Frolik Decompositions for Lattice-ordered Groups

Gerard Buskes and Robert Redfield

University of Mississippi and Hamilton College

Positivity IX, July 2017

- In 1967 Katětov proved a "purely combinatorial" and "possibly well-known" result.

- In 1967 Katětov proved a "purely combinatorial" and "possibly well-known" result.
- (Katětov in 1967) "However, not having found an explicit reference, the present author preferred publishing a well-known result to undertaking a long search".

- In 1967 Katětov proved a "purely combinatorial" and "possibly well-known" result.
- (Katětov in 1967) "However, not having found an explicit reference, the present author preferred publishing a well-known result to undertaking a long search".
- (Krawczyk, Steprāns): "...., which has been variously attributed to Banach, Erdös and de Bruijn, and Katětov ......" (Continuous colorings of closed graphs, Topology and its Applications 51 (1993),13-26).

- In 1967 Katětov proved a "purely combinatorial" and "possibly well-known" result.
- (Katětov in 1967) "However, not having found an explicit reference, the present author preferred publishing a well-known result to undertaking a long search".
- (Krawczyk, Steprāns): "...., which has been variously attributed to Banach, Erdös and de Bruijn, and Katětov ......" (Continuous colorings of closed graphs, Topology and its Applications 51 (1993),13-26).
- Hell, Nešetřil, Raspaud, and Sopena (2000): "due to Erdös, de Bruijn, Katětov, and Kasteleijn...".

- In 1967 Katětov proved a "purely combinatorial" and "possibly well-known" result.
- (Katětov in 1967) "However, not having found an explicit reference, the present author preferred publishing a well-known result to undertaking a long search".
- (Krawczyk, Steprāns): "...., which has been variously attributed to Banach, Erdös and de Bruijn, and Katětov ......" (Continuous colorings of closed graphs, Topology and its Applications 51 (1993),13-26).
- Hell, Nešetřil, Raspaud, and Sopena (2000): "due to Erdös, de Bruijn, Katětov, and Kasteleijn...".

- In 1967 Katětov proved a "purely combinatorial" and "possibly well-known" result.
- (Katětov in 1967) "However, not having found an explicit reference, the present author preferred publishing a well-known result to undertaking a long search".
- (Krawczyk, Steprāns): "...., which has been variously attributed to Banach, Erdös and de Bruijn, and Katětov ......" (Continuous colorings of closed graphs, Topology and its Applications 51 (1993),13-26).
- Hell, Nešetřil, Raspaud, and Sopena (2000): "due to Erdös, de Bruijn, Katětov, and Kasteleijn...".

Theorem

(Katětov) Let X *be a set and let* $T : X \to X$ *be a map such that* $T(x) = x$ *for no* $x \in X$*. Then there exist pairwise disjoint sets* A_1, A_2, A_3 *such that* $A_1 \cup A_2 \cup A_3 = X$ *and, for all* $i \in \{1, 2, 3\}$, $T(A_i) \cap A_i = \varnothing$.

- In 1968, Frolík, a student of Katětov, provided a model for a similar result under the presence of more structure.

- In 1968, Frolík, a student of Katětov, provided a model for a similar result under the presence of more structure.

- In 1968, Frolǐk, a student of Katětov, provided a model for a similar result under the presence of more structure.

Theorem

If X is a Hausdorff space that is compact, extremally disconnected, and regular and if $T : X \to X$ is a homeomorphism, then there exist pairwise disjoint clopen subsets A_0, A_1, A_2, A_3 such that (a) $A_0 \cup A_1 \cup A_2 \cup A_3 = X$, (b) for all $i \in \{1, 2, 3\}$, $T(A_i) \cap A_i = \varnothing$, and (c) A_0 equals the set of fixed points of T.

- In 1968, Frolı̌k, a student of Katětov, provided a model for a similar result under the presence of more structure.

Theorem

If X is a Hausdorff space that is compact, extremally disconnected, and regular and if $T : X \to X$ is a homeomorphism, then there exist pairwise disjoint clopen subsets A_0, A_1, A_2, A_3 such that (a) $A_0 \cup A_1 \cup A_2 \cup A_3 = X$, (b) for all $i \in \{1, 2, 3\}$, $T(A_i) \cap A_i = \varnothing$, and (c) A_0 equals the set of fixed points of T.

- In 1968 as well, Katětov added a footnote to *his* Theorem in another paper: "As I have learned, it was found earlier by H. Kenyon and published as research problem (American Mathematical Monthly 70 (1963), p. 216); the solution appeared in Vol 71 (1964), p.219)". (with the names of 15 other solvers including Kenyon; the published solution was by I.N. Baker).

- By using Stone's Theorem one can translate Frolík's result into a result for vector lattices as follows.

- By using Stone's Theorem one can translate Frolǐk's result into a result for vector lattices as follows.

- By using Stone's Theorem one can translate Frolík's result into a result for vector lattices as follows.

Theorem

For a topological space X to which Frolík's Theorem applies and for a vector lattice isomorphism $T : C(X) \to C(X)$ there exist pairwise disjoint projection bands B_0, B_1, B_2, B_3 such that (a) $B_0 \vee B_1 \vee B_2 \vee B_3 = C(X)$ in the Boolean algebra of disjoint complements in $C(X)$, (b) $T(B_i) \subseteq B_i^{\perp}$ for all $i \in \{1, 2, 3\}$, and (c) $T(P) \subseteq P$ for each disjoint complement P in B_0.

- By using Stone's Theorem one can translate Frolǐk's result into a result for vector lattices as follows.

Theorem

For a topological space X to which Frolǐk's Theorem applies and for a vector lattice isomorphism $T : C(X) \to C(X)$ there exist pairwise disjoint projection bands B_0, B_1, B_2, B_3 such that (a) $B_0 \vee B_1 \vee B_2 \vee B_3 = C(X)$ in the Boolean algebra of disjoint complements in $C(X)$, (b) $T(B_i) \subseteq B_i^{\perp}$ for all $i \in \{1, 2, 3\}$, and (c) $T(P) \subseteq P$ for each disjoint complement P in B_0.

- T in the above Theorem composes continuous functions with the homeomorphism of Frolik's Theorem.

- By using Stone's Theorem one can translate Frolík's result into a result for vector lattices as follows.

Theorem

For a topological space X to which Frolík's Theorem applies and for a vector lattice isomorphism $T : C(X) \to C(X)$ there exist pairwise disjoint projection bands B_0, B_1, B_2, B_3 such that (a) $B_0 \vee B_1 \vee B_2 \vee B_3 = C(X)$ in the Boolean algebra of disjoint complements in $C(X)$, (b) $T(B_i) \subseteq B_i^{\perp}$ for all $i \in \{1, 2, 3\}$, and (c) $T(P) \subseteq P$ for each disjoint complement P in B_0.

- T in the above Theorem composes continuous functions with the homeomorphism of Frolik's Theorem.
- As such, T has a host of properties: it is order continuous, bijective, bi-disjointness-preserving, order bounded, and it has the Maharam property as well.

- By using Stone's Theorem one can translate Froliǩ's result into a result for vector lattices as follows.

Theorem

For a topological space X to which Froliǩ's Theorem applies and for a vector lattice isomorphism $T: C(X) \to C(X)$ there exist pairwise disjoint projection bands B_0, B_1, B_2, B_3 such that (a) $B_0 \vee B_1 \vee B_2 \vee B_3 = C(X)$ in the Boolean algebra of disjoint complements in $C(X)$, (b) $T(B_i) \subseteq B_i^{\perp}$ for all $i \in \{1, 2, 3\}$, and (c) $T(P) \subseteq P$ for each disjoint complement P in B_0.

- T in the above Theorem composes continuous functions with the homeomorphism of Frolik's Theorem.
- As such, T has a host of properties: it is order continuous, bijective, bi-disjointness-preserving, order bounded, and it has the Maharam property as well.
- In addition, $C(X)$ is Dedekind complete.

- As a consequence, the result below of de Pagter and Schep (2000) extends the result.

- As a consequence, the result below of de Pagter and Schep (2000) extends the result.

- As a consequence, the result below of de Pagter and Schep (2000) extends the result.

Theorem

If E is a Dedekind complete vector lattice and $T : E \to E$ is a linear transformation that is order-bounded, disjointness preserving, Maharam, and perpendicular to the identity transformation, then there exist pairwise disjoint polars B_1, B_2, B_3 such that (a) $B_1 \vee B_2 \vee B_3 = E$ in the Boolean algebra of polars of E, and (b) $T(B_i) \subseteq B_i^{\perp}$ for all $i \in \{1, 2, 3\}$.

- As a consequence, the result below of de Pagter and Schep (2000) extends the result.

Theorem

If E is a Dedekind complete vector lattice and $T : E \to E$ is a linear transformation that is order-bounded, disjointness preserving, Maharam, and perpendicular to the identity transformation, then there exist pairwise disjoint polars B_1, B_2, B_3 such that (a) $B_1 \vee B_2 \vee B_3 = E$ in the Boolean algebra of polars of E, and (b) $T(B_i) \subseteq B_i^{\perp}$ for all $i \in \{1, 2, 3\}$.

- QUESTION: Are there similar decompositions for more general vector lattices E and linear maps $T : E \to E$.?

Definition

Let E be a partially ordered set as well as a group. We call E a partially ordered group if whenever $g_1 \leq g_2$ and $x, y \in E$ then $xg_1y \leq xg_2y$. A partially ordered group E is called a lattice ordered group if E is a lattice under the given ordering.

Definition

Let E be a partially ordered set as well as a group. We call E a partially ordered group if whenever $g_1 \leq g_2$ and $x, y \in E$ then $xg_1y \leq xg_2y$. A partially ordered group E is called a lattice ordered group if E is a lattice under the given ordering.

- From here on E will be a lattice ordered group and we will use additive notation for the group operation. For the identity element of G we will use 0.

Definition

For $A \subseteq E$ we say that $A^{\perp} := \{g \in G : |g| \wedge |a| = 0 \text{ for all } a \in A\}$ is the polar of A.

Definition

For $A \subseteq E$ we say that $A^{\perp} := \{g \in G : |g| \wedge |a| = 0 \text{ for all } a \in A\}$ is the polar of A.

- The set of polars form a Boolean algebra.

Definition

For $A \subseteq E$ we say that $A^{\perp} := \{g \in G : |g| \wedge |a| = 0 \text{ for all } a \in A\}$ is the polar of A.

- The set of polars form a Boolean algebra.

Definition

For $A \subseteq E$ we say that $A^{\perp} := \{g \in G : |g| \wedge |a| = 0 \text{ for all } a \in A\}$ is the polar of A.

- The set of polars form a Boolean algebra.

