جامعة محمد الخامس بالرباط

Université Mohammed V de Rabat

ON THE ADJOINTS OF SOME OPERATORS ON BANACH LATTICES

Belmesnaoui AQZZOUZ

Positivity IX, University of Alberta, Edmonton

Canada, July 17-21, 2017

Introduction

In this talk, we give some necessary and some sufficient conditions on Banach lattices E and F for the following conditions to hold :

i) if $T : E \to F$ has a property "P", then $T' : F' \to E'$ has also the same property "P" and

ii) if $T' : F' \to E'$ has a property "P", then $T : E \to F$ has also the same property "P".

where the property "P" is AM-compact (resp. semi-compact, b-weakly compact, Almost Dunford-Pettis, order weakly compact).
This problem is studied for the class of AM-compact operators and the class of semi-compact operators by Zaanen in his book $["RieszSpacesII]$".

Introduction

In this talk I am interesting by the duality property of the following classes of operators :

1. The class of AM-compact operators.
2. The class of semi-compact operators.
3. The class of b-weakly compact operators.
4. The class of Almost Dunford-Pettis operators.
5. The class of order weakly compact operators.

If $T : E \to F$ is an operator, i.e. continuous linear mapping, between two Banach lattices, then its adjoint or dual $T' : F' \to E'$ is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.
For terminology concerning Banach lattice theory and positive operators, we use the excellent book of ([$Aliprantis\text{-}Burkinshaw$], **Positive operators, 2006**).

Introduction

In this talk I am interesting by the duality property of the following classes of operators :

1. The class of AM-compact operators.
2. The class of semi-compact operators.
3. The class of b-weakly compact operators.
4. The class of Almost Dunford-Pettis operators.
5. The class of order weakly compact operators.

If $T : E \to F$ is an operator, i.e. continuous linear mapping, between two Banach lattices, then its adjoint or dual $T' : F' \to E'$ is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.
For terminology concerning Banach lattice theory and positive operators, we use the excellent book of ($[Aliprantis\text{-}Burkinshaw]$, **Positive operators, 2006**).

Introduction

In this talk I am interesting by the duality property of the following classes of operators :

1. The class of AM-compact operators.
2. The class of semi-compact operators.
3. The class of b-weakly compact operators.
4. The class of Almost Dunford-Pettis operators.
5. The class of order weakly compact operators.

If $T : E \to F$ is an operator, i.e. continuous linear mapping, between two Banach lattices, then its adjoint or dual $T' : F' \to E'$ is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.
For terminology concerning Banach lattice theory and positive operators, we use the excellent book of ($[Aliprantis\text{-}Burkinshaw]$, **Positive operators, 2006**).

The class of AM-compact operators.

The class of AM-compact operators is introduced by Fremlin in his paper **(Riesz spaces with the order continuity property I. Proc. Cambr. Phil. Soc. 81 (1977))**.

An operator T from a vector lattice E into a Banach space F is said to be AM-compact if the image of each order bounded subset of E is relatively compact in F.

It is easy to see that each compact operator from a Banach lattice into a Banach space is AM-compact but the converse is false in general. In fact, the identity operator of the Banach lattice ℓ^1 is AM-compact but it is not compact.

However if E is an AM-space with unit, the class of AM-compact operators on E coincides with that of compact operators on E.

The class of AM-compact operators.

The class of AM-compact operators is introduced by Fremlin in his paper **(Riesz spaces with the order continuity property I. Proc. Cambr. Phil. Soc. 81 (1977))**.

An operator T from a vector lattice E into a Banach space F is said to be AM-compact if the image of each order bounded subset of E is relatively compact in F.

It is easy to see that each compact operator from a Banach lattice into a Banach space is AM-compact but the converse is false in general. In fact, the identity operator of the Banach lattice ℓ^1 is AM-compact but it is not compact.

However if E is an AM-space with unit, the class of AM-compact operators on E coincides with that of compact operators on E.

The class of AM-compact operators.

There exist AM-compact operators whose dual operators are not AM-compact, and conversely, there exist operators which are not AM-compact but their dual operators are AM-compact.
In fact, the identity operator of the Banach lattice ℓ^1 is AM-compact but its dual operator, which is the identity operator of ℓ^∞, is not AM-compact.
Conversely, the identity operator of the Banach lattice of all convergent sequences c is not AM-compact but its dual operator, which is the identity operator of the Banach lattice c', is AM-compact where c' is the topological dual of c.

