New Examples of Non-reflexive Grothendieck Spaces

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1. Definition and Examples

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- $\ell_\infty \hat{\otimes}_\pi \ell_p$, $2 < p < \infty$ and $\ell_\infty \hat{\otimes}_\pi T^*$, T^* is the original Tsirelson space (González and Gutiérrez, 1995).

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- $\ell_\infty \hat{\otimes}_\pi \ell_p$, $2 < p < \infty$ and $\ell_\infty \hat{\otimes}_\pi T^*$, T^* is the original Tsirelson space (González and Gutiérrez, 1995).
- the weak L^p-space $L^{p,\infty}$, $1 < p < \infty$ (Lotz, 2010).

2. Grothendieck and Fremlin

Projective Tensor Products

2. Grothendieck Tensor Product

Definition:

Let E and F be Banach spaces. The projective tensor norm on $E \otimes F$ is defined by

$$\|u\|_\pi = \inf\left\{\sum_{k=1}^{n} \|x_k\| \cdot \|y_k\| : x_k \in E, y_k \in F, u = \sum_{k=1}^{n} x_k \otimes y_k\right\}.$$

Let $E \hat{\otimes}_\pi F$ denote the completion of $E \otimes F$ with respect to $\| \cdot \|_\pi$, called the Grothendieck projective tensor product.

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- E, F are Banach lattices $\not\Rightarrow E \hat{\otimes}_\pi F$ is a Banach lattice.
- $\ell_2 \hat{\otimes}_\pi \ell_2$ is not a Banach lattice.

2. Fremlin Tensor Product

Definition (Fremlin, 1974):

Let E and F be Banach lattices, $E\bar{\otimes}F$ be the Risez space tensor product of E and F with the positive cone

$$C_p = \Big\{ \sum_{k=1}^{n} x_k \otimes y_k : n \in \mathbb{N}, x_k \in E^+, y_k \in F^+ \Big\}.$$

The positive projective tensor norm on $E\bar{\otimes}F$ is defined by

$$\|u\|_{|\pi|} = \inf \Big\{ \sum_{k=1}^{n} \|x_k\| \cdot \|y_k\| : x_k \in E^+, y_k \in F^+, |u| \leqslant \sum_{k=1}^{n} x_k \otimes y_k \Big\}.$$

Let $E\hat{\otimes}_{|\pi|}F$ denote the completion of $E\bar{\otimes}F$ with respect to $\|\cdot\|_{|\pi|}$, called the Fremlin projective tensor product.

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Theorem (Cartwright and Lotz, 1975):

Let E and F be Banach lattices. If $\mathcal{L}(E, F)$ and $\mathcal{L}^r(E, F)$ are isometrically isomorphic, then either E is an AL-space or F is an AM-space.

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- $(E\hat{\otimes}_\pi F)^* = \mathcal{L}(E, F^*), \quad (E\hat{\otimes}_{|\pi|} F)^* = \mathcal{L}^r(E, F^*)$.

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- $(E\hat{\otimes}_\pi F)^* = \mathcal{L}(E, F^*), \quad (E\hat{\otimes}_{|\pi|}F)^* = \mathcal{L}^r(E, F^*)$.
- If E and F are Banach lattices, then $E\hat{\otimes}_\pi F$ is isometrically isomorphic to $E\hat{\otimes}_{|\pi|}F$ if and only if either E or F is isometrically isomorphic to an AL-space.

3. Grothendieck Projective Tensor Product being a Grothendieck space

3. Grothendieck Tensor Product being a G-space

Theorem (González and Gutiérrez, 1995):

- Let E and F be Banach spaces. If E is a G-space, F is reflexive, and $\mathcal{L}(E, F^*) = \mathcal{K}(E, F^*)$, then $E \hat{\otimes}_\pi F$ is a G-space.