Theorem

The polars of E form a complete Boolean algebra. The infimum and supremum of a collection of polars are given by the familiar formulas:

$$\bigwedge A_\lambda = \bigcap A_\lambda,\ \bigvee A_\lambda = \left(\bigcup A_\lambda\right)^{\perp\perp},\ \text{and } A^c = A^{\perp}.$$

Definition

For $A \subseteq E$ we say that $A^\perp := \{g \in G : |g| \wedge |a| = 0 \text{ for all } a \in A\}$ is the polar of A.

- The set of polars form a Boolean algebra.

Theorem

The polars of E form a complete Boolean algebra. The infimum and supremum of a collection of polars are given by the familiar formulas:

$$\bigwedge A_\lambda = \bigcap A_\lambda,\ \bigvee A_\lambda = \left(\bigcup A_\lambda\right)^{\perp\perp},\ \textit{and } A^c = A^\perp.$$

- In spite of trying to be careful, we also dictate that the word "band" is an equivalent for the word "polar". A is a polar if and only if $A = A^{\perp\perp}$. Polars are, in particular, convex subgroups.

We will often use the following formula:

$$x = y + z \text{ and } |y| \wedge |z| = 0 \text{ then } |x| = |y| + |z|.$$

Definition

A convex l-subgroup A of an l-group E is called a **cardinal summand** of E if there exists a convex l-subgroup P of E such that $E = A + P$ and $A \cap P = \{0\}$. In that case P is the polar of A.

- Surprisingly, we found that Frolǐk decompositions occur in the setting of quite general lattice ordered groups.

- Surprisingly, we found that Froľik decompositions occur in the setting of quite general lattice ordered groups.

- Surprisingly, we found that Frolík decompositions occur in the setting of quite general lattice ordered groups.

Theorem

If E is a lattice ordered group and $T : E \to E$ is a group homomorphism such that

(1) $T(E)^{\perp\perp}$ *is a* **cardinal summand** *of* E,

(2) $T(E)$ *is a* **polar-dense** l**-subgroup** *of* E,

(3) $|T(x)| \wedge |T(y)| = 0$ *if and only if* $|x| \wedge |y| = 0$ *[i.e.* T **is bi-disjointness-preserving***], and*

(4) *if* B *is a polar and* $x \notin B^{\perp}$, *then* $x = y + z$ *for* $0 \neq y \in B$ *and* $|y| \wedge |z| = 0$ *[*E **has CFC***],*

then there exist pairwise disjoint polars P_0, P_1, P_2, P_3 *such that (a)* $P_0 \vee P_1 \vee P_2 \vee P_3 = E$ *in the Boolean algebra of disjoint complements in* E, *(b)* $T(P_i) \subseteq P_i^{\perp}$ *for all* $i \in \{1, 2, 3\}$, *and* $T(L) \subseteq L$ *for each polar* L *of* P_0.

.

- The lattice-ordered groups E do not need to be Archimedean (nor commutative).

.

- The lattice-ordered groups E do not need to be Archimedean (nor commutative).
- The map T does not need to be order bounded.

.

- The lattice-ordered groups E do not need to be Archimedean (nor commutative).
- The map T does not need to be order bounded.
- T is **bi**-disjointness-preserving, not merely disjointness preserving.

Organization of the talk:

- (I) Definitions and their context needed for The Theorem.

Organization of the talk:

- (I) Definitions and their context needed for The Theorem.
- (II) Proof of The Theorem.

Organization of the talk:

- (I) Definitions and their context needed for The Theorem.
- (II) Proof of The Theorem.
- (III) Application of The Theorem to a Frolík result for **all** Archimedean vector lattices.

Organization of the talk:

- (I) Definitions and their context needed for The Theorem.
- (II) Proof of The Theorem.
- (III) Application of The Theorem to a Frolík result for **all** Archimedean vector lattices.
- (iV) Extensions of The Theorem to situations in which it does not apply.

Organization of the talk:

- (I) Definitions and their context needed for The Theorem.
- (II) Proof of The Theorem.
- (III) Application of The Theorem to a Frolĭk result for **all** Archimedean vector lattices.
- (iV) Extensions of The Theorem to situations in which it does not apply.
- (V) Examples as illustration.

- Definitions of *cardinal summand* and *bi-disjointness-preserving* have already been given.

- Definitions of *cardinal summand* and *bi-disjointness-preserving* have already been given.
- A is called a convex l-subgroup of E when $x \leq y \leq z$ and $x, z \in A$ imply that $y \in A$.

- CONDITION (2): $T(E)$ is a polar dense l-subgroup of E.

- CONDITION (2): $T(E)$ is a polar dense l-subgroup of E.

- CONDITION (2): $T(E)$ is a polar dense l-subgroup of E.

Definition

Let E be an l-group and let A be an l-subgroup of E. We say that A is polar-dense in E if for all $0 < g \in A^{\perp\perp}$ there exists $0 < a \in A$ such that $a^{\perp\perp} \subseteq g^{\perp\perp}$.

Note that:

- Polars are polar dense.

- CONDITION (2): $T(E)$ is a polar dense l-subgroup of E.

Definition

Let E be an l-group and let A be an l-subgroup of E. We say that A is polar-dense in E if for all $0 < g \in A^{\perp\perp}$ there exists $0 < a \in A$ such that $a^{\perp\perp} \subseteq g^{\perp\perp}$.

Note that:

- Polars are polar dense.
- A convex l-subgroup is order dense if and only if $A^{\perp} = \{0\}$.

- CONDITION (2): $T(E)$ is a polar dense l-subgroup of E.

Definition

Let E be an l-group and let A be an l-subgroup of E. We say that A is polar-dense in E if for all $0 < g \in A^{\perp\perp}$ there exists $0 < a \in A$ such that $a^{\perp\perp} \subseteq g^{\perp\perp}$.

Note that:

- Polars are polar dense.
- A convex l-subgroup is order dense if and only if $A^{\perp} = \{0\}$.
- Every order dense l-subgroup is polar dense; the converse of the last statement does not hold:

- CONDITION (2): $T(E)$ is a polar dense l-subgroup of E.

Definition

Let E be an l-group and let A be an l-subgroup of E. We say that A is polar-dense in E if for all $0 < g \in A^{\perp\perp}$ there exists $0 < a \in A$ such that $a^{\perp\perp} \subseteq g^{\perp\perp}$.

Note that:

- Polars are polar dense.
- A convex l-subgroup is order dense if and only if $A^{\perp} = \{0\}$.
- Every order dense l-subgroup is polar dense; the converse of the last statement does not hold:
- $\mathbb{Z}$ is a polar dense l-subgroup of $\mathbb{R}$, but it is not order dense in $\mathbb{R}$.

- CONDITION (4) We say that E has *CFC* (acronym for *Cofinal Family of Components*) when the following holds. If B is a polar of E and $x \notin B^{\perp}$, then $x = y + z$ for $0 \neq y \in B$ and $|y| \wedge |z| = 0$.

- CONDITION (4) We say that E has *CFC* (acronym for *Cofinal Family of Components*) when the following holds. If B is a polar of E and $x \notin B^{\perp}$, then $x = y + z$ for $0 \neq y \in B$ and $|y| \wedge |z| = 0$.
- To illustrate the relative strength of Condition (4), consider the following implications for vector lattices:

$$DC \Longrightarrow PP \Longrightarrow PPP \Longrightarrow SMP \Longrightarrow WFP \Longrightarrow CFC \not\Rightarrow \text{Archimedean}$$

- CONDITION (4) We say that E has *CFC* (acronym for *Cofinal Family of Components*) when the following holds. If B is a polar of E and $x \notin B^{\perp}$, then $x = y + z$ for $0 \neq y \in B$ and $|y| \wedge |z| = 0$.
- To illustrate the relative strength of Condition (4), consider the following implications for vector lattices:

$$DC \Longrightarrow PP \Longrightarrow PPP \Longrightarrow SMP \Longrightarrow WFP \Longrightarrow CFC \not\Rightarrow \text{Archimedean}$$

- *DC*: *Dedekind complete*; *every subset of E that is bounded above has a least upper bound in E.*

- CONDITION (4) We say that E has *CFC* (acronym for *Cofinal Family of Components*) when the following holds. If B is a polar of E and $x \notin B^{\perp}$, then $x = y + z$ for $0 \neq y \in B$ and $|y| \wedge |z| = 0$.
- To illustrate the relative strength of Condition (4), consider the following implications for vector lattices:

$$DC \Longrightarrow PP \Longrightarrow PPP \Longrightarrow SMP \Longrightarrow WFP \Longrightarrow CFC \not\Rightarrow \text{Archimedean}$$

- *DC*: *Dedekind complete*; *every subset of E that is bounded above has a least upper bound in E.*
- *PP*: *Projection Property*; *every polar in E is a cardinal summand in E.*

- *PPP*: *Principal Projection Property*; every principal polar is a cardinal summand.

- *PPP*: *Principal Projection Property*; every principal polar is a cardinal summand.
- *SMP*: *Sufficiently many projections*; Every nonzero band in E contains a nonzero projection band.

- *PPP*: *Principal Projection Property*; every principal polar is a cardinal summand.
- *SMP*: *Sufficiently many projections*; Every nonzero band in E contains a nonzero projection band.
- *WFP*: *Weak Freudenthal Property*; For every $e \in E$, every x in the principal ideal generated by e can be e-uniformly approximated by components of e.

- *PPP*: *Principal Projection Property*; every principal polar is a cardinal summand.
- *SMP*: *Sufficiently many projections*; Every nonzero band in E contains a nonzero projection band.
- *WFP*: *Weak Freudenthal Property*; For every $e \in E$, every x in the principal ideal generated by e can be e-uniformly approximated by components of e.
- *CFC*: *Cofinal Family of Components*; appears for the first time in an analysis of disjointness preserving operators by Abramovich and Kitover (2005).

- *PPP*: *Principal Projection Property*; every principal polar is a cardinal summand.
- *SMP*: *Sufficiently many projections*; Every nonzero band in E contains a nonzero projection band.
- *WFP*: *Weak Freudenthal Property*; For every $e \in E$, every x in the principal ideal generated by e can be e-uniformly approximated by components of e.
- *CFC*: *Cofinal Family of Components*; appears for the first time in an analysis of disjointness preserving operators by Abramovich and Kitover (2005).
- $DC \implies PP \implies PPP \implies SMP$ is ancient history ([L,Z] and [Z]).

