States in some ordered structures and axioms of choice

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Aim

Axiom of Choice

AC : Given an infinite family $(A_i)_{i \in I}$ of non-empty sets, the product $\prod_{i \in I} A_i$ is non-empty.

We work in **ZF**, set theory **without the Axiom of Choice**.

We consider the **Hahn-Banach axiom** (**HB**), a weak form of the Axiom of Choice. Remark : in **ZF**, **HB** is not provable and **HB** does not imply **AC** (see Howard and Rubin's book, [3]).

Theorem 1 : in ZF, HB implies the following statement $\mathbf{S_g}$

$\mathbf{S_g}$: For every abelian ordered group G with a positive order unit e and every subgroup H of G such that $e \in H$, every e-state on H can be extended into a e-state on G.

Remark : the converse statement $\mathbf{S_g} \Rightarrow \mathbf{HB}$ also holds in **ZF**.

Hahn-Banach Axiom

Hahn-Banach Axiom **HB** : a weak form of the **AC**

HB : Given a real vector space E, a sublinear mapping $p : E \to \mathbb{R}$ (*i.e.* a subadditive mapping such that for all $t \in \mathbb{R}^+$ and for all $x \in E, p(tx) = tp(x)$), a vector subspace S of E and a linear mapping $f : S \to \mathbb{R}$ such that $f \leq p_{|S}$, there exists a linear mapping $g : E \to \mathbb{R}$ extending f such that $g \leq p$.

Corollary 1 **:** in **ZF**, **HB** implies the classical following statement

Given a real normed vector space $(E, || \, ||)$ and $a \in E \setminus \{0\}$, there exists a linear form $\varphi : E \to \mathbb{R}$ continuous of norm 1 such that $\varphi(a) = ||a||$.

Proof : apply **HB** to the sublinear mapping $p := || \, ||$, the vector subspace $S := \text{Vect}(a)$ and the linear form $f : \begin{array}{ccc} S & \to & \mathbb{R} \\ \lambda a & \mapsto & \lambda ||a|| \end{array}$.

Order unit e

1. **On partially ordered groups :**
   Let G be an abelian ordered group. A non-zero element e of G is an **order unit** of G if : $\forall x \in G \ \exists k \in \mathbb{Z} \ \ -ke \leq x \leq ke$.

2. **On partially ordered vector spaces :**
   Let E be an ordered vector space over $\mathbb{R}$.
   - An element $e \in E \setminus \{0\}$ is an *order unit* of E if it is an order unit of the ordered group $(E, +)$. For an order unit e of E : $e \in E^+$ or $-e \in E^+$.
   - Given a positive order unit $e \in E^+$ we associate a *semi-norm* $|| \ ||_e$ defined by :

   $$\forall x \in E \ \ ||x||_e := \inf\{t \in \mathbb{R}^+, -te \leq x \leq te\}$$

   - The semi-norms associated to two positive order units are equivalent and then, they define the same topology on E.

Positive morphisms

1. **On ordered groups :**
   Let G be an abelian ordered group. A group morphism $f : G \to \mathbb{R}$ is *positive* if it is *increasing i.e.* $\forall x, y \in G, \big(x \leq y \Rightarrow f(x) \leq f(y)\big)$.

2. **On ordered vector spaces :**

Lemma 1 (Characterisation)

Let $f : E \to \mathbb{R}$ be a linear form on an ordered vector space E with an order unit $e \in E^+$. Then :

f is positive (i.e. increasing) if and only if f is continuous of norm $f(e)$.

Proof :

- Assume that f is positive. Let $x \in E$: there exists $s \in \mathbb{R}_+^*$ such that $-se \leq x \leq se$, so $|f(x)| \leq s|f(e)|$ and then f is continuous and of norm $f(e)$.
- Now assume that f is continuous of norm $f(e)$. Let $x \in E^+$, show that $f(x) \geq 0$:
  - If $x \leq e$ then $0 \leq e - x \leq e$ so $f(e-x) \leq ||f||.||e-x||_e \leq f(e)$ and finally $f(x) \geq 0$.
  - If $x \nleq e$, there exists $s \in \mathbb{R}_+^*$ such that $-se \leq x \leq se$ then, apply the previous case to $\frac{1}{s}x$.