The class of AM-compact operators.

In his book ($[Zaanen]$, **Riesz spaces II, 1983**), Zaanen studied the duality problem of AM-compact operators on Banach lattices. He proved that

Théorème 1 (Zaanen)

Let E and F be two Banach lattices and T be a regular operator from E into F ;

1. *If E' has an order continuous norm and T is AM-compact, then the dual operator T' is AM-compact from F' into E'.*
2. *If F has an order continuous norm, then T is AM-compact whenever its dual operator T' from F' into E' is AM-compact.*

The class of AM-compact operators

The proofs of Zaanen are long and very difficult. By using some results of ($[Aliprantis$-$Burkinshaw]$, **Positive compact operators on Banach lattices (1980)**) and ($[Aliprantis$-$Burkinshaw$-$Duhoux]$, **Compactness properties of abstract kernel operators. Pacific J. Math. (1982)**). In the following, we give an easy and simple proof of this Theorem. Our Proof uses arguments which are different from those of Zaanen.

The class of AM-compact operators.

Remarque 1

The sufficient conditions of Zaanen are not necessaries. In fact,

1. *If we take E be a Banach lattice such that the norm of E' is not order continuous (for example ℓ^1) and F is a finite-dimensional space, then it is clear that each operator T from E into F is AM-compact and its dual operator T' from F' into E' is also AM-compact.*
2. *If we take F be a Banach lattice such that its norm is not order continuous (for example ℓ^∞) and E is a finite-dimensional space, then each regular operator T from E into F is AM-compact and its dual operator T' from F' into E' is also AM-compact.*

The class of AM-compact operators.

Now, we state the converse of Zaanen's Theorem. For the converse of the second result of Zaanen we obtain.

Théorème 2

Let E and F be two Banach lattices. If each positive operator $S : E \to F$ is AM-compact whenever its adjoint $S' : F' \to E'$ is AM-compact, then one of the following statements is valid :

1. *the norm of F is order continuous.*
2. *E' is discrete.*

Proof : Assume by way of contradiction that the conditions 1) and 2) fails.
-Since **the norm of F is not order continuous**, it follows from Theorem 2.4.2 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) the existence of $y \in F^+$ and a disjoint sequence (y_n) in F such that $0 \leq y_n \leq y$ and $\|y_n\| = 1$ for all n.

The class of AM-compact operators.

Now, we state the converse of Zaanen's Theorem. For the converse of the second result of Zaanen we obtain.

Théorème 2

Let E and F be two Banach lattices. If each positive operator $S : E \to F$ is AM-compact whenever its adjoint $S' : F' \to E'$ is AM-compact, then one of the following statements is valid :

1. *the norm of F is order continuous.*
2. *E' is discrete.*

Proof : Assume by way of contradiction that the conditions 1) and 2) fails.
-Since **the norm of F is not order continuous**, it follows from Theorem 2.4.2 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) the existence of $y \in F^+$ and a disjoint sequence (y_n) in F such that $0 \leq y_n \leq y$ and $\|y_n\| = 1$ for all n.

Hence, by a consequence of Theorem 39.3 of ($[Zaanen]$, **Riesz spaces II, 1983**) there exists a positive disjoint sequence (g_n) of F' such that

$$\|g_n\| \leq 1,\ g_n(y_n) = 1 \text{ for all } n \text{ and } g_n(y_m) = 0 \text{ for } n \neq m. \qquad (*)$$

-As E' **is not discrete**, Theorem 3.1 of ($[Chen\text{-}Wickstead]$, **Some applications of Rademacher sequences in Banach lattices, Positivity (1998)**) implies the existence of a sequence $(f_n) \subset E'$ such that $f_n \to 0$ for $\sigma(E', E)$ and $|f_n| = f > 0$ for all n and some $f \in E'$.
-We consider the positive operator $S : E \to F$ defined by

$$S(x) = \left(\sum_{n=1}^{\infty} f_n(x) y_n\right) + f(x)y \text{ for all } x \in E.$$

in Theorem 1 of ($[Wickstead]$, **Converses for the Dodds-Fremlin and Kalton-Saab Theorems, (1996)**).
4- We have to prove that the positive operator $S : E \to F$ is not AM-compact and its adjoint $S' : F' \to E'$ is AM-compact.
This gives a contradiction