- *PPP*: *Principal Projection Property*; every principal polar is a cardinal summand.
- *SMP*: *Sufficiently many projections*; Every nonzero band in E contains a nonzero projection band.
- *WFP*: *Weak Freudenthal Property*; For every $e \in E$, every x in the principal ideal generated by e can be e-uniformly approximated by components of e.
- *CFC*: *Cofinal Family of Components*; appears for the first time in an analysis of disjointness preserving operators by Abramovich and Kitover (2005).
- $DC \implies PP \implies PPP \implies SMP$ is ancient history ([L,Z] and [Z]).
- $SMP \implies WFP$ (Wojtowicz [1992]).

- *PPP*: *Principal Projection Property*; every principal polar is a cardinal summand.
- *SMP*: *Sufficiently many projections*; Every nonzero band in E contains a nonzero projection band.
- *WFP*: *Weak Freudenthal Property*; For every $e \in E$, every x in the principal ideal generated by e can be e-uniformly approximated by components of e.
- *CFC*: *Cofinal Family of Components*; appears for the first time in an analysis of disjointness preserving operators by Abramovich and Kitover (2005).
- $DC \implies PP \implies PPP \implies SMP$ is ancient history ([L,Z] and [Z]).
- $SMP \implies WFP$ (Wojtowicz [1992]).
- $WFP \implies CFC$ (Abramovich, Kitover [2005]).

Example of a space that has *CFC*.

Example

Let E be the set of all functions $f : [0,1) \to \mathbb{R}$ for which there exists a partition $[0,1) = \bigcup_\alpha [p_\alpha, q_\alpha)$ with the property: for each α there exist $a_\alpha, b_\alpha \in \mathbb{R}$ such that $f(x) = a_\alpha x + b_\alpha$ for all $x \in [p_\alpha, q_\alpha)$: the piecewise linear functions. This E has *CFC* but does not have the Projection Property.

- The decomposition that we will create is relative to a convex l-subgroup that is maximal with respect to the following property.

- The decomposition that we will create is relative to a convex l-subgroup that is maximal with respect to the following property.

- The decomposition that we will create is relative to a convex l-subgroup that is maximal with respect to the following property.

Definition

Suppose that E is an l-group and that $T : E \to E$ is a group homomorphism. We say that a convex subgroup I of E is T-polarizing if $T(I) \subset I^{\perp}$.

- The decomposition that we will create is relative to a convex l-subgroup that is maximal with respect to the following property.

Definition

Suppose that E is an l-group and that $T : E \to E$ is a group homomorphism. We say that a convex subgroup I of E is T-polarizing if $T(I) \subset I^{\perp}$.

Definition

If A is an l-subgroup of an l-group E then we write for a subset X of E.

$$X^{\perp_A} = X^{\perp} \cap A.$$

Lemma

$A^{\perp\perp}$ is T-polarizing if A is T-polarizing and T is bi-disjointness preserving.

Proof.

- First we show that for any subset U of E we have that $T(U^{\perp}) = T(U)^{\perp_{T(E)}}$. Suppose that $x \in T(U)^{\perp} \cap T(E)$.

Lemma

$A^{\perp\perp}$ is T-polarizing if A is T-polarizing and T is bi-disjointness preserving.

Proof.

- First we show that for any subset U of E we have that $T(U^{\perp}) = T(U)^{\perp_{T(E)}}$. Suppose that $x \in T(U)^{\perp} \cap T(E)$.
- Then $x = T(h)$ for some $h \in E$ and for all $u \in U$ we have that $|T(u)| \wedge |T(h)| = |T(u)| \wedge |T(x)| = 0$. Then $|u| \wedge |h| = 0$ and hence $h \in U^{\perp}$.

Lemma

$A^{\perp\perp}$ is T-polarizing if A is T-polarizing and T is bi-disjointness preserving.

Proof.

- First we show that for any subset U of E we have that $T(U^{\perp}) = T(U)^{\perp_{T(E)}}$. Suppose that $x \in T(U)^{\perp} \cap T(E)$.
- Then $x = T(h)$ for some $h \in E$ and for all $u \in U$ we have that $|T(u)| \wedge |T(h)| = |T(u)| \wedge |T(x)| = 0$. Then $|u| \wedge |h| = 0$ and hence $h \in U^{\perp}$.
- So $x \in T(U^{\perp})$.

Lemma

$A^{\perp\perp}$ is T-polarizing if A is T-polarizing and T is bi-disjointness preserving.

Proof.

- First we show that for any subset U of E we have that $T(U^{\perp}) = T(U)^{\perp_{T(E)}}$. Suppose that $x \in T(U)^{\perp} \cap T(E)$.
- Then $x = T(h)$ for some $h \in E$ and for all $u \in U$ we have that $|T(u)| \wedge |T(h)| = |T(u)| \wedge |T(x)| = 0$. Then $|u| \wedge |h| = 0$ and hence $h \in U^{\perp}$.
- So $x \in T(U^{\perp})$.
- Conversely, suppose that $h \in U^{\perp}$. Then $|u| \wedge |h| = 0$ for all $u \in U$. Then $|T(u)| \wedge |T(h)| = 0$ since T is disjointness preserving. Then $T(h) \in T(E) \cap T(U)^{\perp}$.

Lemma

$A^{\perp\perp}$ is T-polarizing if A is T-polarizing and T is bi-disjointness preserving.

Proof.

- First we show that for any subset U of E we have that $T(U^{\perp}) = T(U)^{\perp_{T(E)}}$. Suppose that $x \in T(U)^{\perp} \cap T(E)$.
- Then $x = T(h)$ for some $h \in E$ and for all $u \in U$ we have that $|T(u)| \wedge |T(h)| = |T(u)| \wedge |T(x)| = 0$. Then $|u| \wedge |h| = 0$ and hence $h \in U^{\perp}$.
- So $x \in T(U^{\perp})$.
- Conversely, suppose that $h \in U^{\perp}$. Then $|u| \wedge |h| = 0$ for all $u \in U$. Then $|T(u)| \wedge |T(h)| = 0$ since T is disjointness preserving. Then $T(h) \in T(E) \cap T(U)^{\perp}$.
- Now we use the latter observation to prove the Lemma. By applying it twice we get that

$$T(A^{\perp\perp}) = T(A)^{\perp_{T(E)}\perp_{T(E)}}.$$

Lemma

$A^{\perp\perp}$ is T-polarizing if A is T-polarizing and T is bi-disjointness preserving.

Proof.

- First we show that for any subset U of E we have that $T(U^{\perp}) = T(U)^{\perp_{T(E)}}$. Suppose that $x \in T(U)^{\perp} \cap T(E)$.
- Then $x = T(h)$ for some $h \in E$ and for all $u \in U$ we have that $|T(u)| \wedge |T(h)| = |T(u)| \wedge |T(x)| = 0$. Then $|u| \wedge |h| = 0$ and hence $h \in U^{\perp}$.
- So $x \in T(U^{\perp})$.
- Conversely, suppose that $h \in U^{\perp}$. Then $|u| \wedge |h| = 0$ for all $u \in U$. Then $|T(u)| \wedge |T(h)| = 0$ since T is disjointness preserving. Then $T(h) \in T(E) \cap T(U)^{\perp}$.
- Now we use the latter observation to prove the Lemma. By applying it twice we get that

$$T(A^{\perp\perp}) = T(A)^{\perp_{T(E)}\perp_{T(E)}}.$$

- The formal definition of an n-decomposition.

- The formal definition of an n-decomposition.

- The formal definition of an n-decomposition.

Definition

Let E be an l-group; let $T : E \longrightarrow E$ be a group homomorphism; let n be a positive integer; then E is n-decomposable with respect to T if there exist pairwise disjoint polars $P_0, ..., P_n$ of E such that

(1) $E = P_0 \vee ... \vee P_n$ in the Boolean algebra of polars of E,
(2) for all $i = 1, ..., n,\ T(P_i) \subseteq P_i^d$,
(3) T is polar preserving on P_0.

- The polars in the previous definition are called an n-decomposition of E with respect to T.

- The polars in the previous definition are called an n-decomposition of E with respect to T.
- Alternatively to (1) in that definition one can also, equivalently, use:

$$E = (P_0 + \ldots + P_n)^{\perp\perp}.$$

- The polars in the previous definition are called an n-decomposition of E with respect to T.
- Alternatively to (1) in that definition one can also, equivalently, use:

$$E = (P_0 + ... + P_n)^{\perp\perp}.$$

- Here is a list of easy properties that n-decompositions have:

- The polars in the previous definition are called an n-decomposition of E with respect to T.
- Alternatively to (1) in that definition one can also, equivalently, use:

$$E = (P_0 + ... + P_n)^{\perp\perp}.$$

- Here is a list of easy properties that n-decompositions have:
- (1) $T(P_0) \subseteq P_0$,

- The polars in the previous definition are called an *n*-decomposition of E with respect to T.
- Alternatively to (1) in that definition one can also, equivalently, use:

$$E = (P_0 + ... + P_n)^{\perp\perp}.$$

- Here is a list of easy properties that *n*-decompositions have:
- (1) $T(P_0) \subseteq P_0$,
- (2) $P_0 = (P_1 + ... + P_n)^{\perp}$,

- The polars in the previous definition are called an *n*-decomposition of E with respect to T.
- Alternatively to (1) in that definition one can also, equivalently, use:

$$E = (P_0 + ... + P_n)^{\perp\perp}.$$

- Here is a list of easy properties that *n*-decompositions have:
- (1) $T(P_0) \subseteq P_0$,
- (2) $P_0 = (P_1 + ... + P_n)^{\perp}$,
- (3) if $|x| \wedge |y| = 0$ for $x, y \in P_0$ then $|T(x)| \wedge |y| = 0$ as well, and

- The polars in the previous definition are called an n-decomposition of E with respect to T.
- Alternatively to (1) in that definition one can also, equivalently, use:

$$E = (P_0 + ... + P_n)^{\perp\perp}.$$

- Here is a list of easy properties that n-decompositions have:
- (1) $T(P_0) \subseteq P_0$,
- (2) $P_0 = (P_1 + ... + P_n)^{\perp}$,
- (3) if $|x| \wedge |y| = 0$ for $x, y \in P_0$ then $|T(x)| \wedge |y| = 0$ as well, and
- (4) if T is nonzero, then $P_i \neq E$ for all $i \in \{1, ..., n\}$.

- Note that (3) above does not imply that T is an orthomorphism. We will later give an example of a non-order bounded T on an Archimedean vector lattice E and an operator T on E such that E is 1-decomposable with respect to T but T is not order bounded.

- Note that (3) above does not imply that T is an orthomorphism. We will later give an example of a non-order bounded T on an Archimedean vector lattice E and an operator T on E such that E is 1-decomposable with respect to T but T is not order bounded.
- Looking at the result by de Pagter and Schep that we mentioned, we illustrate both, some similarity as well as some difference, between results that they obtained versus our result.

