, p)\), is there a ‘smallest’ 1-injective asymmetrically normed space \((Y, q)\) ‘containing’ \(X\)?

More precisely: Is there a 1-injective asymmetrically normed space \((Y, q)\) and an isometric isomorphism \(\Psi\) from \(X\) into \(Y\) such that there is no proper 1-injective subspace of \(Y\) containing \(\Psi(X)\)?

If such a pair \((Y, \Psi)\) exists, we call \(Y\) (somewhat loosely) an injective hull of \(X\).
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Injective hulls

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Some historical remarks about injective hulls

Existence for metric spaces: Isbell (1964)

Existence for Banach spaces: Cohen (1964)

Isbell also showed (non-constructively) that the metric injective hull has the structure of a Banach space.

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Question: Can an injective hull for asymmetrically normed spaces be constructed?
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Question: Can an injective hull for asymmetrically normed spaces be constructed?
The construction: function pairs

\((X, p)\) is an asymmetrically normed space. A function pair \(f = (f_1, f_2)\), where \(f_i : X \to [0, \infty)\) for \(i = 1, 2\), is called \textbf{ample} if \(p(y - x) \leq f_2(x) + f_1(y)\).

\(f\) is \textbf{minimal} whenever \(g = (g_1, g_2)\) is an ample pair such that if \(g_1 \leq f_1, g_2 \leq f_2\), then \(g_1 = f_1, g_2 = f_2\).

The set of all minimal function pairs on \(X\) will be denoted by \(\mathcal{E}(X, p)\).

For every \(z \in X\), we define the minimal function pair \(f_z = (f_{z,1}, f_{z,2})\) by \(f_{z,1}(x) = p(x - z), \quad f_{z,2}(x) = p(z - x)\).

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Asymmetric norms and injectivity
Jurie Conradie
Hahn-Banach theorems
Injective normed spaces
Injective asymmetrically normed spaces
Injective partially ordered normed spaces
Links
The injective hull of an asymmetrically normed space
Scalar multiplication in $\mathcal{E}(X, p)$

Scalar multiplication:
For $\lambda \in \mathbb{R}$ and $f \in \mathcal{E}(X, p)$, we define the function pair $f^\lambda = (f_1^\lambda, f_2^\lambda)$ by

$$f_1^\lambda(x) = \begin{cases} 
\lambda f_1(\lambda^{-1}x) & \text{if } \lambda > 0, \\
p(x) & \text{if } \lambda = 0, \text{ and} \\
|\lambda| f_2(\lambda^{-1}x) & \text{if } \lambda < 0
\end{cases}$$

$$f_2^\lambda(x) = \begin{cases} 
\lambda f_2(\lambda^{-1}x) & \text{if } \lambda > 0, \\
p(-x) & \text{if } \lambda = 0, \\
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Now define $\lambda f = f^\lambda$.

The mapping $x \mapsto f_x$ preserves scalar multiplication.
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Addition in $\mathcal{E}(X, p)$

If $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{E}(X, p), x \in X$ we put
\[ f \oplus g = ((f \oplus g)_1, (f \oplus g)_2), \]
where
\[
(f \oplus g)_1(x) = \sup\{(f_1(x - s) - g_2(s))^+ : s \in X\}
\]
\[
(f \oplus g)_2(x) = \sup\{(f_2(x - s) - g_1(s))^+ : s \in X\}
\]

The map $x \mapsto f_x$ preserves addition.

The only candidate for the additive identity is $f^0 = (f^0_1, f^0_2)$, with
\[ f^0_1(x) = p(x), f^0_2(x) = p(-x). \]

The only candidate for the additive inverse of $f = (f_1, f_2)$ is
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**Asymmetric norms and injectivity**

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The injective hull of an asymmetrically normed space
\( \mathcal{E}(X, p) \) is an Isbell-convex hull of \( X \)

**Theorem**

*If scalar multiplication on \( \mathcal{E}(X, p) \) and addition \( \oplus \) are defined as above, then \( \mathcal{E}(X, p) \) is a vector space and the map \( x \mapsto f_x \) is a linear isomorphism of \( X \) into \( \mathcal{E}(X, p) \).*

**Proposition**

The function \( \tilde{p} : \mathcal{E}(X, p) \to [0, \infty) \) defined by \( \tilde{p}(f) = \tilde{p}((f_1, f_2)) = f_2(0) \) is an asymmetric norm on \( \mathcal{E}(X, p) \) and the map \( x \mapsto f_x \) is an isometry.

**Proposition**

\( (\mathcal{E}(X, p), \tilde{p}) \) is an Isbell-convex asymmetrically normed space containing an isometrically isomorphic copy of \( X \).
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The second question answered

Theorem

An 1-injective asymmetrically normed space \((X, p)\) is Isbell-convex.
The order structure of the injective hull

Proposition

If \((X, p)\) is an asymmetrically normed space and \(\mathcal{E}(X, p)\) its injective hull, equipped with the asymmetric norm \(\tilde{p}\), then the order \(\leq \tilde{p}\) on \(\mathcal{E}(X, p)\) is given by

\[
f \leq \tilde{p} g \iff f_1(x) \geq g_1(x) \text{ for every } x \in X
\]

\[
\iff f_2(x) \leq g_2(x) \text{ for every } x \in X,
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where \(f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{E}(X, p)\).

Theorem

If \((X, p)\) is an asymmetrically normed space, then with the above order, \(\mathcal{E}(X, p)\) is a Dedekind complete vector lattice, and the asymmetric norm \(\tilde{p}\) on \(\mathcal{E}(X, p)\) is maximal.
It follows from the previous theorem that if the asymmetrically normed space \((X, p)\) is 1-injective, \(p\) must be maximal.

This answers Question 4 in the affirmative.

Combined with the previous partial answer to Question 1, this gives

**Theorem**

If the asymmetrically normed space \((Y, q)\) is 1-injective, there is an extremally disconnected compact Hausdorff space \(K\) such that \((Y, q)\) is asymmetrically isomorphic to \((C(K), p)\), where

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The last two questions answered

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