Lp-Spaces with respect to conditional expectation on Riesz spaces

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Some notations

We consider
a Dedekind complete Riesz space E
with weak order unit e, and
a conditional expectation T.
Here $T : E \longrightarrow E$ satisfies the following conditions

1. positive projection,
2. order continuous
3. $Te = e$,
4. T is *strictly positive* (i.e., $Tx > 0$ whenever $x > 0$),
5. $R(T)$ is a Dedekind complete Riesz subspace of E.

The sup-completion of a Riesz space

This notion is introduced by Donner in 1982

E_s plays the same role for E as $\mathbb{R}_\infty : \mathbb{R} \cup \{\infty\}$ does for $\mathbb{R}$.

We recall that E_s is a Dedekind complete lattice cone which satisfies the following conditions.

1. E is an ordered subset of E_s,
2. E_s has a biggest element,
3. If $x \in E_s$ then $x = \sup \{y \in E : y \leq x\}$,
4. if $y \leq x$ with $x \in E$ and $y \in E_s$ then $y \in E$.

Examples

1. If $E = \mathbb{R}$ then $E_s = \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$
2. More generally if $E = \mathbb{R}^n$ then $E_s = \mathbb{R}^n_\infty$.
3. If $E = L^p$ then $E_s = \{f$ measurable: $f \geq g$ for some $g \in L^p\}$.

Functional Calculus

If f is a real function and $x \in E$,
what is the meaning of $f(x)$?
We will use two kinds of functional calculus.

1. In the sense of Buskes, de Pagter, and von Rooij (1991).
   For $f \in \mathbb{R}^{\mathbb{R}}$, the equality $b = f(x)$ in E means that there exists a Riesz subspace V of E such that
   - $b, x \in V$;
   - $H(V)$ separates the points of V;
   - $\omega(b) = f(\omega(x)) \quad$ for all $\omega \in H(V)$.
2. In the sense of Grobler
   Here we use the Daniell Integral on Riesz spaces.

First kind

- Let $\mathrm{H}(E)$ be set of all Riesz homomorphisms on E.
- The Riesz subspace of E generated by a subset $A \subset E$ is denoted by $\langle A \rangle_E$.
- If A is finite and $V = \langle A \rangle_E$ then $\mathrm{H}(V)$ *separates the points* of V.
- If such a b exists, it is unique.
- If, in addition, E is an f-algebra, we use $H_m(V)$ rather than $H(V)$. Here, $H_m(V)$ is the subset of $H(V)$ of all multiplicative.

Second kind

For $x \in E$ and $t \in \mathbb{R}$, let
$p_t = e - P_{(x-te)^+}e$, $t \in \mathbb{R}$
$L = span\left\{\chi_{(a,b]}\right\} \subset \mathbb{R}^{\mathbb{R}}$.
Define $f(x)$ by

1. $f(x) = p_b - p_a$ if $f = \chi_{(a,b]}$
2. the definition is extended *via* a linearity process on L.
3. If $f_n \in L$, $f_n \geq 0$ and $f_n \uparrow f$ in $\mathbb{R}^{\mathbb{R}}$, we put $f(x) = \sup f_n(x) \in E_s$, (This is well defined)
4. If $f^+(x)$ and $f^-(x)$ $\exists$ and $f^-(x) \in E$ we put $f(x) = f^+(x) - f^-(x) \in E_s$.

Some results

- If $f_n \longrightarrow f$ uniformly and $f_n^D(x)$ and $f^D(x)$ exist in E. Then $f_n^D(x) \longrightarrow f^D(x)$ in order in E.
- If f is continuous and $f \circ g$ is well-defined and $g^D(x) \in E$. Then

$$(f \circ g)^D(x) = f^D\left(g^D(x)\right) \in E_s.$$

- f^D and f^H coincide on I_e.
- If f is increasing and continuous and $f^D(x) \in E$ then $f^D(x \wedge ne) \uparrow f^D(x)$ in E and if $f(x \vee -ke) \in E$ for some $k \in \mathbb{N}$ then $f^D(x \vee -ne) \downarrow f^D(x)$ in E.
- If f is continuous and $f^H(x)$ and $f^H(y)$ exist then
  1. f increasing $\implies f^H(x \wedge y) = f^H(x) \wedge f^H(y)$.
  2. $f(0) = 0$ and $x \perp y \implies f^H(x+y) = f^H(x) + f^H(y)$.