- Note that (3) above does not imply that T is an orthomorphism. We will later give an example of a non-order bounded T on an Archimedean vector lattice E and an operator T on E such that E is 1-decomposable with respect to T but T is not order bounded.
- Looking at the result by de Pagter and Schep that we mentioned, we illustrate both, some similarity as well as some difference, between results that they obtained versus our result.

- Note that (3) above does not imply that T is an orthomorphism. We will later give an example of a non-order bounded T on an Archimedean vector lattice E and an operator T on E such that E is 1-decomposable with respect to T but T is not order bounded.
- Looking at the result by de Pagter and Schep that we mentioned, we illustrate both, some similarity as well as some difference, between results that they obtained versus our result.

Theorem

(de Pagter, Schep; 2000) Let E be a Dedekind complete vector lattice and let $T : E \to E$ be an operator with the following properties: T is order bounded, disjointness preserving, order continuous, and Maharam, and for all $0 \leq z \in E$, $\inf\{T(x) + z - x : 0 \leq x \leq z\} = 0$. Then there exist mutually disjoint bands B_1, B_2, and B_3 such that $B_1 \vee B_2 \vee B_3 = E$ and $T(B_i) \subseteq B_i^{\perp}$ for $1 \leq i \leq 3$.

- CONDITION 2. We now continue to illustrate the conditions of our main result before we start the proof.

- CONDITION 2. We now continue to illustrate the conditions of our main result before we start the proof.

- CONDITION 2. We now continue to illustrate the conditions of our main result before we start the proof.

Definition

Let E be an l-group and let A be an l-subgroup of E. We say that A is polar-dense in E if for all $0 < g \in A^{\perp\perp}$ there exists $0 < a \in A$ such that $a^{\perp\perp} \subseteq g^{\perp\perp}$.

- Of course polars are polar dense, a convex l-subgroup is order dense if and only if $A^{\perp} = \{0\}$, and every order dense l-subgroup is polar dense; the converse of the last statement does not hold: $\mathbb{Z}$ is a polar dense l-subgroup of $\mathbb{R}$, but it is not order dense in $\mathbb{R}$.

SET-UP for the PROOF: Easy facts and a definition.
Let E be an l-group and let $T : E \longrightarrow E$ be a bi-disjointness-preserving group homomorphism. The following facts are easy.

- If A is a T-polarizing convex l-subgroup of E then $A^{\perp\perp}$ also is T-polarizing.

SET-UP for the PROOF: Easy facts and a definition.
Let E be an l-group and let $T : E \longrightarrow E$ be a bi-disjointness-preserving group homomorphism. The following facts are easy.

- If A is a T-polarizing convex l-subgroup of E then $A^{\perp\perp}$ also is T-polarizing.
- The convex l-subgroup $\langle \mathcal{K}(T[T(E)^{\perp}])\rangle$ generated by $T[T(E)^{\perp}]$ is a T-polarizing convex l-subgroup of $T(E)^{\perp\perp}$.

SET-UP for the PROOF: Easy facts and a definition.
Let E be an l-group and let $T : E \longrightarrow E$ be a bi-disjointness-preserving group homomorphism. The following facts are easy.

- If A is a T-polarizing convex l-subgroup of E then $A^{\perp\perp}$ also is T-polarizing.
- The convex l-subgroup $\langle \mathcal{K}(T[T(E)^{\perp}]) \rangle$ generated by $T[T(E)^{\perp}]$ is a T-polarizing convex l-subgroup of $T(E)^{\perp\perp}$.

SET-UP for the PROOF: Easy facts and a definition.
Let E be an l-group and let $T : E \longrightarrow E$ be a bi-disjointness-preserving group homomorphism. The following facts are easy.

- If A is a T-polarizing convex l-subgroup of E then $A^{\perp\perp}$ also is T-polarizing.
- The convex l-subgroup $\langle \mathcal{K}(T[T(E)^{\perp}]) \rangle$ generated by $T[T(E)^{\perp}]$ is a T-polarizing convex l-subgroup of $T(E)^{\perp\perp}$.

Definition: Since $\{0\}$ clearly is a T-polarizing subgroup, we can use the Axiom of Choice to pick a maximal chain $\mathcal{C}$ of T-polarizing convex subgroups of $T(E)^{\perp\perp}$. We define $I_0 = \bigcup \mathcal{C}$.

SET-UP for the PROOF: Easy facts and a definition.
Let E be an l-group and let $T : E \to E$ be a bi-disjointness-preserving group homomorphism. The following facts are easy.

- If A is a T-polarizing convex l-subgroup of E then $A^{\perp\perp}$ also is T-polarizing.
- The convex l-subgroup $\langle \mathcal{K}(T[T(E)^{\perp}]) \rangle$ generated by $T[T(E)^{\perp}]$ is a T-polarizing convex l-subgroup of $T(E)^{\perp\perp}$.

Definition: Since $\{0\}$ clearly is a T-polarizing subgroup, we can use the Axiom of Choice to pick a maximal chain $\mathcal{C}$ of T-polarizing convex subgroups of $T(E)^{\perp\perp}$. We define $I_0 = \bigcup \mathcal{C}$.

- If $T(E)^{\perp\perp}$ is a cardinal summand of E and $\langle \mathcal{K}(T[T(E)^{\perp}] \rangle \in \mathcal{C}$ then $\mathcal{C}$ is a maximal chain of T-polarizing convex l-subgroups of E.

Assuming E is an l-group and T is a bi-disjointness-preserving group homomorphism.

- (Fact 1) I_0 is a T-polarizing convex l-subgroup.

Assuming E is an l-group and T is a bi-disjointness-preserving group homomorphism.

- (Fact 1) I_0 is a T-polarizing convex l-subgroup.
- (Fact 2) I_0 is a polar in E.

Assuming E is an l-group and T is a bi-disjointness-preserving group homomorphism.

- (Fact 1) I_0 is a T-polarizing convex l-subgroup.
- (Fact 2) I_0 is a polar in E.
- (Fact 3) $I_0 \cap T(I_0) = \{0\} = I_0 \cap T^{-1}(I_0)$.

Assuming E is an l-group and T is a bi-disjointness-preserving group homomorphism.

- (Fact 1) I_0 is a T-polarizing convex l-subgroup.
- (Fact 2) I_0 is a polar in E.
- (Fact 3) $I_0 \cap T(I_0) = \{0\} = I_0 \cap T^{-1}(I_0)$.
- (Fact 4) (Assuming that $T(E)$ is a polar-dense l-subgroup.) $T(I_0)$ is a polar in $T(E)$.

Assuming E is an l-group and T is a bi-disjointness-preserving group homomorphism.

- (Fact 1) I_0 is a T-polarizing convex l-subgroup.
- (Fact 2) I_0 is a polar in E.
- (Fact 3) $I_0 \cap T(I_0) = \{0\} = I_0 \cap T^{-1}(I_0)$.
- (Fact 4) (Assuming that $T(E)$ is a polar-dense l-subgroup.) $T(I_0)$ is a polar in $T(E)$.
- (Fact 5) (Assuming that $T(E)$ is polar-dense l-subgroup.) $T^{-1}(I_0)$ is a polar in E and for every polar B in $T(E)^{\perp\perp}$it follows that $T^{-1}(B^{\perp}) = T^{-1}(B)^{\perp}$.

Assuming E is an l-group and T is a bi-disjointness-preserving group homomorphism.

- (Fact 1) I_0 is a T-polarizing convex l-subgroup.
- (Fact 2) I_0 is a polar in E.
- (Fact 3) $I_0 \cap T(I_0) = \{0\} = I_0 \cap T^{-1}(I_0)$.
- (Fact 4) (Assuming that $T(E)$ is a polar-dense l-subgroup.) $T(I_0)$ is a polar in $T(E)$.
- (Fact 5) (Assuming that $T(E)$ is polar-dense l-subgroup.) $T^{-1}(I_0)$ is a polar in E and for every polar B in $T(E)^{\perp\perp}$it follows that $T^{-1}(B^{\perp}) = T^{-1}(B)^{\perp}$.
- (Fact 6) T is one-to-one.

Assuming E is an l-group and T is a bi-disjointness-preserving group homomorphism.

- (Fact 1) I_0 is a T-polarizing convex l-subgroup.
- (Fact 2) I_0 is a polar in E.
- (Fact 3) $I_0 \cap T(I_0) = \{0\} = I_0 \cap T^{-1}(I_0)$.
- (Fact 4) (Assuming that $T(E)$ is a polar-dense l-subgroup.) $T(I_0)$ is a polar in $T(E)$.
- (Fact 5) (Assuming that $T(E)$ is polar-dense l-subgroup.) $T^{-1}(I_0)$ is a polar in E and for every polar B in $T(E)^{\perp\perp}$it follows that $T^{-1}(B^{\perp}) = T^{-1}(B)^{\perp}$.
- (Fact 6) T is one-to-one.
- (Fact 7) (If $T(E)$ is a polar dense l-subgroup and E has CFC.) $T(E)$ has CFC.

Assuming E is an l-group and T is a bi-disjointness-preserving group homomorphism.

- (Fact 1) I_0 is a T-polarizing convex l-subgroup.
- (Fact 2) I_0 is a polar in E.
- (Fact 3) $I_0 \cap T(I_0) = \{0\} = I_0 \cap T^{-1}(I_0)$.
- (Fact 4) (Assuming that $T(E)$ is a polar-dense l-subgroup.) $T(I_0)$ is a polar in $T(E)$.
- (Fact 5) (Assuming that $T(E)$ is polar-dense l-subgroup.) $T^{-1}(I_0)$ is a polar in E and for every polar B in $T(E)^{\perp\perp}$it follows that $T^{-1}(B^{\perp}) = T^{-1}(B)^{\perp}$.
- (Fact 6) T is one-to-one.
- (Fact 7) (If $T(E)$ is a polar dense l-subgroup and E has CFC.) $T(E)$ has CFC.
- (Fact 8) $\langle \mathcal{K}(T[T(E)^{\perp}]) \rangle$ is a T-polarizing convex l-subgroup of $T(E)^{\perp\perp}$.

Assuming E is an l-group and T is a bi-disjointness-preserving group homomorphism.