Concurrent relations (Luxemburg, [4])

Let X and Y be two sets. Given a binary relation $\mathcal{R}$ on $X \times Y$, for every $x \in X$, we define $\mathcal{R}(x) := \{y \in Y \mid x\mathcal{R}y\}$.

- The relation $\mathcal{R}$ is *concurrent* if for every finite subset $F := \{x_1, \ldots, x_n\}$ of X, the intersection $\mathcal{R}(x_1) \cap \cdots \cap \mathcal{R}(x_n)$ is non-empty.

- If $\mathcal{R}$ is a concurrent relation on $X \times Y$, we can define the *filter* $\mathcal{F}$ on Y generated by the sets $\mathcal{R}(x)$, $x \in X$:

$$\mathcal{F} := \{A \subseteq Y \mid \exists x_1, \ldots, x_n \in X \ \ \mathcal{R}(x_1) \cap \cdots \cap \mathcal{R}(x_n) \subseteq A\}$$

Reduced power (Luxemburg, [4])

Definition

Let E be a real vector space. Consider a set T and $\mathcal{F}$ a filter over T. Denote by Z the following vector subspace of the vector space E^T :

$$Z := \{(x_t)_{t\in T} \in E^T \mid \{t \in T \mid x_t = 0\} \in \mathcal{F}\}$$

The **reduced power** $E^T/\mathcal{F}$ of E by the filter $\mathcal{F}$ is the quotient vector space E^T/Z. We denote by $\overline{z}$ the class of an element $z \in E^T$ and we consider the canonical embedding : $\begin{array}{rccc} \text{can}: & E & \to & E^T/\mathcal{F} \\ & x & \mapsto & \overline{(x)_{t\in T}} \end{array}$

Remarks : if E is an ordered vector space :

- The vector space E^T endowed with the product order is an ordered vector space and the vector subspace Z is **order-convex** *i.e.* for every $v, w \in F, [v,w] := \{x \in E, v \le x \le w\} \subseteq F$.
  Thus, the reduced power $E^T/\mathcal{F}$ is also an ordered vector space.
- Moreover, if E has an order unit e, then the set :
  $$\mathcal{L}_0(E^T/\mathcal{F}\} := \{z \in E^T/\mathcal{F} \mid \exists\alpha, \beta \in \mathbb{R} \;\; \alpha\,\text{can}(e) \le z \le \beta\,\text{can}(e)\}$$
  is an ordered vector space with order unit can(e).

HB $\Rightarrow$ $\mathbf{S_g}$

A group morphism $f : G \to \mathbb{R}$ on an abelian ordered group G with positive order unit e is a *e-state* if f is positive and $f(e) = 1$.

We want to prove the following result :

Theorem 1 : in **ZF**, **HB** implies the following statement $\mathbf{S_g}$

$\mathbf{S_g}$: For every abelian ordered group G with a positive order unit e and every subgroup H of G such that $e \in H$, every e-state on H can be extended into a e-state on G.

The proof is in two steps : first, extending to "one dimension" and then extending to G.

Step 1 : extending to one dimension, in **ZF**

Let G be an abelian ordered group, H be a subgroup of G and $f : H \to \mathbb{R}$ a positive group morphism on H.

Extending to one dimension : If H is **cofinal** (*i.e.* for every $x \in G$, there exists $y \in H$ such that $x \leq y$) and if $x \in G$, we consider :

- $p_H(x) = \sup \left\{ \frac{f(y)}{m} \mid m \in \mathbb{N}^*, y \in H, y \leq mx \right\} \in \mathbb{R}$
- $r_H(x) = \inf \left\{ \frac{f(z)}{n} \mid n \in \mathbb{N}^*, z \in H, nx \leq z \right\} \in \mathbb{R}$

Remark : $p_H(x) \leq r_H(x)$.

Lemma 2 (Goodearl [2], extending to one dimension)

Let G be an abelian ordered group, H be a cofinal subgroup of G and $f : H \to \mathbb{R}$ a positive group morphism on H. Let $x \in G$:

1. *For every positive group morphism $g : H + \mathbb{Z}x \to \mathbb{R}$ extending f, we have : $p_H(x) \leq g(x) \leq r_H(x)$.*
2. *For every $t \in [p_H(x), r_H(x)]$, it is possible to extend f to a positive group morphism $g : H + \mathbb{Z}x \to \mathbb{R}$ such that $g(x) = t$.*

Corollary 2 **:** extending to a finite number of dimensions

Let G be an abelian ordered group, H be a cofinal subgroup of G and $f : H \to \mathbb{R}$ a positive group morphism on H. Let $x_1, \dots, x_n \in G$. There exists a positive group morphism $g : H + \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$ extending f such that $p_H \leq g \leq r_H$.