Some results

- If f is increasing and continuous and $f^D(x)$ and $f^D(y)$ exist in E then
  1. $f^D(x \wedge y) = f^D(x) \wedge f^D(y)$.
  2. If $x \perp y$ then $f^D(x+y) = f^D(x) + f^D(y) - f(0)e$.
- Assume that E is in addition an f-algebra with e as multiplicative identity.
  Let $x \in E^+$ and $f, g \in \mathbb{R}^{\mathbb{R}}$ be continuous functions on $\mathbb{R}^+$ of bounded variation on each closed interval $[0, a]$ with $a \in (0, \infty)$. If $f^D(x), g^D(x), (fg)^D(x)$ exist in E then $(fg)^D(x) = f^D(x)g^D(x)$.
- Let f be a convex increasing real-valued function on $[0, \infty)$ and (x_α) be an increasing net in E^+ with $x = \sup x_\alpha$. If $f(x) \in E$. then $f(x_\alpha) \uparrow f(x)$.

Convex functions

A function $f : C \longrightarrow E$ is said to be

- convex if
$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(t\right)$$
for all $x, y \in C,\ t \in \left[0, 1\right]$.

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- *positively homogeneous* if (C is a cone and)
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- *positively homogeneous* if (C is a cone and)
$$f\left(tx\right) = tf\left(x\right) \quad \text{for all } x \in C \text{ and } t \in [0, \infty)$$
- *sub-additive* if (C is a cone and)
$$f\left(x+y\right) \leq f\left(x\right) + f\left(y\right) \quad \text{for all } x, y \in E.$$

Two Theorems

We define the *lower-level set* is meant any subset of C of the form

$$L(f,a) = \{x \in C : f(x) \leq a\}, \quad \in E \tag{1}$$

Theorem

Let C be a cone in E. A positively homogeneous function $f : C \longrightarrow E$ is convex if and only if f is sub-additive.

Theorem

Let C be a cone in the Euclidean Riesz space $\mathbb{R}^n$ and $f : C \to \mathbb{R}^+$ be a positively homogeneous function. If $L(f,1)$ is convex then f is a convex function.

Generalization

Theorem

Let C be a cone in E. A positively homogeneous function $f : C \longrightarrow E_+$ is convex if and only if f is sub-additive.

Theorem

Let C be a cone C in E and $f : C \rightarrow E^+$ be a positively homogeneous function. Then f is convex if and only if the $L(f, e)$ is a convexset.

- Is it true?

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- YES

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- Is it true?
- YES
- NO

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Theorem

An increasing positively homogeneous function $f : C \longrightarrow E^{+}$ *is convex if and only if* $L(f, a)$ *is convex for all* $a \in E$.

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Proof.

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Proof.

- It is enough to show that f is sub-additive.
- Let $x, y \in C$ and put $z = |x| + |y| + |f(x)| + |f(y)| + |f(x+y)|$.

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Theorem

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- Let $x, y \in C$ and put $z = |x| + |y| + |f(x)| + |f(y)| + |f(x+y)|$.
- The ideal E_z is an AM-space with z as a strong order unit.

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- By Kakutani Representation Theorem we may assume that $E_z = C(K)$, where K is compact and Hausdorff and $z = 1_K$.

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- ....

□

Convex function still convex

Theorem

Let $f \in \mathbb{R}^{\mathbb{R}}$ be a convex function and C be a sublattice cone in E^{+} which contains e. If $f(x)$ exists in E for all $x \in C$ then f is convex on C.

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Theorem (Kuo-Labushagne-Watson, 2005)

Let E be a Dedekind complete f-algebra with order unit e and T be a conditional expectation operator T on E with $Te = e$. Then T is an averaging operator, i. e., $T(xy) = xTy$, for $y \in E$, $x \in R(T)$.