Hence, by a consequence of Theorem 39.3 of ($[Zaanen]$, **Riesz spaces II, 1983**) there exists a positive disjoint sequence (g_n) of F' such that

$$\|g_n\| \leq 1,\ g_n(y_n) = 1 \text{ for all } n \text{ and } g_n(y_m) = 0 \text{ for } n \neq m. \qquad (*)$$

-As E' **is not discrete**, Theorem 3.1 of ($[Chen\text{-}Wickstead]$, **Some applications of Rademacher sequences in Banach lattices, Positivity (1998)**) implies the existence of a sequence $(f_n) \subset E'$ such that $f_n \to 0$ for $\sigma(E', E)$ and $|f_n| = f > 0$ for all n and some $f \in E'$.
-We consider the positive operator $S : E \to F$ defined by

$$S(x) = (\textstyle\sum_{n=1}^{\infty} f_n(x) y_n) + f(x) y \text{ for all } x \in E.$$

in Theorem 1 of ($[Wickstead]$, **Converses for the Dodds-Fremlin and Kalton-Saab Theorems, (1996)**).
4- We have to prove that the positive operator $S : E \to F$ is not AM-compact and its adjoint $S' : F' \to E'$ is AM-compact.
This gives a contradition

The class of AM-compact operators.

Question. Is the second necessary condition of Theorem 1.2 sufficient. The answer is no.

In fact, if we take $E = F = c$, the Banach lattice of all convergent sequences, it is clear that the identity operator of E is not AM-compact but its dual operator, which is the identity of the dual topological c', is AM-compact. However the Banach lattice c' is discrete.

The class of AM-compact operators.

Question. Is the second necessary condition of Theorem 1.2 sufficient. The answer is no.
In fact, if we take $E = F = c$, the Banach lattice of all convergent sequences, it is clear that the identity operator of E is not AM-compact but its dual operator, which is the identity of the dual topological c', is AM-compact. However the Banach lattice c' is discrete.

The class of AM-compact operators.

Now, we establish the converse of the first result of Zaanen.

Théorème 3

Let E and F be two Banach lattices. Then the following statements are equivalent :

1. *Each regular AM-compact operator $T : E \to F$ has an AM-compact adjoint $T' : F' \longrightarrow E'$.*
2. *One of the following conditions is valid :*
   a) *the norm of E' is order continuous.*
   b) *F' is discrete and its norm is order continuous.*

Proof. $2.a) \Rightarrow (1)$ It is just Theorem 125.6 (i) of ($[Zaanen]$, **Riesz spaces II, 1983**).
$2.b) \Rightarrow (1)$ Assume that F' is discrete and its norm is order continuous. By Theorem 6.1 of ($[Wnuk]$, **Banach lattices with order continuous norms, 1999**), each order interval of F' is norm compact. Then in this case each operator from F' into E' is AM-compact.

The class of AM-compact operators.

Now, we establish the converse of the first result of Zaanen.

Théorème 3

Let E and F be two Banach lattices. Then the following statements are equivalent :

1. *Each regular AM-compact operator $T : E \to F$ has an AM-compact adjoint $T' : F' \longrightarrow E'$.*
2. *One of the following conditions is valid :*
   a) *the norm of E' is order continuous.*
   b) *F' is discrete and its norm is order continuous.*

Proof. $2.a) \Rightarrow (1)$ It is just Theorem 125.6 (i) of ($[Zaanen]$, **Riesz spaces II, 1983**).
$2.b) \Rightarrow (1)$ Assume that F' is discrete and its norm is order continuous. By Theorem 6.1 of ($[Wnuk]$, **Banach lattices with order continuous norms, 1999**), each order interval of F' is norm compact. Then in this case each operator from F' into E' is AM-compact.