- (Fact 1) I_0 is a T-polarizing convex l-subgroup.
- (Fact 2) I_0 is a polar in E.
- (Fact 3) $I_0 \cap T(I_0) = \{0\} = I_0 \cap T^{-1}(I_0)$.
- (Fact 4) (Assuming that $T(E)$ is a polar-dense l-subgroup.) $T(I_0)$ is a polar in $T(E)$.
- (Fact 5) (Assuming that $T(E)$ is polar-dense l-subgroup.) $T^{-1}(I_0)$ is a polar in E and for every polar B in $T(E)^{\perp\perp}$it follows that $T^{-1}(B^{\perp}) = T^{-1}(B)^{\perp}$.
- (Fact 6) T is one-to-one.
- (Fact 7) (If $T(E)$ is a polar dense l-subgroup and E has CFC.) $T(E)$ has CFC.
- (Fact 8) $\langle \mathcal{K}(T[T(E)^{\perp}]) \rangle$ is a T-polarizing convex l-subgroup of $T(E)^{\perp\perp}$.
- (Fact 9) If $T(E)^{\perp\perp}$ is a cardinal summand of E and $\langle \mathcal{K}(T[T(E)^{\perp}] \rangle \in \mathcal{C}$ then $\mathcal{C}$ is a maximal chain of T-polarizing convex l-subgroups of E.

Beginning of the Proof of the main result.

Proof.

- $\langle \mathcal{K}(T[T(E)^{\perp}]) \rangle$ is a T-polarizing convex l-subgroup of $T(E)^{\perp\perp}$ by Fact 8. Then choose a maximal chain $\mathcal{C}$ of T-polarizing convex l-subgroups of $T(E)^{\perp\perp}$ that contains $\langle \mathcal{K}(T[T(E)^{\perp}]) \rangle$. By fact 9, $\mathcal{C}$ is a maximal chain of T-polarizing convex l-subgroups of E. We let now l_0 be the union of this chain.

$\square$

Beginning of the Proof of the main result.

Proof.

- $\langle \mathcal{K}(T[T(E)^{\perp}]) \rangle$ is a T-polarizing convex l-subgroup of $T(E)^{\perp\perp}$ by Fact 8. Then choose a maximal chain $\mathcal{C}$ of T-polarizing convex l-subgroups of $T(E)^{\perp\perp}$ that contains $\langle \mathcal{K}(T[T(E)^{\perp}]) \rangle$. By fact 9, $\mathcal{C}$ is a maximal chain of T-polarizing convex l-subgroups of E. We let now l_0 be the union of this chain.
- We will base our 3-decomposition of E on the following subsets.

$$M = l_0 + T(l_0)^{\perp\perp} + (T^{-1}(l_0) \cap T(l_0)^{\perp}) \text{ and}$$
$$F(T) = \{f \in E : T(g) \in g^{\perp\perp} \text{ for all } g \in f^{\perp\perp}\}.$$

$\square$

Theorem

Under the conditions of our main Theorem we have that $M^{\perp} = F(T)$.

Proof in 3 steps:

- STEP 1: For all $f \in F(T)$, $T(f)^{\perp\perp} = f^{\perp\perp}$.

Proof in 3 steps:

- STEP 1: For all $f \in F(T)$, $T(f)^{\perp\perp} = f^{\perp\perp}$.
- STEP 2: $M^{\perp} \supseteq F(T)$.

Proof in 3 steps:

- STEP 1: For all $f \in F(T)$, $T(f)^{\perp\perp} = f^{\perp\perp}$.
- STEP 2: $M^{\perp} \supseteq F(T)$.
- STEP 3: $M^{\perp} \subseteq F(T)$.

PROOF OF STEP 1: We show $T(f)^{\perp\perp} = f^{\perp\perp}$ for all $f \in F(T)$. Let $f \in F(T)$.

- Since $T(f) \in f^{\perp\perp}$ by definition of $F(T)$, it follows that $T(f)^{\perp\perp} \subseteq f^{\perp\perp}$. Thus if $f \in T(f)^{\perp\perp}$ then $T(f)^{\perp\perp} = f^{\perp\perp}$. Thus assume, reasoning by contradiction, that $f \notin T(f)^{\perp\perp}$.

PROOF OF STEP 1: We show $T(f)^{\perp\perp} = f^{\perp\perp}$ for all $f \in F(T)$. Let $f \in F(T)$.

- Since $T(f) \in f^{\perp\perp}$ by definition of $F(T)$, it follows that $T(f)^{\perp\perp} \subseteq f^{\perp\perp}$. Thus if $f \in T(f)^{\perp\perp}$ then $T(f)^{\perp\perp} = f^{\perp\perp}$. Thus assume, reasoning by contradiction, that $f \notin T(f)^{\perp\perp}$.
- Then since E has CFC, there exits g_1 and g_2 where $0 \neq g_1 \in T(f)^{\perp}$, $f = g_1 + g_2$, and $|g_1| \wedge |g_2| = 0$.

PROOF OF STEP 1: We show $T(f)^{\perp\perp} = f^{\perp\perp}$ for all $f \in F(T)$. Let $f \in F(T)$.

- Since $T(f) \in f^{\perp\perp}$ by definition of $F(T)$, it follows that $T(f)^{\perp\perp} \subseteq f^{\perp\perp}$. Thus if $f \in T(f)^{\perp\perp}$ then $T(f)^{\perp\perp} = f^{\perp\perp}$. Thus assume, reasoning by contradiction, that $f \notin T(f)^{\perp\perp}$.
- Then since E has CFC, there exits g_1 and g_2 where $0 \neq g_1 \in T(f)^{\perp}$, $f = g_1 + g_2$, and $|g_1| \wedge |g_2| = 0$.
- Since T is disjointness preserving, $T(f) = T(g_1) + T(g_2)$ and $|T(g_1)| \wedge |T(g_2)| = 0$. Then $|T(f)| = |T(g_1)| + |T(g_2)| \geq |T(g_1)|$ by simple-lattice-arithmetic. But then $T(g_1) \in T(f)^{\perp\perp}$.

PROOF OF STEP 1: We show $T(f)^{\perp\perp} = f^{\perp\perp}$ for all $f \in F(T)$. Let $f \in F(T)$.

- Since $T(f) \in f^{\perp\perp}$ by definition of $F(T)$, it follows that $T(f)^{\perp\perp} \subseteq f^{\perp\perp}$. Thus if $f \in T(f)^{\perp\perp}$ then $T(f)^{\perp\perp} = f^{\perp\perp}$. Thus assume, reasoning by contradiction, that $f \notin T(f)^{\perp\perp}$.
- Then since E has CFC, there exits g_1 and g_2 where $0 \neq g_1 \in T(f)^{\perp}$, $f = g_1 + g_2$, and $|g_1| \wedge |g_2| = 0$.
- Since T is disjointness preserving, $T(f) = T(g_1) + T(g_2)$ and $|T(g_1)| \wedge |T(g_2)| = 0$. Then $|T(f)| = |T(g_1)| + |T(g_2)| \geq |T(g_1)|$ by simple-lattice-arithmetic. But then $T(g_1) \in T(f)^{\perp\perp}$.
- Similarly, $|f| = |g_1| + |g_2| \geq |g_1|$, and since $f \in F(T)$ and $g_1 \in T(f)^{\perp}$we have $T(g_1) \in g_1^{\perp\perp} \subseteq T(f)^{\perp}$.So $T(g_1) \in T(f)^{\perp} \cap T(f)^{\perp\perp}$ and then $T(g_1) = 0$. Then, since T is one-to-one by FACT 6, we have that $g_1 = 0$, which is a contradiction. Thus $T(f)^{\perp\perp} = f^{\perp\perp}$.

- STEP 2: $M^{\perp} \supseteq F(T)$. Let $f \in F(T)$. We will prove that

- STEP 2: $M^{\perp} \supseteq F(T)$. Let $f \in F(T)$. We will prove that
- $$f \in I_0^{\perp} \cap T(I_0)^{\perp} \cap (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}.$$

- STEP 2: $M^{\perp} \supseteq F(T)$. Let $f \in F(T)$. We will prove that
- $$f \in I_0^{\perp} \cap T(I_0)^{\perp} \cap (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}.$$
- **We first show that $f \in I_0^{\perp}$.**

- STEP 2: $M^{\perp} \supseteq F(T)$. Let $f \in F(T)$. We will prove that
- $$f \in I_0^{\perp} \cap T(I_0)^{\perp} \cap (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}.$$
- **We first show that $f \in I_0^{\perp}$.**
- Suppose that $g \in I_0$. Then $|f| \wedge |g| \in I_0$ since I_0 is convex (Fact 1), and since I_0 is T-polarizing (also Fact 1), it follows that $T(|f| \wedge |g|) \in I_0^{\perp}$.

- STEP 2: $M^{\perp} \supseteq F(T)$. Let $f \in F(T)$. We will prove that
- $$f \in I_0^{\perp} \cap T(I_0)^{\perp} \cap (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}.$$
- **We first show that** $f \in I_0^{\perp}$.
- Suppose that $g \in I_0$. Then $|f| \wedge |g| \in I_0$ since I_0 is convex (Fact 1), and since I_0 is T-polarizing (also Fact 1), it follows that $T(|f| \wedge |g|) \in I_0^{\perp}$.
- But also $|f| \wedge |g| \in f^{\perp\perp}$ and thus as well (because $f \in F(T)$), $T(|f| \wedge |g|) \in (|f| \wedge |g|)^{\perp\perp} \subseteq I_0^{\perp\perp}$. Then $T(|f| \wedge |g|) = 0$ and by the injectivity of T (Fact 6), $|f| \wedge |g| = 0$.**So f** $\in \mathbf{I}_0^{\perp}$.

We next show that $f \in T(I_0)^{\perp}$.

- Suppose that $f \notin T(I_0)^{\perp}$. Then $|f| \wedge |T(g)| > 0$ for some $g \in I_0$. Since $T(f)^{\perp\perp} = f^{\perp\perp}$(by Step 1), it then follows that $|T(f)| \wedge |T(g)| > 0$. Since T is bi-disjointness-preserving we have that $|f| \wedge |g| > 0$. Then $g \notin f^{\perp} = \left(f^{\perp\perp}\right)^{\perp}$ by simple-polar-reasoning.

We next show that $f \in T(I_0)^{\perp}$.

- Suppose that $f \notin T(I_0)^{\perp}$. Then $|f| \wedge |T(g)| > 0$ for some $g \in I_0$. Since $T(f)^{\perp\perp} = f^{\perp\perp}$(by Step 1), it then follows that $|T(f)| \wedge |T(g)| > 0$. Since T is bi-disjointness-preserving we have that $|f| \wedge |g| > 0$. Then $g \notin f^{\perp} = \left(f^{\perp\perp}\right)^{\perp}$ by simple-polar-reasoning.
- Since E has *CFC* it follows that $g = g_1 + g_2$ for $0 \neq g_1 \in f^{\perp\perp}$ and $g_2 \in E$ with $|g_1| \wedge |g_2| = 0$.

We next show that $f \in T(I_0)^{\perp}$.