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Theorem (K-L-W, 2005)

Let E be a Dedekind complete Riesz space with weak order unit and T a conditional expectation on E. Then extension $T : L^1(T) \to L^1(T)$ is an averaging operator, i. e.,

$$T(xy) = xT(y) \text{ for all } x \in R(T) \text{ and } y \in L^1(T) \text{ with } xy \in L^1(T).$$

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1. *The range $R(T)$ of T is an f-subalgebra of $L^1(T)^u$.*
2. $R(T)L^1(T) \subseteq L^1(T)$.

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$$L^p(T) = \left\{ x \in L^1(T) : |x|^p \in L^1(T) \right\}.$$
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Let T be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p < \infty$. Then

$$N_p(x+y) \leq N_p(x) + N_p(y) \text{ for all } x, y \in L^p(T).$$

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- **Lyapunov Inequality**

$$L^p(T) \subset L^q(T) \text{ and } N_q(x) \leq N_p(x) \text{ for all } x \in L^p(T).$$

More inequalities

Again T be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p < \infty$.

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Theorem

Let $0 \leq x \in L^p(T)$ and $u \in R(T)$. Then the following holds

$$u^p TP_{(x-u)^+} e \leq Tx^p.$$

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Theorem

Let $0 \leq x \in L^p(T)$ *and* $u \in R(T)$. *Then the following holds*

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- **Another one**

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Theorem

Under the same assumptions we have

$$u^{p-1} TP_{(x-u)^+} x \leq Tx^p.$$

T-uniform family

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Definition (Kuo-Vardy-Watson, 2013)

A family (x_α) in E is called T-uniform if

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Theorem

Let T be a conditional expectation with natural domain $L^1(T)$ and $1 < p < \infty$. Let $(x_\alpha)_{\alpha \in \Lambda}$ be a family in $L^1(T)$ which is bounded in $L^p(T)$, i.e., there exists $y \in L^1(T)$ such that

$$T(|x_\alpha|^p) \leq y \text{ for all } \alpha \in \Lambda.$$

Then (x_α) is T-uniform.

Theorem

Let T be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p < \infty$. A locally bounded net (x_α) in $L^1(T)$ converges to x in $L^p(T)$ if and only if $(|x_\alpha|^p)$ has a T-uniform tail and converges to x in T-conditionally probability.

The space L_∞.

- If $(\Omega, \mathcal{F}, \mu)$ is a probability space, then $L^\infty(\mu)$ is given by

$$L^\infty(\mu) = \left\{ f \in L^1(\mu) : |f| \leq \lambda \text{ for some } \lambda \in \mathbb{R} \right\}.$$

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- One of the classical results stipulates that

$$L^\infty(\mu) = \left\{ f \in \bigcap_{1 \leq p < \infty} L^p(\mu) : \lim_{p \longrightarrow \infty} \|f\|_p < \infty \right\} \tag{2}$$

and

$$\|f\|_\infty = \lim_{p \longrightarrow \infty} \|f\|_p \text{ for all } f \in L^\infty(\mu). \tag{3}$$

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- The vector-valued norm N_∞ should be defined:

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- But,... there is a problem !!

The "right" definition

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- Does it exist?
- Yes,

$$L^{\infty}(T) = \left\{ x \in L^{1}(T) : |f| \leq u \text{ for some } u \in R(T) \right\}.$$

and

$$N_{\infty}(x) = \inf \left\{ u \in R(T) : |x| \leq u \right\}, \qquad x \in L^{\infty}(T).$$

Properties of $L_\infty(T)$

Let T be a conditional expectation with natural domain $L^1(T)$

Theorem

The following hold

1. $L^\infty(T)$ *is an* f*-subalgebra of* $L^1(T)^u$.
2. $L^\infty(T)\, L^p(T) \subset L^p(T)$ *for* $p \in [1, \infty]$.

Theorem

Let $x \in L^1(T)$.

1. *The following are equivalent*
    1. $x \in L^\infty(T)$;
    2. $x \in \bigcap_{1 \leq p < \infty} L^p(T)$ *and* $\{N_p(x)\}_{p \in [1,\infty)}$ *is bounded in* $L^1(T)$.
2. *In this case we havethe following formula*

$$N_\infty(x) = \sup\{N_p(x) : p \in [1, \infty)\}.$$

Thank you