$(1) \Rightarrow (2)$ We have just to prove that if E' does not have an order continuous norm, then F' is discrete and its norm is order continuous.
A- Indeed, suppose that E' does not have an order continuous norm, by Theorem 2.4.2 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**), there is a positive order bounded disjoint sequence (f_n) of E' satisfying $\|f_n\| = 1$ for all n.
B- Let $f = \vee_{n=1}^{\infty} f_n$ in E', and define a positive operator $S_1 : E \to l^1$ by

$$S_1(x) = (f_n(x))_{n=1}^{\infty} \text{ for all } x \in E.$$

which is AM-compact.
C- To finish the proof, we have to show that F' is discrete and its norm is order continuous. Otherwise,
1- by Theorem 6.1 of ($[Wnuk]$, **Banach lattices with order continuous norms, Warsaw 1999**), F' has an order interval $[0, g]$ which is not norm compact. Thus for some $\varepsilon > 0$ we may choose a sequence (g_n) in $[0, g]$ such that $\|g_n - g_m\| \geq \varepsilon > 0$ for all $n \neq m$.

2- Let $\phi : N \times N \setminus \{(n,n) : n \in \mathbf{N}\} \to N$ be a bijection and choose a sequence of elements $y_n \in F$ with $\|y_n\| = 1$ and $\left|(g_n - g_m)\left(y_{\phi(n,m)}\right)\right| \geq \frac{1}{2}\|g_n - g_m\| \geq \frac{\varepsilon}{2}$ for all $n \neq m$.
(see proof of Theorem 2.2 of ($[Wickstead]$, **Positive compact operators on Banach lattices : some loose ends ; Positivity (2000)**)).
3- Now consider the regular operator $S_2 : l^1 \to F$ defined by

$$S_2\left((a_n)\right) = \sum_{n=1}^{\infty} a_n y_n$$

for all $(a_n) \in \ell^1$.
4- Since S_1 is AM-compact, the composed operator $S = S_2 \circ S_1 : E \to F$ defined by

$$S(x) = \sum_{n=1}^{\infty} f_n(x) y_n \text{ for all } x \in E,$$

is regular and AM-compact.
But its adjoint operator $S' : F' \longrightarrow E'$ defined by

$$S'(h) = \sum_{n=1}^{\infty} h(y_n) f_n \text{ for all } h \in F',$$

is not AM-compact.
This is a contradiction and completes the proof of $1) \Rightarrow 2)$.

The class of semi-compact operators.

Zaanen (**Riesz spaces II, 1983**), section 126, p. 543) defined semi-compact operators for order bounded operators, but Aliprantis-Burkinshaw (**Positive operators, 2006**) gave a more general definition (see also ([Meyer-Nieberg], **Banach lattices, 1991**), Definition 3.6.9, p. 213).

Définition 1

An operator T from a Banach space E into a Banach lattice F is said to be semi-compact if for each $\varepsilon > 0$, there exists some $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$ where B_H is the closed unit ball of $H = E, F$.

Note that every compact operator T from a Banach space E into a Banach lattice F is semi-compact, but the converse is false in general. In fact, since the Banach lattice ℓ^∞ is an AM-space with unit, its identity operator $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is semi-compact but it is not compact.

The class of semi-compact operators.

Zaanen (**Riesz spaces II, 1983**), section 126, p. 543) defined semi-compact operators for order bounded operators, but Aliprantis-Burkinshaw (**Positive operators, 2006**) gave a more general definition (see also ([Meyer-Nieberg], **Banach lattices, 1991**), Definition 3.6.9, p. 213).

Définition 1

An operator T from a Banach space E into a Banach lattice F is said to be semi-compact if for each $\varepsilon > 0$, there exists some $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$ where B_H is the closed unit ball of $H = E, F$.

Note that every compact operator T from a Banach space E into a Banach lattice F is semi-compact, but the converse is false in general. In fact, since the Banach lattice ℓ^∞ is an AM-space with unit, its identity operator $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is semi-compact but it is not compact.

The class of semi-compact operators.

This class of operators does not satisfy the duality problem. For examples,

- the identity operator $Id_c : c \longrightarrow c$ of the Banach lattice of all convergent sequences c is semi-compact but its adjoint $Id_{c'} : c' \longrightarrow c'$ is not semi-compact.
- the identity operator $Id_{\ell^1} : \ell^1 \longrightarrow \ell^1$ is not semi-compact but its adjoint $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is semi-compact.

Without any hypotheses on Banach lattices E and F, Meyer-Nieberg (**Banach lattices, 1991**), Proposition 3.6.18) established the following properties :

Théorème 4 (Meyer-Nieberg)

Let E and F be two Banach lattices.