- Suppose that $f \notin T(I_0)^{\perp}$. Then $|f| \wedge |T(g)| > 0$ for some $g \in I_0$. Since $T(f)^{\perp\perp} = f^{\perp\perp}$(by Step 1), it then follows that $|T(f)| \wedge |T(g)| > 0$. Since T is bi-disjointness-preserving we have that $|f| \wedge |g| > 0$. Then $g \notin f^{\perp} = \left(f^{\perp\perp}\right)^{\perp}$ by simple-polar-reasoning.
- Since E has CFC it follows that $g = g_1 + g_2$ for $0 \neq g_1 \in f^{\perp\perp}$ and $g_2 \in E$ with $|g_1| \wedge |g_2| = 0$.
- Then $|T(g_1)| \wedge |T(g_2)| = 0$ and $T(g) = T(g_1) + T(g_2)$.

We next show that $f \in T(I_0)^{\perp}$.

- Suppose that $f \notin T(I_0)^{\perp}$. Then $|f| \wedge |T(g)| > 0$ for some $g \in I_0$. Since $T(f)^{\perp\perp} = f^{\perp\perp}$(by Step 1), it then follows that $|T(f)| \wedge |T(g)| > 0$. Since T is bi-disjointness-preserving we have that $|f| \wedge |g| > 0$. Then $g \notin f^{\perp} = \left(f^{\perp\perp}\right)^{\perp}$ by simple-polar-reasoning.
- Since E has *CFC* it follows that $g = g_1 + g_2$ for $0 \neq g_1 \in f^{\perp\perp}$ and $g_2 \in E$ with $|g_1| \wedge |g_2| = 0$.
- Then $|T(g_1)| \wedge |T(g_2)| = 0$ and $T(g) = T(g_1) + T(g_2)$.
- Since $g_1 \in f^{\perp\perp}$ and $f \in F(T)$, we obtain that $T(g_1) \in g_1^{\perp\perp}$. By simple lattice arithmetic, $|g| = |g_1| + |g_2|$, and then $|g| \geq |g_1|$ and, as well, since I_0 is convex, $g_1 \in I_0$. By Fact 1, $T(g_1) \in I_0^{\perp}$. But since $T(g_1) \in g_1^{\perp\perp}$ and I_0 is a polar (Fact 2), we have that $T(g_1) \in I_0$. Then $T(g_1) \in I_0 \cap I_0^{\perp}$. So $T(g_1) = 0$ and (T is injective) $g_1 = 0$; contradiction so $f \in T(I_0)^{\perp}$.

We next show that $f \in T(I_0)^{\perp}$.

- Suppose that $f \notin T(I_0)^{\perp}$. Then $|f| \wedge |T(g)| > 0$ for some $g \in I_0$. Since $T(f)^{\perp\perp} = f^{\perp\perp}$(by Step 1), it then follows that $|T(f)| \wedge |T(g)| > 0$. Since T is bi-disjointness-preserving we have that $|f| \wedge |g| > 0$. Then $g \notin f^{\perp} = \left(f^{\perp\perp}\right)^{\perp}$ by simple-polar-reasoning.
- Since E has *CFC* it follows that $g = g_1 + g_2$ for $0 \neq g_1 \in f^{\perp\perp}$ and $g_2 \in E$ with $|g_1| \wedge |g_2| = 0$.
- Then $|T(g_1)| \wedge |T(g_2)| = 0$ and $T(g) = T(g_1) + T(g_2)$.
- Since $g_1 \in f^{\perp\perp}$ and $f \in F(T)$, we obtain that $T(g_1) \in g_1^{\perp\perp}$. By simple lattice arithmetic, $|g| = |g_1| + |g_2|$, and then $|g| \geq |g_1|$ and, as well, since I_0 is convex, $g_1 \in I_0$. By Fact 1, $T(g_1) \in I_0^{\perp}$. But since $T(g_1) \in g_1^{\perp\perp}$ and I_0 is a polar (Fact 2), we have that $T(g_1) \in I_0$. Then $T(g_1) \in I_0 \cap I_0^{\perp}$. So $T(g_1) = 0$ and (T is injective) $g_1 = 0$; contradiction so $f \in T(I_0)^{\perp}$.
- Finally, **we show that** $f \in (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}$.

- Suppose that $f \notin (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}$.

- Suppose that $f \notin (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}$.
- By FACT 5, we know that $T^{-1}(I_0)$ is a polar and then $T^{-1}(I_0) \cap T(I_0)^{\perp}$ is a polar as well.

- Suppose that $f \notin (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}$.
- By FACT 5, we know that $T^{-1}(I_0)$ is a polar and then $T^{-1}(I_0) \cap T(I_0)^{\perp}$ is a polar as well.
- Since E has CFC, $f = g_1 + g_2$ where $0 \neq g_1 \in T^{-1}(I_0) \cap T(I_0)^{\perp}$ and $|g_1| \wedge |g_2| = 0$. Then $g_1 \in T^{-1}(I_0)$ and $g_1 \in T(I_0)^{\perp}$, and $|T(g_1)| \wedge |T(g_2)| = 0$ since T is disjointness preserving.

- Suppose that $f \notin (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}$.
- By FACT 5, we know that $T^{-1}(I_0)$ is a polar and then $T^{-1}(I_0) \cap T(I_0)^{\perp}$ is a polar as well.
- Since E has CFC, $f = g_1 + g_2$ where $0 \neq g_1 \in T^{-1}(I_0) \cap T(I_0)^{\perp}$ and $|g_1| \wedge |g_2| = 0$. Then $g_1 \in T^{-1}(I_0)$ and $g_1 \in T(I_0)^{\perp}$, and $|T(g_1)| \wedge |T(g_2)| = 0$ since T is disjointness preserving.
- Of course $T(f) = T(g_1) + T(g_2)$ and $|T(f)| = |T(g_1)| + |T(g_2)|$. Thus $|T(f)| \geq |T(g_1)|$.

- Suppose that $f \notin (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}$.
- By FACT 5, we know that $T^{-1}(I_0)$ is a polar and then $T^{-1}(I_0) \cap T(I_0)^{\perp}$ is a polar as well.
- Since E has CFC, $f = g_1 + g_2$ where $0 \neq g_1 \in T^{-1}(I_0) \cap T(I_0)^{\perp}$ and $|g_1| \wedge |g_2| = 0$. Then $g_1 \in T^{-1}(I_0)$ and $g_1 \in T(I_0)^{\perp}$, and $|T(g_1)| \wedge |T(g_2)| = 0$ since T is disjointness preserving.
- Of course $T(f) = T(g_1) + T(g_2)$ and $|T(f)| = |T(g_1)| + |T(g_2)|$. Thus $|T(f)| \geq |T(g_1)|$.
- Since $f \in F(T)$ (still), we have $T(f) \in f^{\perp\perp}$ and then by the previous line and convexity $T(g_1) \in f^{\perp\perp}$.

- Suppose that $f \notin (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}$.
- By FACT 5, we know that $T^{-1}(I_0)$ is a polar and then $T^{-1}(I_0) \cap T(I_0)^{\perp}$ is a polar as well.
- Since E has CFC, $f = g_1 + g_2$ where $0 \neq g_1 \in T^{-1}(I_0) \cap T(I_0)^{\perp}$ and $|g_1| \wedge |g_2| = 0$. Then $g_1 \in T^{-1}(I_0)$ and $g_1 \in T(I_0)^{\perp}$, and $|T(g_1)| \wedge |T(g_2)| = 0$ since T is disjointness preserving.
- Of course $T(f) = T(g_1) + T(g_2)$ and $|T(f)| = |T(g_1)| + |T(g_2)|$. Thus $|T(f)| \geq |T(g_1)|$.
- Since $f \in F(T)$ (still), we have $T(f) \in f^{\perp\perp}$ and then by the previous line and convexity $T(g_1) \in f^{\perp\perp}$.
- But from STEP 1, $f \in I_0^{\perp}$, so $f^{\perp\perp} \subseteq I_0^{\perp}$and thus $T(g_1) \in I_0^{\perp}$ but also $T(g_1) \in I_0$ (since $g_1 \in T^{-1}(I_0) \cap T(I_0)^{\perp}$. Then $T(g_1) = 0$ and by injectivity $g_1 = 0$, a contradiction.

- Suppose that $f \notin (T^{-1}(I_0) \cap T(I_0)^{\perp})^{\perp}$.
- By FACT 5, we know that $T^{-1}(I_0)$ is a polar and then $T^{-1}(I_0) \cap T(I_0)^{\perp}$ is a polar as well.
- Since E has CFC, $f = g_1 + g_2$ where $0 \neq g_1 \in T^{-1}(I_0) \cap T(I_0)^{\perp}$ and $|g_1| \wedge |g_2| = 0$. Then $g_1 \in T^{-1}(I_0)$ and $g_1 \in T(I_0)^{\perp}$, and $|T(g_1)| \wedge |T(g_2)| = 0$ since T is disjointness preserving.
- Of course $T(f) = T(g_1) + T(g_2)$ and $|T(f)| = |T(g_1)| + |T(g_2)|$. Thus $|T(f)| \geq |T(g_1)|$.
- Since $f \in F(T)$ (still), we have $T(f) \in f^{\perp\perp}$ and then by the previous line and convexity $T(g_1) \in f^{\perp\perp}$.
- But from STEP 1, $f \in I_0^{\perp}$, so $f^{\perp\perp} \subseteq I_0^{\perp}$and thus $T(g_1) \in I_0^{\perp}$ but also $T(g_1) \in I_0$ (since $g_1 \in T^{-1}(I_0) \cap T(I_0)^{\perp}$. Then $T(g_1) = 0$ and by injectivity $g_1 = 0$, a contradiction.
- Thus

$$f \in I_0^{\perp} \cap \left[T(I_0)^{\perp}\right]^{\perp\perp} \cap \left[T^{-1}(I_0) \cap T(I_0)^{\perp}\right]^{\perp} = \ldots$$

$$= \left[I_0 + T(I_0)^{\perp\perp} + (T^{-1}(I_0) \cap T(I_0)^{\perp})\right]^{\perp},$$

- and $\left[I_0 + T(I_0)^{\perp\perp} + (T^{-1}(I_0) \cap T(I_0)^{\perp})\right]^{\perp} = M^{\perp}$.

- and $\left[I_0 + T(I_0)^{\perp\perp} + (T^{-1}(I_0) \cap T(I_0)^{\perp})\right]^{\perp} = M^{\perp}$.
- STEP 3: $M^{\perp} \subseteq F(T)$. Suppose that there exists $f \in M^{\perp}$ that is not in $F(T)$. We will arrive at a contradiction by showing that, under this assumption, $\mathcal{C}$ is not a maximal chain of T-polarizing convex l-subgroups of E.

