Automatic regularity of algebra homomorphisms

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The problem

Given Banach lattice algebras $\mathcal{A}$ & $\mathcal{B}$ and a continuous algebra homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$, find conditions on $\mathcal{A}$ and $\mathcal{B}$ that imply regularity of Θ.

There are a few results in the case where $\mathcal{A}$ & $\mathcal{B}$ are semisimple f-algebras.

We shall be interested in the case in which $\mathcal{A}$ & $\mathcal{B}$ are Riesz subalgebras of $\mathcal{L}^r(X)$ & $\mathcal{L}^r(Y)$, respectively, with X & Y Banach lattices.

A simple case:

Let $\mathcal{A}^r(\ell_2)$ be the closure of the finite rank operators in $\mathcal{L}^r(\ell_2)$ and let $\Theta : \mathcal{A}^r(\ell_2) \to \mathcal{A}^r(\ell_2)$ be an algebra automorphism. Then Θ is regular.

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- There is $(n_i) \subset \mathbb{N}$ strictly increasing such that if P_1 is the natural projection onto $[e_{n_i}]$ then UP_1 is regular.
- Similarly, there is an infinite natural projection $P_2 : \ell_2 \to \ell_2$ such that $P_2 U^{-1}$ is regular.
- There are $R_i, S_i \in \mathcal{L}^r(\ell_2)$ such that $R_i S_i = \mathrm{id}$ & $S_i R_i = P_i$ $(i = 1, 2)$.

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- It only remains to note all the following are regular:
  $T \mapsto S_1 T R_2$;
  $T \mapsto \Theta(P_1 T P_2)$;
  $T \mapsto \lim_m \lim_n \Theta(R_1 E_m) T \Theta(E_n S_2)$.

Let $\mathcal{A}$ be a Riesz and topological algebra. We shall say:

- $(a_n) \subset \mathcal{A}$ is **convergence preserving** if $\forall (b_n) \subset \mathcal{A}$ convergent, $(a_n b_n)$ & $(b_n a_n)$ converge.
- $(a_n) \subset \mathcal{A}$ **factors through** $(b_n) \subset \mathcal{A}$ if there are bounded, convergence preserving sequences $(u_n), (v_n) \subset \mathcal{A}_+$ such that $a_n = u_n b_{k_n} v_n$ $(n \in \mathbb{N})$, where $(k_n) \subset \mathbb{N}$ is increasing.
- an idempotent $p \in \mathcal{A}$ is **o-minimal** if it is positive and $p\mathcal{A}p = \mathbb{K}p$.

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Recall $\mathcal{A}$ is **semiprime** if $\{0\}$ is the only two-sided ideal $\mathcal{I}$ of $\mathcal{A}$ such that $\mathcal{I}^2 = \{0\}$.

Theorem

Let $\mathcal{A}$ & $\mathcal{B}$ be Riesz and topological algebras, and $\Theta : \mathcal{A} \to \mathcal{B}$ a continuous algebra homomorphism.
Suppose
– $\mathcal{A}_+$ is closed and $\mathcal{A}$ is semiprime;
– there is an order complete, locally solid topological Riesz algebra $\widetilde{\mathcal{B}}$ such that $\mathcal{B}$ is an ideal in $\widetilde{\mathcal{B}}$ and $\mathcal{A} \to \widetilde{\mathcal{B}}$, $a \mapsto \Theta(a)$, is compact;
– there are a bounded approx. identity $(e_n) \subset \mathcal{A}$, a sequence of mutually orthogonal o-minimal idempotents $(p_i) \subset \mathcal{A}$, and disjoint sequences $(P_i) \subset \mathcal{M}_l(\widetilde{\mathcal{B}})$ & $(Q_i) \subset \mathcal{M}_r(\widetilde{\mathcal{B}})$ of continuous band projections such that

- *$(\sum_{i=1}^{n} p_i)$ is bounded, convergence preserving and (e_n) factors through it;*
- *$\sum_i |P_i(\Theta(p_i)) - \Theta(p_i)||\Theta(p_i)|$ & $\sum_i |\Theta(p_i)||Q_i(\Theta(p_i)) - \Theta(p_i)|$ exist in $\mathcal{B}$.*

Then Θ is regular.