1. *If an operator $T : E \longrightarrow F$ is semi-compact, then its adjoint $T' : F' \longrightarrow E'$ is order weakly compact.*
2. *If an operator $T : E \longrightarrow F$ is such that its adjoint $T' : F' \longrightarrow E'$ is semi-compact, then T is order weakly compact.*

The class of semi-compact operators.

This class of operators does not satisfy the duality problem. For examples,

- the identity operator $Id_c : c \longrightarrow c$ of the Banach lattice of all convergent sequences c is semi-compact but its adjoint $Id_{c'} : c' \longrightarrow c'$ is not semi-compact.
- the identity operator $Id_{\ell^1} : \ell^1 \longrightarrow \ell^1$ is not semi-compact but its adjoint $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is semi-compact.

Without any hypotheses on Banach lattices E and F, Meyer-Nieberg (**Banach lattices, 1991**), Proposition 3.6.18) established the following properties :

Théorème 4 (Meyer-Nieberg)

Let E and F be two Banach lattices.

1. *If an operator $T : E \longrightarrow F$ is semi-compact, then its adjoint $T' : F' \longrightarrow E'$ is order weakly compact.*
2. *If an operator $T : E \longrightarrow F$ is such that its adjoint $T' : F' \longrightarrow E'$ is semi-compact, then T is order weakly compact.*

The class of semi-compact operators.

But before the results of Meyer-Nieberg, Zaanen (**Riesz spaces II, 1983**) started a study on the duality problem for the class of semi-compact operators on Banach lattices. He proved that

Théorème 5 (ZaanenII)

Let E and F be two Banach lattices.

1. *If the norm of F is order continuous, then each order bounded semi-compact operator $T : E \longrightarrow F$ has a semi-compact adjoint operator $T' : F' \longrightarrow E'$ (Theorem 127.1 of ([Zaanen],* **Riesz spaces II, 1983***)).*
2. *If the norm of E' is order continuous and F is Dedekind complete, then each order bounded operator $T : E \longrightarrow F$, with a semi-compact adjoint operator $T' : F' \longrightarrow E'$, is semi-compact. (Theorem 127.3 of ($[Zaanen]$,* **Riesz spaces II, 1983***)).*

The class of semi-compact operators.

But before the results of Meyer-Nieberg, Zaanen (**Riesz spaces II, 1983**) started a study on the duality problem for the class of semi-compact operators on Banach lattices. He proved that

Théorème 5 (ZaanenII)

Let E and F be two Banach lattices.

1. *If the norm of F is order continuous, then each order bounded semi-compact operator $T : E \longrightarrow F$ has a semi-compact adjoint operator $T' : F' \longrightarrow E'$ (Theorem 127.1 of ([Zaanen],* **Riesz spaces II, 1983***)).*
2. *If the norm of E' is order continuous and F is Dedekind complete, then each order bounded operator $T : E \longrightarrow F$, with a semi-compact adjoint operator $T' : F' \longrightarrow E'$, is semi-compact. (Theorem 127.3 of ($[Zaanen]$,* **Riesz spaces II, 1983***)).*

The class of semi-compact operators.

As a consequence, Zaanen obtained

Théorème 6 (Theorem 127.4 of [ZaanenII)

*]. Let E and F be two Banach lattices.
If the norms of E' and F are order continuous, then each regular operator $T : E \longrightarrow F$ is semi-compact if and only if its adjoint $T' : F' \longrightarrow E'$ is semi-compact.*

As for AM-compact operators, The proofs of Theorem 5 are long and very difficult.
By using the class of L-weakly compact operators and the class of M-weakly compact operators introduced by Meyer-Nieberg (**Banach lattices, 1991**), and Theorem 3.6.2, Proposition 3.6.11 and Corollary 3.6.14 of [Meyer-Nieberg], **Banach lattices, 1991**), we give a simple proof to these results of Zaanen.

The class of semi-compact operators.

As a consequence, Zaanen obtained

Théorème 6 (Theorem 127.4 of [ZaanenII)

]. Let E and F be two Banach lattices.
If the norms of E' and F are order continuous, then each regular operator $T : E \longrightarrow F$ is semi-compact if and only if its adjoint $T' : F' \longrightarrow E'$ is semi-compact.