Proof : apply the preceding Lemma and remark that if H_1 is a subgroup of G such that $H \subseteq H_1$, $p_H \leq p_{H_1} \leq r_{H_1} \leq r_H$.

Extending to G : Let G be an abelian ordered group with positive order unit e, H a subgroup of G such that $e \in H$ (then H is cofinal) and $f : H \to \mathbb{R}$ a e-state on H.

1. Concurrent relation :

- Denote by $\mathcal{P}_{fin}(G)$ the set of finite subsets of G and $T := \{g \in \mathbb{R}^G \mid p_H \leq g \leq r_H\}$.
- Let $\mathcal{R}_f$ be the binary relation defined by $\forall (F, g) \in \mathcal{P}_{fin}(G) \times T$:

$$\mathcal{R}_f(F,g) : \begin{cases} \forall a, b \in F \ \ (a+b \in F \Rightarrow g(a+b) = g(a) + g(b)) \\ \forall a \in F \ (-a \in F \Rightarrow g(-a) = -g(a)) \\ \forall a, b \in F \ (a \leq b \Rightarrow g(a) \leq g(b)) \\ \forall a \in F \ (a \in H \Rightarrow g(a) = f(a)) \\ \forall a \in F \ p_H(a) \leq g(a) \leq r_H(a) \end{cases}$$

- Using Corollary 2, we prove that if $F \in \mathcal{P}_{fin}(G)$ there exists $g \in T$ extending f ; then $\mathcal{R}_f(F) \neq \emptyset$.
- $\mathcal{R}_f$ is concurrent because if $F_1, \ldots, F_n \in \mathcal{P}_{fin}(G)$ then $\emptyset \neq \mathcal{R}_f(F_1 \cup \cdots \cup F_n) \subseteq \mathcal{R}_f(F_1) \cap \cdots \cap \mathcal{R}_f(F_n)$.
- Thus, we consider the filter $\mathcal{F}$ on T generated by the sets $\mathcal{R}_f(F)$, $F \in \mathcal{P}_{fin}(G)$.

2. Reduced power of $\mathbb{R}$:

- Consider the reduced power $\mathbb{R}^T/\mathcal{F}$ (if $z \in \mathbb{R}^T$, we note $\overline{z}$ the class of z in $\mathbb{R}^T/\mathcal{F}$) and can : $\mathbb{R} \to \mathbb{R}^T/\mathcal{F}$ the canonical embedding.

- Consider $\begin{array}{cccc} \varphi: & G & \to & \mathbb{R}^T/\mathcal{F} \\ & x & \mapsto & \overline{(g(x))_{g\in T}} \end{array}$: φ is a positive group morphism.

- For all $x \in G, \varphi(x) \in \mathcal{L}_0(\mathbb{R}^T/\mathcal{F})$ because :

$$\forall g \in T \ \ r_H \leq g \leq p_H$$

Then :

$$\forall x \in G \ \ r_H(x)\,\mathrm{can}(1) \leq \varphi(x) \leq p_H(x)\,\mathrm{can}(1)$$

First positive group morphism

$$\varphi : G \to \mathcal{L}_0(\mathbb{R}^T/\mathcal{F})$$

3. Use of HB :

- **Normed vector space $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N$:**
  - $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})$ is an ordered vector space with an order unit $e_1 := \mathrm{can}(1)$.
  - Thus it is endowed with a semi-norm $|| \ ||_{e_1}$.
  - Let N be the vector subspace $N := \{x \in \mathcal{L}_0(\mathbb{R}^T/\mathcal{F}) \mid ||x||_{e_1} = 0\}$.
  - The quotient vector space $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N$ is endowed with the associated quotient norm.
- Apply **HB** (Corollary 1) to $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N$: there exists a linear form $\psi : \mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N \to \mathbb{R}$ continuous of norm 1 such that $\psi(e_1 + N) = ||e_1 + N|| = 1$.