- and $\left[I_0 + T(I_0)^{\perp\perp} + (T^{-1}(I_0) \cap T(I_0)^{\perp})\right]^{\perp} = M^{\perp}$.
- STEP 3: $M^{\perp} \subseteq F(T)$. Suppose that there exists $f \in M^{\perp}$ that is not in $F(T)$. We will arrive at a contradiction by showing that, under this assumption, $\mathcal{C}$ is not a maximal chain of T-polarizing convex l-subgroups of E.
- There exists $b \in f^{\perp\perp}$ such that $T(b) \notin b^{\perp\perp}$. Then $T(b) \notin b^{\perp_{T(E)}\perp_{T(E)}} = b^{\perp\perp} \cap T(E)$ by an earlier observation in this talk.

- and $\left[I_0 + T(I_0)^{\perp\perp} + (T^{-1}(I_0) \cap T(I_0)^{\perp})\right]^{\perp} = M^{\perp}$.
- STEP 3: $M^{\perp} \subseteq F(T)$. Suppose that there exists $f \in M^{\perp}$ that is not in $F(T)$. We will arrive at a contradiction by showing that, under this assumption, $\mathcal{C}$ is not a maximal chain of T-polarizing convex l-subgroups of E.
- There exists $b \in f^{\perp\perp}$ such that $T(b) \notin b^{\perp\perp}$. Then $T(b) \notin b^{\perp_{T(E)}\perp_{T(E)}} = b^{\perp\perp} \cap T(E)$ by an earlier observation in this talk.
- From FACT 7 we know that $T(E)$ has CFC. Then there exist $r, s \in E$ with $0 \neq T(r) \in b^{\perp_{T(E)}}$ and $|T(r)| \wedge |T(s)| = 0$ and $T(r) + T(s) = T(b)$. Since T is injective, $b = r + s$. Since T is bi-disjointness-preserving $|r| \wedge |s| = 0$. Then

$$|b| \geq |r| \geq r \geq -|r| \geq -|b|$$

and $r \in b^{\perp\perp} \subseteq f^{\perp\perp} \subseteq M^{\perp}$.

- and $\left[I_0 + T(I_0)^{\perp\perp} + (T^{-1}(I_0) \cap T(I_0)^{\perp})\right]^{\perp} = M^{\perp}$.
- STEP 3: $M^{\perp} \subseteq F(T)$. Suppose that there exists $f \in M^{\perp}$ that is not in $F(T)$. We will arrive at a contradiction by showing that, under this assumption, $\mathcal{C}$ is not a maximal chain of T-polarizing convex l-subgroups of E.
- There exists $b \in f^{\perp\perp}$ such that $T(b) \notin b^{\perp\perp}$. Then $T(b) \notin b^{\perp_{T(E)}\perp_{T(E)}} = b^{\perp\perp} \cap T(E)$ by an earlier observation in this talk.
- From FACT 7 we know that $T(E)$ has CFC. Then there exist $r, s \in E$ with $0 \neq T(r) \in b^{\perp_{T(E)}}$ and $|T(r)| \wedge |T(s)| = 0$ and $T(r) + T(s) = T(b)$. Since T is injective, $b = r + s$. Since T is bi-disjointness-preserving $|r| \wedge |s| = 0$. Then

$$|b| \geq |r| \geq r \geq -|r| \geq -|b|$$

  and $r \in b^{\perp\perp} \subseteq f^{\perp\perp} \subseteq M^{\perp}$.
- Since $r^{\perp\perp} \subseteq b^{\perp\perp}$ it follows that $r^{\perp} = r^{\perp\perp\perp} \supseteq b^{\perp\perp\perp} = b^{\perp}$ and $T(r) \in b^{\perp_{T(E)}} \subseteq b^{\perp} \subseteq r^{\perp}$.

Now define

$$J = (I_0 \cup (r^{\perp\perp}))^{\perp\perp}.$$

We will show that $J \neq I_0$ and that J is T-polarizing. Indeed, since $b \in f^{\perp\perp}$ and $f \in M^{\perp}$ it follows that $b \in M^{\perp}$. As $I_0 \subseteq M$, we conclude that $I_0^{\perp} \supseteq M^{\perp}$ and thus $b \in I_o^{\perp}$. Since $|b| \geq |r|$, also $r \in I_o^{\perp}$ and thus $J \neq I_0$.

- To prove that J is T-polarizing, we need to show that $T(J) \subseteq J^{\perp}$. Since $J^{\perp} = I_0^{\perp} \cap r^{\perp}$, the observations that $T(r) \in I_0^{\perp} \cap r^{\perp}$ and $T(I_0) \subseteq I_0^{\perp} \cap r^{\perp}$ will suffice. Most of that is straightforward, except for $T(r) \in I_0^{\perp}$, which we will show next.

- We know that $T(I_0)^{\perp\perp}$ is a polar and we have shown that I_0 and $T^{-1}(I_0)$ are polars. Then

- We know that $T(I_0)^{\perp\perp}$ is a polar and we have shown that I_0 and $T^{-1}(I_0)$ are polars. Then
-

$$
\begin{aligned}
M^{\perp\perp} &\supseteq T^{-1}(I_0) \cap M^{\perp\perp} \\
&= T^{-1}(I_0) \cap \left[I_0 + T(I_0)^{\perp\perp} + (T^{-1}(I_0) \cap T(I_0)^{\perp}) \right]^{\perp\perp} \\
&= T^{-1}(I_0) \cap \left[I_0 \vee T(I_0)^{\perp\perp} \vee (T^{-1}(I_0) \cap T(I_0)^{\perp}) \right] \\
&= \left[T^{-1}(I_0) \cap T(I_0)^{\perp\perp} \right] \vee \left[T^{-}(I_0) \cap (T^{-1}(I_0) \cap T(I_0)^{\perp}) \right] \\
&= \left[T^{-1}(I_0) \cap T(I_0)^{\perp\perp} \right] \vee \left[T^{-}(I_0) \cap T(I_0)^{\perp} \right] \\
&= T^{-1}(I_0) \cap \left[T(I_0)^{\perp\perp} \vee T(I_0)^{\perp} \right] = T^{-1}(I_0),
\end{aligned}
$$

- We know that $T(I_0)^{\perp\perp}$ is a polar and we have shown that I_0 and $T^{-1}(I_0)$ are polars. Then
-

$$
\begin{aligned}
M^{\perp\perp} &\supseteq T^{-1}(I_0) \cap M^{\perp\perp} \\
&= T^{-1}(I_0) \cap \left[I_0 + T(I_0)^{\perp\perp} + (T^{-1}(I_0) \cap T(I_0)^{\perp}) \right]^{\perp\perp} \\
&= T^{-1}(I_0) \cap \left[I_0 \vee T(I_0)^{\perp\perp} \vee (T^{-1}(I_0) \cap T(I_0)^{\perp}) \right] \\
&= \left[T^{-1}(I_0) \cap T(I_0)^{\perp\perp} \right] \vee \left[T^{-}(I_0) \cap (T^{-1}(I_0) \cap T(I_0)^{\perp}) \right] \\
&= \left[T^{-1}(I_0) \cap T(I_0)^{\perp\perp} \right] \vee \left[T^{-}(I_0) \cap T(I_0)^{\perp} \right] \\
&= T^{-1}(I_0) \cap \left[T(I_0)^{\perp\perp} \vee T(I_0)^{\perp} \right] = T^{-1}(I_0),
\end{aligned}
$$

- where we have used Fact 3 in going from line 3 to line 4.

- We know that $T(I_0)^{\perp\perp}$ is a polar and we have shown that I_0 and $T^{-1}(I_0)$ are polars. Then
-

$$
\begin{aligned}
M^{\perp\perp} &\supseteq T^{-1}(I_0) \cap M^{\perp\perp} \\
&= T^{-1}(I_0) \cap \left[I_0 + T(I_0)^{\perp\perp} + (T^{-1}(I_0) \cap T(I_0)^{\perp}) \right]^{\perp\perp} \\
&= T^{-1}(I_0) \cap \left[I_0 \vee T(I_0)^{\perp\perp} \vee (T^{-1}(I_0) \cap T(I_0)^{\perp}) \right] \\
&= \left[T^{-1}(I_0) \cap T(I_0)^{\perp\perp} \right] \vee \left[T^{-}(I_0) \cap (T^{-1}(I_0) \cap T(I_0)^{\perp}) \right] \\
&= \left[T^{-1}(I_0) \cap T(I_0)^{\perp\perp} \right] \vee \left[T^{-}(I_0) \cap T(I_0)^{\perp} \right] \\
&= T^{-1}(I_0) \cap \left[T(I_0)^{\perp\perp} \vee T(I_0)^{\perp} \right] = T^{-1}(I_0),
\end{aligned}
$$

- where we have used Fact 3 in going from line 3 to line 4.
- and then $M^{\perp} \subseteq T^{-1}(I_0)^{\perp}$. Since $r \in M^{\perp}$ it follows that $r \in T^{-1}(I_0)^{\perp}$. From FACT 5, $r \in T^{-1}(I_0^{\perp})$ and then $T(r) \in I_0^{\perp}$, which is what we wanted to show.

- We conclude that J is a T-polarizing ideal that strictly contains I_0.Then $\mathcal{C} \cup \{J\}$is a chain of of T-polarizing convex l-subgroups of E, which is a contradiction. Thus $M^\perp \subset F(T)$.

- We are now in a position to phrase the Frolĭk Theorem for bi-disjointness-preserving operators more precisely than before as follows.

- We are now in a position to phrase the Frolík Theorem for bi-disjointness-preserving operators more precisely than before as follows.

- We are now in a position to phrase the Frolǐk Theorem for bi-disjointness-preserving operators more precisely than before as follows.

Theorem

Let E be a lattice ordered group and $T : E \to E$ a group homomorphism with the following conditions:

(1) $T(E)^{\perp\perp}$ *is a* **cardinal summand** *of* E*;*

(2) $T(E)$ *is a* **polar-dense** l**-subgroup** *of* E*;*

(3) $|T(x)| \wedge |T(y)| = 0$ *if and only if* $|x| \wedge |y| = 0$*;[i.e.* T **is bi-disjointness preserving***]*

(4) *if* B *is a polar and* $x \notin B^{\perp}$*, then* $x = y + z$ *for* $0 \neq y \in B$ *and* $|y| \wedge |z| = 0$*.[*E **has CFC***]*

Then the subsets

$$P_0 = F(T),\ P_1 = I_0,\ P_2 = T(I_0)^{\perp\perp},\ \text{and}\ P_3 = T^{-1}(I_0) \cap T(I_0$$

form a 3-decomposition of E *with respect to* T*.*

Proof of the main Theorem.

- We now know that $P_0 = F(T)$ is a polar.

Proof of the main Theorem.

- We now know that $P_0 = F(T)$ is a polar.
- Of course P_2 is a polar and

Proof of the main Theorem.