As for AM-compact operators, The proofs of Theorem 5 are long and very difficult.
By using the class of L-weakly compact operators and the class of M-weakly compact operators introduced by Meyer-Nieberg (**Banach lattices, 1991**), and Theorem 3.6.2, Proposition 3.6.11 and Corollary 3.6.14 of [Meyer-Nieberg], **Banach lattices, 1991**), we give a simple proof to these results of Zaanen.

The class of semi-compact operators.

Recall that a semi-compact (resp. order bounded) operator is not necessary order weakly compact. In fact, the identity operator $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is semi-compact (resp. order bounded) but it is not order weakly compact.

To prove our first result on the converse of Zaanen, we need the following result.

Théorème 7

Let E and F be two Banach lattices such that F is Dedekind σ-complete. Then the following assertions are equivalent :

1. *Each semi-compact operator T from E into F is order weakly compact.*
2. *Each positive semi-compact operator T from E into F is order weakly compact.*
3. *One of the following conditions is valid :*
   a) *the norm of E is order continuous.*
   b) *the norm of F is order continuous.*

The class of semi-compact operators.

Recall that a semi-compact (resp. order bounded) operator is not necessary order weakly compact. In fact, the identity operator $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is semi-compact (resp. order bounded) but it is not order weakly compact.

To prove our first result on the converse of Zaanen, we need the following result.

Théorème 7

Let E and F be two Banach lattices such that F is Dedekind σ-complete. Then the following assertions are equivalent :

1. *Each semi-compact operator T from E into F is order weakly compact.*
2. *Each positive semi-compact operator T from E into F is order weakly compact.*
3. *One of the following conditions is valid :*
   a) *the norm of E is order continuous.*
   b) *the norm of F is order continuous.*

The class of semi-compact operators.

Remarque 2

The assumption "F is Dedekind σ-complete" is essential for Theorem 7.
In fact, if we take $E = l^\infty$ and $F = c$. It follows from the proof of Proposition 1 of Wnuk ($[Wnuk]$, **Remarks on J. R. Holubs paper concerning Dunford-Pettis operators. Math. Japon., (1993)***) that each operator $T : l^\infty \to c$ is weakly compact (and hence is order weakly compact). Then the assertions 1), 2) and 3) of Theorem 7 hold, but the assertion 4) of Theorem 7 is false.*

Now, we use Theorem 7 to give a converse of Theorem 127.1 of ($[Zaanen]$, **Riesz spaces II, 1983**) about the duality problem of semi-compact operators.

The class of semi-compact operators.

Remarque 2

The assumption "F is Dedekind σ-complete" is essential for Theorem 7.
In fact, if we take $E = l^\infty$ and $F = c$. It follows from the proof of Proposition 1 of Wnuk ($[Wnuk]$, **Remarks on J. R. Holubs paper concerning Dunford-Pettis operators. Math. Japon., (1993)***) that each operator $T : l^\infty \to c$ is weakly compact (and hence is order weakly compact). Then the assertions 1), 2) and 3) of Theorem 7 hold, but the assertion 4) of Theorem 7 is false.*

Now, we use Theorem 7 to give a converse of Theorem 127.1 of ($[Zaanen]$, **Riesz spaces II, 1983**) about the duality problem of semi-compact operators.

The class of semi-compact operators.

Théorème 8

Let E and F be two Banach lattices such that F is Dedekind σ-complete. If each positive semi-compact operator $T : E \longrightarrow F$ admits a semi-compact adjoint $T' : F' \longrightarrow E'$, then one of the following properties is valid :

1. *E is a KB-space.*
2. *The norm of F is order continuous.*

Proof. In fact, it suffices to establish that if the norm of F is not order continuous, then E is a KB-space.
Indeed, suppose that F does not have an order continuous norm.
Step 1. The norm of E is order continuous. Otherwise, by Theorem 7 there is a positive semi-compact operator $T : E \longrightarrow F$ which is not order weakly compact. And Proposition 3.6.18 (i) of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) implies that the adjoint $T' : F' \longrightarrow E'$ is not semi-compact, which is in contradiction with our hypothesis. Therefore the norm of E is order continuous.

The class of semi-compact operators.