In fact, if $(u_i),(v_i) \subset \mathcal{A}_+$ are convergence preserving such that $e_i = u_i \pi_{k_i} v_i$ $(i \in \mathbb{N})$, where $\pi_n = \sum_{i=1}^{n} p_i$ $(n \in \mathbb{N})$, then there are $\widetilde{b}_u, \widetilde{b}_v \in \widetilde{\mathcal{B}}$ such that

$$\Theta(a) = \widetilde{b}_u \Psi\Big(\lim_i \lim_j v_i a u_j \Big) \widetilde{b}_v, \forall\, a \in \mathcal{A},$$

where

$$\Psi : \mathcal{A} \to \mathcal{B},\ a \mapsto \Theta\Big(\lim_m \lim_n \pi_m a \pi_n \Big).$$

Now let $\mathcal{A} \subseteq \mathcal{L}^r(X)$ and $\mathcal{B} \subseteq \mathcal{L}^r(Y)$, with X and Y Banach lattices.

Given such $\mathcal{A}$ and $\mathcal{B}$, we should like to find conditions (as weak as possible) on X and Y that force an algebra homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$ (possibly with some additional properties) to be regular.

We shall start with the 'atomic' case and then extend the results to the 'non-atomic' case by means of 'direct limits'.

The atomic case

We shall say a separable atomic Banach lattice X satisfies $\star$ if there is $\mu \geq 1$ and an arrangement of its normalized atoms $\{x_i : i \in \mathbb{N}\}$ such that, $\forall n \in \mathbb{N}$, if $l_1 < l_2 < \cdots < l_n$ satisfy $(x_i)_{i=1}^n \sim (x_{l_i})_{i=1}^n$ then $\forall k$ there is $l_{n+1} \geq l_n + k$ such that $(x_i)_{i=1}^{n+1} \sim (x_{l_i})_{i=1}^{n+1}$.

Here $(x_i)_{i=1}^n \sim (y_i)_{i=1}^n$ stands for

$$\frac{1}{\mu}\Big\|\sum_1^n a_i x_i\Big\| \leq \Big\|\sum_1^n a_i y_i\Big\| \leq \mu\Big\|\sum_1^n a_i x_i\Big\|$$

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Example: Any Banach lattice of the form $\left(\bigoplus_{i=1}^\infty X_i\right)_{(e_i)}$, with (e_i) a 1-unconditional basis and (X_i) a sequence of Banach lattices with subsymmetric bases, satisfies $\star$.

We shall say

- $\mathcal{A} \subseteq \mathcal{L}^r(X)$ is a **Riesz operator subalgebra** if it is a subalgebra and a Riesz subspace of $\mathcal{L}^r(X)$ containing $\mathcal{F}(X)$;
- $\mathcal{B} \subseteq \mathcal{L}^r(Y)$ is a **Riesz algebra ideal** if it is an algebra ideal and a Riesz subspace of $\mathcal{L}^r(Y)$.

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Theorem

Let X be a separable atomic Banach lattice satisfying $\star$ and let Y be atomic and reflexive. Let $\mathcal{A} \subseteq \mathcal{L}^r(X)$ be a Riesz operator subalgebra, let $\mathcal{B} \subseteq \mathcal{L}^r(Y)$ be a Riesz algebra ideal and let $\Theta : \mathcal{A} \to \mathcal{B}$ be a continuous injective algebra homomorphism such that $\overline{\Theta(\mathcal{A})} \cap \mathcal{F}(Y) \neq \{0\}$. Then Θ is regular.

About the pf:

We take in $\mathcal{A}$ the topology generated by the system of seminorms $(\tau_x)_{x \in X}$, where

$$\tau_x(T) := \big\| |T|(|x|) \big\| \quad (T \in \mathcal{A}).$$

We let $\widetilde{\mathcal{B}} := \mathcal{L}^r(Y)$ with the weak*-top. and endow $\mathcal{B}$ with the subspace top.

In this situation, provided Θ is injective, one can show Θ is τ-weak* continuous.

If (x_i) is the sequence of normalized atoms of X in the 'right' order, then (p_i), defined by $p_i := x_i^* \otimes x_i$ $(i \in \mathbb{N})$, is a sequence of mutually orthogonal o-minimal idempotents.