- We now know that $P_0 = F(T)$ is a polar.
- Of course P_2 is a polar and
- from FACT 2, I_0 is a polar and from FACT 7, $T^{-1}(I_0)$ is a polar and then so is $T^{-1}(I_0) \cap T(I_0)^{\perp}$.

Proof of the main Theorem.

- We now know that $P_0 = F(T)$ is a polar.
- Of course P_2 is a polar and
- from FACT 2, I_0 is a polar and from FACT 7, $T^{-1}(I_0)$ is a polar and then so is $T^{-1}(I_0) \cap T(I_0)^{\perp}$.
- We have seen that P_0 is disjoint with each of the P_i with $i \in \{1, 2, 3\}$.

Proof of the main Theorem.

- We now know that $P_0 = F(T)$ is a polar.
- Of course P_2 is a polar and
- from FACT 2, I_0 is a polar and from FACT 7, $T^{-1}(I_0)$ is a polar and then so is $T^{-1}(I_0) \cap T(I_0)^{\perp}$.
- We have seen that P_0 is disjoint with each of the P_i with $i \in \{1, 2, 3\}$.
- One has to check that others are pairwise disjoint as well.

Proof of the main Theorem.

- We now know that $P_0 = F(T)$ is a polar.
- Of course P_2 is a polar and
- from FACT 2, I_0 is a polar and from FACT 7, $T^{-1}(I_0)$ is a polar and then so is $T^{-1}(I_0) \cap T(I_0)^{\perp}$.
- We have seen that P_0 is disjoint with each of the P_i with $i \in \{1, 2, 3\}$.
- One has to check that others are pairwise disjoint as well.
- That $P_1 \vee P_2 \vee P_3 = M^{\perp\perp}$ follows from the way we have defined M (and polar arithmetic) and then
$P_0 \vee P_1 \vee P_2 \vee P_3 = F(T) \vee M^{\perp\perp} = E$.

- There are some remaining details.

- There are some remaining details.
- We know that $T(I_0) \subset I_0^{\perp}$. Then $T(P_1) = T(I_0) \subseteq T(I_0)^{\perp\perp} \subseteq I_0^{\perp\perp\perp} = P_1^{\perp}$.

- There are some remaining details.
- We know that $T(I_0) \subset I_0^{\perp}$. Then $T(P_1) = T(I_0) \subseteq T(I_0)^{\perp\perp} \subseteq I_0^{\perp\perp\perp} = P_1^{\perp}$.
- Similar exercises with polar arithmetic and the definitions lead to $T(P_2) \subseteq P_2^{\perp}$ and $T(P_3) \subseteq P_3^{\perp}$.

- There are some remaining details.
- We know that $T(I_0) \subset I_0^{\perp}$. Then $T(P_1) = T(I_0) \subseteq T(I_0)^{\perp\perp} \subseteq I_0^{\perp\perp\perp} = P_1^{\perp}$.
- Similar exercises with polar arithmetic and the definitions lead to $T(P_2) \subseteq P_2^{\perp}$ and $T(P_3) \subseteq P_3^{\perp}$.
- To show that T is polar preserving on P_0, assume that B is a polar in P_0. Let $g \in P_0$. Then $g \in g^{\perp\perp}$.

- There are some remaining details.
- We know that $T(I_0) \subset I_0^{\perp}$. Then $T(P_1) = T(I_0) \subseteq T(I_0)^{\perp\perp} \subseteq I_0^{\perp\perp\perp} = P_1^{\perp}$.
- Similar exercises with polar arithmetic and the definitions lead to $T(P_2) \subseteq P_2^{\perp}$ and $T(P_3) \subseteq P_3^{\perp}$.
- To show that T is polar preserving on P_0, assume that B is a polar in P_0. Let $g \in P_0$. Then $g \in g^{\perp\perp}$.
- Since B is a polar in P_0 and $g \in F(T)$ then $T(g) \in g^{\perp\perp} \subset B^{\perp\perp} = B$. So $T(B) \subset B$ and T is polar preserving on P_0.

- It is easy to derive Frolǐk's original Theorem from our result. The following consequence might well be easier to remember.

- It is easy to derive Frolík's original Theorem from our result. The following consequence might well be easier to remember.

- It is easy to derive Frolǐk's original Theorem from our result. The following consequence might well be easier to remember.

Theorem

Let E be any Archimedean vector lattice and let $T : E \to E$ be an order continuous d-isomorphism. Then there exists a 3-decomposition of E with respect to T.

- **Proof**:

- It is easy to derive Frolĭk's original Theorem from our result. The following consequence might well be easier to remember.

Theorem

Let E be any Archimedean vector lattice and let $T : E \to E$ be an order continuous d-isomorphism. Then there exists a 3-decomposition of E with respect to T.

- **Proof**:
- Every order continuous d-isomorphism is order bounded.

- It is easy to derive Froľik's original Theorem from our result. The following consequence might well be easier to remember.

Theorem

Let E be any Archimedean vector lattice and let $T : E \to E$ be an order continuous d-isomorphism. Then there exists a 3-decomposition of E with respect to T.

- **Proof**:
- Every order continuous d-isomorphism is order bounded.
- Every order continuous (hence order bounded) d-isomorphism extends uniquely to a d-isomorphism on the Dedekind completion E^{δ} of E from a well-known result by Veksler.

- It is easy to derive Frolǐk's original Theorem from our result. The following consequence might well be easier to remember.

Theorem

Let E be any Archimedean vector lattice and let $T : E \to E$ be an order continuous d-isomorphism. Then there exists a 3-decomposition of E with respect to T.

- **Proof**:
- Every order continuous d-isomorphism is order bounded.
- Every order continuous (hence order bounded) d-isomorphism extends uniquely to a d-isomorphism on the Dedekind completion E^{δ} of E from a well-known result by Veksler.
- The conditions of our Frolik l-group result are satisfied.

- It is easy to derive Frolík's original Theorem from our result. The following consequence might well be easier to remember.

Theorem

Let E be any Archimedean vector lattice and let $T : E \to E$ be an order continuous d-isomorphism. Then there exists a 3-decomposition of E with respect to T.

- **Proof**:
- Every order continuous d-isomorphism is order bounded.
- Every order continuous (hence order bounded) d-isomorphism extends uniquely to a d-isomorphism on the Dedekind completion E^δ of E from a well-known result by Veksler.
- The conditions of our Frolik l-group result are satisfied.
- The intersection of the decomposition of E^δ with E provides the decomposition for E.

We present just one example of many opportunities to use the Theorem where it does not immediately apply. Then we present a couple of examples as food for thought.

Theorem

Let E be an l-group. Suppose that $T : E \to E$ is a bi-disjointness-preserving group homomorphism such that $T(E)$ is a polar dense l-subgroup of E. If E has a polar dense l-subgroup A such that $T(A) \subseteq A$, $A^{\perp\perp} = E$, and A is 3-decomposable with respect to $T_{|A}$ then E is 3-decomposable with respect to T.

Example 1:

- There exists a vector lattice E that is not Archimedean but it does have CFC, together with a bi-disjointness preserving linear bijection $T : E \longrightarrow E$ that is not order bounded and our Theorem applies:

Example 1:

- There exists a vector lattice E that is not Archimedean but it does have CFC, together with a bi-disjointness preserving linear bijection $T : E \to E$ that is not order bounded and our Theorem applies:
- The vector lattice we consider is the lexicographically ordered

$$E = \sum_{\mathbb{Q}^{>}} \mathbb{R}$$

Example 1:

- There exists a vector lattice E that is not Archimedean but it does have CFC, together with a bi-disjointness preserving linear bijection $T : E \longrightarrow E$ that is not order bounded and our Theorem applies:
- The vector lattice we consider is the lexicographically ordered

$$E = \sum_{\mathbb{Q}^{>}} \mathbb{R}$$

- and $T : E \longrightarrow E$ is defined by $T(f)_q = f_{q^{-1}}$.

Example 1:

- There exists a vector lattice E that is not Archimedean but it does have CFC, together with a bi-disjointness preserving linear bijection $T : E \longrightarrow E$ that is not order bounded and our Theorem applies:
- The vector lattice we consider is the lexicographically ordered

$$E = \sum_{\mathbb{Q}^{>}} \mathbb{R}$$

- and $T : E \longrightarrow E$ is defined by $T(f)_q = f_{q^{-1}}$.
- T is a linear bijection, so $T(E) = E$ and $T(E)^{\perp\perp}$ is a cardinal summand.

Example 1:

- There exists a vector lattice E that is not Archimedean but it does have CFC, together with a bi-disjointness preserving linear bijection $T : E \to E$ that is not order bounded and our Theorem applies:
- The vector lattice we consider is the lexicographically ordered

$$E = \sum_{\mathbb{Q}^>} \mathbb{R}$$

- and $T : E \to E$ is defined by $T(f)_q = f_{q^{-1}}$.
- T is a linear bijection, so $T(E) = E$ and $T(E)^{\perp\perp}$ is a cardinal summand.
- It is easy to see that E has CFC since it is totally ordered. Our decomposition result applies but T is easily seen not to be order bounded.

- There exists a disjointness preserving map T on the piecewise linear functions for which there is a 1-decomposition, T is not bi-disjointness-preserving, and T is not order bounded, though the piecewise linear functions do have property CFC.

- There exists a disjointness preserving map T on the piecewise linear functions for which there is a 1-decomposition, T is not bi-disjointness-preserving, and T is not order bounded, though the piecewise linear functions do have property CFC.
- The map T is the familiar right-hand derivative

$$T(f)(x) = f'(x).$$

- There exists a disjointness preserving map T on the piecewise linear functions for which there is a 1-decomposition, T is not bi-disjointness-preserving, and T is not order bounded, though the piecewise linear functions do have property CFC.
- The map T is the familiar right-hand derivative

$$T(f)(x) = f^{'}(x).$$

- T is linear and disjointness preserving and $T^2(E) = 0$. Then T is not bi-disjointness preserving. $F(T) = E$ and $P_0 = E$, $P_1 = \{0\}$ form a 1-decomposition.

.

Example 3:

- It is easy to come up with a compact regular topological space X and a homeomorphism $\tau : X \to X$ for which there is no n-decomposition for any n.

.

Example 3:

- It is easy to come up with a compact regular topological space X and a homeomorphism $\tau : X \to X$ for which there is no n-decomposition for any n.
- Take $X = \{\frac{1}{n} : 0 \neq n \in \mathbb{Z}\} \cup \{0\}$ and define $\tau : X \to X$ by $\tau(x) = -x$. Then the set of fixed points is $\{0\}$, which is closed but not open. Frolik's Theorem does not apply.

Thank you!