Théorème 8

Let E and F be two Banach lattices such that F is Dedekind σ-complete. If each positive semi-compact operator $T : E \longrightarrow F$ admits a semi-compact adjoint $T' : F' \longrightarrow E'$, then one of the following properties is valid :

1. *E is a KB-space.*
2. *The norm of F is order continuous.*

Proof. In fact, it suffices to establish that if the norm of F is not order continuous, then E is a KB-space.
Indeed, suppose that F does not have an order continuous norm.
Step 1. The norm of E is order continuous. Otherwise, by Theorem 7 there is a positive semi-compact operator $T : E \longrightarrow F$ which is not order weakly compact. And Proposition 3.6.18 (i) of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) implies that the adjoint $T' : F' \longrightarrow E'$ is not semi-compact, which is in contradiction with our hypothesis. Therefore the norm of E is order continuous.

Step 2. E is a KB-space. Otherwise, it follows from ($[Aqzzouz - Elbour - Hmichane]$. **The duality problem for the class of b-weakly compact operators. Positivity (2009)**) that there exists a positive disjoint sequence (x_n) of E^+ such that $\|x_n\| = 1$ for all n and $0 \leq x_n \leq x''$ for all n and some $0 \leq x'' \in E''$.
Thus, Theorem 39.3 of ($[Zaanen]$, **Riesz spaces II, 1983**) implies that there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| \leq 1$ such that

$$g_n(x_n) = 1 \text{ for all } n \text{ and } g_n(x_m) = 0 \text{ for } n \neq m \qquad (*).$$

A- Since E has an order continuous norm, it follows from Corollary 2.4.3 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) that $g_n \to 0$ for $\sigma(E', E)$.
Hence the positive operator $R : E \to c_0$ defined by

$$R(x) = (g_n(x))_{n=1}^{\infty} \text{ for each } x \in E,$$

is well defined and $R(B_E) \subset B_{c_0}$.

B- Since the norm of F is not order continuous, Theorem 4.14 of ($[Aliprantis\text{-}Burkinshaw]$, **Positive operators, 2006**) implies the existence of some $u \in F^+$ and a disjoint sequence $(u_n) \subset [0, u]$ which does not converge to zero in norm. We may assume that $0 \leq u_n \leq u$ and $\|u_n\| = 1$ for all n. It follows from the proof of Theorem 117.1 of ($[Zaanen]$, **Riesz spaces II, 1983**) that the positive operator

$$S : c_0 \longrightarrow F, \, (\alpha_1, \alpha_2, ...) \longmapsto \sum_{i=1}^{\infty} \alpha_i u_i$$

defines a lattice isomorphism from c_0 into F. From the disjointness of the sequence (u_n) and $0 \leq u_n \leq u$ for all n, we see that $S\left(B_{c_0}\right) \subset [-u, u]$.

C- Next, we consider the positive operator

$$T = S \circ R : E \longrightarrow F, \, x \longmapsto \sum_{i=1}^{\infty} g_i(x) u_i.$$

D- T is semi-compact. But, its adjoint $T' : F' \to E'$ is not semi-compact. So, E is a KB-space.

The class of semi-compact operators.

If in Theorem 8, we replace F is Dedekind σ-complete by the norm of E is order continuous, we obtain

Théorème 9

Let E and F be two Banach lattices such that the norm of E is order continuous. If each positive semi-compact operator $T : E \longrightarrow F$ admits a semi-compact adjoint $T' : F' \longrightarrow E'$, then one of the following properties is valid :

1. *E is a KB-space.*
2. *The norm of F is order continuous.*

Proof. It is exactly Step 2 of Theorem 8.

The class of semi-compact operators.

Remarque 3

i- *The first necessary condition of Theorem 8 (resp. Theorem 9) is not sufficient.*
In fact, if we take $E = \ell^2$ *and* $F = \ell^\infty$*. Since* ℓ^∞ *is an AM-space with unit, the inclusion mapping* $i : \ell^2 \longrightarrow \ell^\infty$ *is semi-compact. But its adjoint* $i' : (\ell^\infty)' \longrightarrow \ell^2$ *is not semi-compact (if not, its adjoint* i' *would be compact, since* ℓ^2 *is discrete and its norm is order continuous). However, the Banach lattice* $E = \ell^2$ *is a KB-space.*
ii- *The assumption "*F *is Dedekind* σ*-complete" (resp. the norm of* E *is order continuous) is essential for Theorem 8 (resp. Theorem 9). In fact, we have just to take the example of the Remark after Theorem 12 or Remark after Theorem 7, let* $T : \ell^\infty \longrightarrow c$ *be an arbitrary operator. Clearly* T *is semi-compact.*

....