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- $\ell_\infty\hat{\otimes}_\pi\ell_p$ is a G-space for $2 < p < \infty$.
- $\ell_\infty\hat{\otimes}_\pi T^*$ is a G-space, T^* is the original Tsirelson space.
- Let T be the dual of T^*. Then $\mathcal{L}(\ell_\infty, T) = \mathcal{K}(\ell_\infty, T)$.

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Question:

For what Banach lattices E and F, the Fremlin projective tensor product $E\hat{\otimes}_{|\pi|}F$ can be a G-space?

4. Fremlin Projective Tensor Product

being a Grothendieck Space

4. Fremlin Tensor Product being a G-space

Let λ be a Banach sequence lattice, X be a Banach lattice. Define

$$\lambda_\varepsilon(X) = \left\{ \bar{x} = (x_i)_i \in X^{\mathbb{N}} : \big(x^*(|x_i|)\big)_i \in \lambda, \ \ \forall\, x^* \in X^{*+} \right\}$$

and

$$\|\bar{x}\|_{\lambda_\varepsilon(X)} = \sup \left\{ \left\| \big(x^*(|x_i|)\big)_i \right\|_\lambda : \ x^* \in B_{X^{*+}} \right\}.$$

Then $\lambda_\varepsilon(X)$ is a Banach lattice. Let

$$\lambda_{\varepsilon,0}(X) = \left\{ \bar{x} \in \lambda_\varepsilon(X) : \ \lim_n \left\| (0, \ldots, 0, x_n, x_{n+1}, \ldots) \right\|_{\lambda_\varepsilon(X)} = 0 \right\}.$$

Then $\lambda_{\varepsilon,0}(X)$ is an ideal of $\lambda_\varepsilon(X)$.

4. Fremlin Tensor Product being a G-space

Let λ' be the Köthe dual of λ. Define

$$\lambda_\pi(X) = \left\{ \bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} x_i^*(|x_i|) < +\infty, \ \forall (x_i^*)_i \in \lambda'_\varepsilon(X^*)^+ \right\}$$

and

$$\|\bar{x}\|_{\lambda_\pi(X)} = \sup \left\{ \sum_{i=1}^{\infty} x_i^*(|x_i|) : \ (x_i^*)_i \in B_{\lambda'_\varepsilon(X^*)^+} \right\}.$$

Then $\lambda_\pi(X)$ is a Banach lattice. Let

$$\lambda_{\pi,0}(X) = \left\{ \bar{x} \in \lambda_\pi(X) : \ \lim_n \left\| (0, \ldots, 0, x_n, x_{n+1}, \ldots) \right\|_{\lambda_\pi(X)} = 0 \right\}.$$

Then $\lambda_{\pi,0}(X)$ is an ideal of $\lambda_\pi(X)$.

4. Fremlin Tensor Product being a G-space

Theorem (Bu and Wong, 2012):

$\lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*)$ and $\lambda_{\pi,0}(X)^* = \lambda'_\varepsilon(X^*)$.

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Theorem (Bu and Wong, 2012):

$\lambda_{\varepsilon,0}(X)^* = \lambda'_\pi(X^*)$ and $\lambda_{\pi,0}(X)^* = \lambda'_\varepsilon(X^*)$.

Lemma 1:

Let λ' be σ-order continuous and let $\bar{x}^{(n)}, \bar{x}^{(0)} \in \lambda_{\varepsilon,0}(X)$. Then $\lim_n \bar{x}^{(n)} = \bar{x}^{(0)}$ weakly in $\lambda_{\varepsilon,0}(X)$ if and only if $\lim_n x_i^{(n)} = x_i^{(0)}$ weakly in X and $\sup_n \|\bar{x}^{(n)}\|_{\lambda_\varepsilon(X)} < \infty$.

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Lemma 2:

Let λ be σ-order continuous and let $\bar{x}^{*(n)}, \bar{x}^{*(0)} \in \lambda_{\pi,0}(X)^*$. Then $\lim_n \bar{x}^{*(n)} = \bar{x}^{*(0)}$ weak* in $\lambda_{\pi,0}(X)^*$ if and only if $\lim_n x_i^{*(n)} = x_i^{*(0)}$ weak* in X^* and $\sup_n \|\bar{x}^{*(n)}\|_{\lambda'_\varepsilon(X^*)} < \infty$.