  Then, $\begin{array}{rccc} \Gamma : & \mathcal{L}_0(\mathbb{R}^T/\mathcal{F}) & \to & \mathbb{R} \\ & z & \mapsto & \psi(z+N) \end{array}$ is continuous of norm 1 and $\Gamma(e_1) = 1$: with Lemma 1, Γ is a e_1-state.
- $\Gamma \circ \mathrm{can} = Id_{\mathbb{R}}$.

Second positive group morphism

$$\Gamma : \mathcal{L}_0(\mathbb{R}^T/\mathcal{F}) \to \mathbb{R}$$

4. Existence of state on G :

Extension of f

$$\tilde{f} := \Gamma \circ \varphi : G \to \mathbb{R}$$

- $\tilde{f}$ is a *e*-state.
- $\tilde{f}$ extends f because if $x \in H$ then :
  - $\tilde{f}(x) = \Gamma \circ \varphi(x) = \Gamma(\overline{(g(x))_{g \in T}})$.
  - But $\overline{(g(x))_{g \in T}} = \mathrm{can}(f(x))$ because $\mathcal{R}_f(\{x\}) \subseteq \{g \in T \mid g(x) = f(x)\} \in \mathcal{F}$.
  - Then $\tilde{f}(x) = \Gamma(\mathrm{can}(f(x)) = f(x)$ because $\Gamma \circ \mathrm{can} = Id_{\mathbb{R}}$

Other structures

We worked on several structures : abelian ordered group with positive order unit, real vector spaces with positive order unit, or unital C^*-algebras.

Given an abelian ordered group G (resp. a real ordered vector space E) with a positive order unit e, a *pure state* on G (resp. on E) is an extreme point of the convex set of e-states on G (resp. on E).

Question

Which consequence of axiom of choice do we need to prove the existence of states or pure states on ordered groups or ordered vector spaces with order unit ?

Other axioms

Consider the two other following weak forms of the Axiom of Choice :

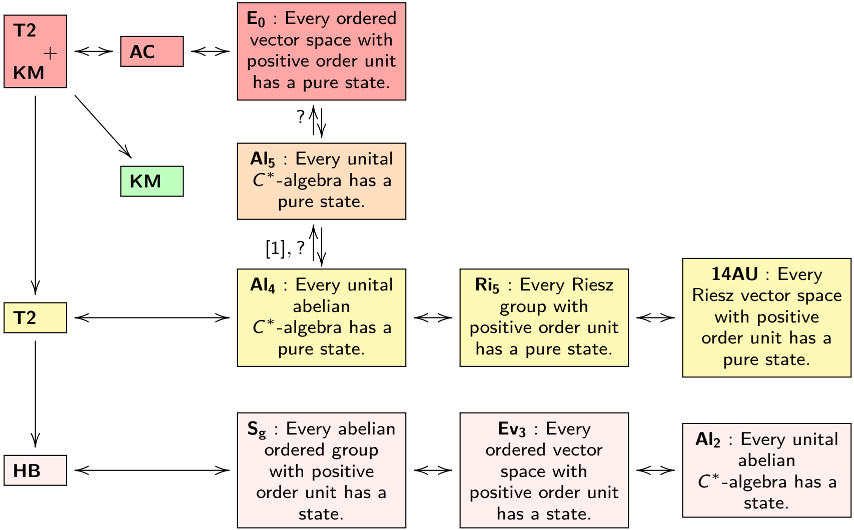
- **KM** (*Krein-Milman axiom*) : Let K be a non-empty compact convex subset of a topological locally convex Haussdorf real vector space X. Then K has an extreme point.
- **T2** (*Tychonov's axiom*) : For every family $(X_i)_{i\in I}$ of compact Haussdorf spaces, the product $\prod\limits_{i\in I} X_i$ is compact.

We have the following diagram :

$$\begin{tikzcd}
 & \mathbf{AC} \arrow[r, leftrightarrow] \arrow[dl] \arrow[dr] & (\mathbf{KM}+\mathbf{T2}) \\
\mathbf{T2} \arrow[ur, bend right, "/" marking] \arrow[d] & & \mathbf{KM} \arrow[ul, bend left, "/" marking] \\
\mathbf{HB} \arrow[u, bend left, "/" marking] & &
\end{tikzcd}$$

We obtained the following results :

Diagram : states and axioms of choice



Thank you for listening.

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