One just needs to choose a 'suitable' subsequence of (p_i) and 'suitable' disjoint sequences of band projections $(P_i) \subset \mathcal{M}_l(\mathcal{L}^r(Y))$ and $(Q_i) \subset \mathcal{M}_r(\mathcal{L}^r(Y))$.

The non-atomic case

We shall assume X is such that there is a bounded set of positive projections $\Pi \subset \mathcal{L}^r(X)$ such that

- $\overline{\bigcup_{\pi \in \Pi} \pi(X)} = X$;
- $\pi(X)$ is a Riesz subspace of X, $\forall \pi \in \Pi$;
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We shall call Π a **directed bounded generating system for X.**

Example: Banach lattices of the form $L^p(\mu, X)$, where X admits a bounded generating system Π with each $\pi(X)$ a Banach lattice satisfying $\star$ for some fixed μ, and rearrangement invariant spaces on $[0, +\infty)$ have the above property.

Given X with a set Π as above, we shall say $\mathcal{A} \subseteq \mathcal{L}^r(X)$ is a Π **hereditary Riesz operator subalgebra** if:

- $\pi \mathcal{A} \pi \subseteq \mathcal{A}$, $\forall \pi \in \Pi$;
- $\{T \in \mathcal{L}^r(\pi(X)) : \imath_\pi T p_\pi \in \mathcal{A}\}$ is a Riesz operator subalgebra of $\mathcal{L}^r(X)$, $\forall \pi \in \Pi$, where $p_\pi : X \to \pi(X)$, $x \mapsto \pi(x)$, and $\imath_\pi : \pi(X) \to X$ is the inclusion map.

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Example: Any order and algebra ideal of $\mathcal{L}^r(X)$ is a Π hereditary Riesz operator subalgebra of $\mathcal{L}^r(X)$ for any directed bounded generating system of X.

We shall call a seminormalized sequence $(x_i) \subset X$ **asymptotically disjoint** (a.d. in short) if for some disjoint sequence $(\xi_i) \subset X$, $\lim_i \|x_i - \xi_i\| = 0$.

Recall E is **Levi** if every increasing norm-bounded net in E has a supremum.

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Theorem

Let X be order continuous with a directed bounded generating system Π such that $\pi(X)$ has property $\star$ $(\pi \in \Pi)$, let Y be reflexive such that any complemented seminormalized unconditional basic sequence, either in Y or in Y', contains an a.d. subsequence. Let $\mathcal{A} \subseteq \mathcal{L}^r(X)$ be a Π hereditary Riesz operator subalgebra, let $\mathcal{B} \subseteq \mathcal{L}^r(Y)$ be a Levi Riesz algebra ideal, and let $\Theta : \mathcal{A} \to \mathcal{B}$ be a continuous injective algebra homomorphism such that $\overline{\Theta(\mathcal{A})} \cap \mathcal{F}(Y) \neq \{0\}$ and $w^*\text{-}\lim_\pi \Theta(\pi T \pi) = \Theta(T)$ $(T \in \mathcal{A})$. Then Θ is regular.

If Θ preserves ranks then the additional assumption on Y can be dropped.

Theorem

Let X be as in the previous theorem and let Y be reflexive. Let $\mathcal{A} \subseteq \mathcal{L}^r(X)$ be a Π hereditary Riesz operator subalgebra, let $\mathcal{B} \subseteq \mathcal{L}^r(Y)$ be a Levi Riesz algebra ideal, and let $\Theta : \mathcal{A} \to \mathcal{B}$ be a continuous injective algebra homomorphism such that $\overline{\Theta(\mathcal{A})} \cap \mathcal{F}(Y)$ contains elements of rank one and $w^*\text{-}\lim_\pi \Theta(\pi T \pi) = \Theta(T)$ $(T \in \mathcal{A})$. Then Θ is regular.

A question of Sourour

Let $\Theta : \mathcal{L}^r(X) \to \mathcal{L}^r(X)$ be an algebra automorphism. Is there $U \in \mathcal{L}^r(X)$ invertible such that $\Theta(T) = UTU^{-1}$ $(T \in \mathcal{L}^r(X))$?

So far, the answer is yes for all reflexive Banach lattices of the kind described earlier.