4. Fremlin Tensor Product being a G-space

Lemma 3:

Let λ be a reflexive Banach sequence lattice. Then $\lambda_{\pi,0}(X)$ is a G-space if and only if X is a G-space and $\lambda'_{\varepsilon}(X^*) = \lambda'_{\varepsilon,0}(X^*)$.

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- If λ is σ-order continuous then $\lambda_{\pi}(X) = \lambda_{\pi,0}(X)$.
- If λ' is σ-order continuous then $\lambda'_{\varepsilon}(X^*) = \lambda'_{\varepsilon,0}(X^*)$ if and only if every positive linear operator from λ_0 to X^* is compact.

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Let λ be a reflexive Banach sequence lattice. Then $\lambda_{\pi,0}(X)$ is a G-space if and only if X is a G-space and every positive linear operator from λ to X^* is compact.

Theorem (Bu and Buskes, 2009):

If λ is σ-order continuous then $\lambda\hat{\otimes}_{|\pi|}X$ is lattice isometric to $\lambda_{\pi,0}(X)$.

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Theorem (Bu and Buskes, 2009):

If λ is σ-order continuous then $\lambda\hat{\otimes}_{|\pi|}X$ is lattice isometric to $\lambda_{\pi,0}(X)$.

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Let λ be a reflexive Banach sequence lattice and X be a Banach lattice. Then $\lambda\hat{\otimes}_{|\pi|}X$ is a G-space if and only if X is a G-space and every positive linear operator from λ to X^* is compact.

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Let T^* be the original Tsirelson space and T be the dual of T^*. Then $\mathcal{L}(\ell_\infty, T) = \mathcal{K}(\ell_\infty, T)$.

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Let T^* be the original Tsirelson space and T be the dual of T^*. Then $\mathcal{L}(\ell_\infty, T) = \mathcal{K}(\ell_\infty, T)$. Thus $\mathcal{L}(T^*, \ell_\infty^*) = \mathcal{K}(T^*, \ell_\infty^*)$.

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New Example 1:

The Fremlin projective tensor product $\ell_\infty\hat{\otimes}_{|\pi|}T^*$ is a G-space.

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Fact:

Let $1 < q < \infty$ and X be an AM-space with an order unit. Then every positive linear operator from X to ℓ_q is compact.

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Let λ be a reflexive Banach sequence lattice and X be a Banach lattice. Then $\lambda \hat{\otimes}_{|\pi|} X$ is a G-space if and only if X is a G-space and every positive linear operator from λ to X^* is compact.

Fact:

Let $1 < q < \infty$ and X be an AM-space with an order unit. Then every positive linear operator from X to ℓ_q is compact.

Theorem 2:

Let $1 < p < \infty$ and X be both an AM-space with an order unit and a G-space. Then $\ell_p \hat{\otimes}_{|\pi|} X$ is a G-space.

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Theorem 2:

Let $1 < p < \infty$ and X be both an AM-space with an order unit and a G-space. Then $\ell_p \hat{\otimes}_{|\pi|} X$ is a G-space.

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If K is a compact stonean space, a compact σ-stonean space, or a F-space, then $C(K)$ is a G-space.

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New Example 2:

Let $1 < p < \infty$ and K be a compact stonean space, a compact σ-stonean space, or a F-space. Then the Fremlin projective tensor product $\ell_p \hat{\otimes}_{|\pi|} C(K)$ is a G-space.

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The Fremlin projective tensor product $\ell_p \hat{\otimes}_{|\pi|} \ell_\infty$ is a G-space for $1 < p < \infty$.

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Ole Example (González and Gutiérrez, 1995):

The Grothendieck projective tensor product $\ell_p \hat{\otimes}_\pi \ell_\infty$ is a G-space if and only if $2 < p < \infty$.

Thank you for your attention!!