We claim that its adjoint T' is semi-compact. In fact, it follows from the proof of Proposition 1 of ($[Wnuk]$, **Remarks on J. R. Holubs paper concerning Dunford-Pettis operators. Math. Japon., (1993)***) that the operator T is weakly compact. Hence, its adjoint $T' : c' \longrightarrow (\ell^\infty)'$ is also weakly compact. Since $(\ell^\infty)'$ has the positive Schur property, it follows from Theorem 3.4 of ($[Chen\text{-}Wickstead]$,* **L-weakly and M-weakly compact operators. Indag. Math. (1999)***) that T' is L-weakly compact . So T' is semi-compact. (See also Theorem 2.5.4 of ($[Meyer\text{-}Nieberg]$,* **Banach lattices, 1991***)). But the conditions 1) and 2) of Theorem 8 (resp. Theorem 9) are not satisfy.*

The class of semi-compact operators.

Now, we give our second characterization

Théorème 10

Let E and F be two Banach lattices. Then the following assertions are equivalent :

1. *Each positive operator from F' into E' is order weakly compact.*
2. *The adjoint of each positive operator from E into F is order weakly compact.*
3. *Each semi-compact operator from F' into E' is order weakly compact.*
4. *Each positive semi-compact operator from F' into E' is order weakly compact.*
5. *If $T : E \longrightarrow F$ is a positive operator such that T' is semi-compact, then T' is order weakly compact.*
6. *One of the following conditions is valid :*
   a) *the norm of E' is order continuous.*
   b) *the norm of F' is order continuous.*

Proof. $1) \Longrightarrow 2) \Longrightarrow 5)$ and $3) \Longrightarrow 4) \Longrightarrow 5)$ are clear.
$5) \Longrightarrow 6)$ Assume by way of contradiction that neither E' nor F' has an order continuous norm, we have to construct a positive operator $T : E \longrightarrow F$ such that T' is semi-compact but T' is not order weakly compact.
For this, we have just to take the same operator constructed in the implication $2) \Longrightarrow 3)$ of Theorem 8.
In fact, if we consider the operator product $T = T_2 \circ T_1 : E \to l^1 \to F$ defined in the proof of the implication $2) \Longrightarrow 3)$ of Theorem 2.5.
Since ℓ^∞ is an AM-space with unit, the positive operator $T_1' : l^\infty \longrightarrow E'$ is semi-compact. Hence $T' = T_1' \circ T_2' : F' \longrightarrow \ell^\infty \longrightarrow E'$ is also semi-compact. But T' is not order weakly compact (see the proof of $(\mathbf{2}) \Longrightarrow (3)$ of Theorem 2.5).
So $5) \Longrightarrow 6)$.
$6) \Longrightarrow 1)$, $6) \Longrightarrow 3)$ By the same proof as the implication $4) \Longrightarrow 1)$, $4) \Longrightarrow 2)$ of Theorem 7 respectively.

The class of b-weakly compact operators

The class of b-weakly compact operator is introduced in ($[Alpay\text{-}Altin\text{-}Tonyali]$, **On property (b) of vector lattices. Positivity, (2003)**).

Recall that

A subset A of a Banach lattice E is called b-order bounded in E if it is order bounded in the topological bidual E''.

Every order bounded subset of E is b-order bounded. The converse is not true in general. In fact, the subset $A = \{e_n : n \in \mathbb{N}\}$ is b-order bounded in the Banach lattice c_0 but A is not order bounded in c_0, where e_n is the sequence of reals numbers with all terms zero except for the n'th which is 1.

Définition 2

An operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact if it maps each b-order bounded subset of E into a relatively weakly compact subset in X.

The class of b-weakly compact operators

The class of b-weakly compact operator is introduced in ($[Alpay\text{-}Altin\text{-}Tonyali]$, **On property (b) of vector lattices. Positivity, (2003)**).

Recall that

A subset A of a Banach lattice E is called b-order bounded in E if it is order bounded in the topological bidual E''.

Every order bounded subset of E is b-order bounded. The converse is not true in general. In fact, the subset $A = \{e_n : n \in \mathbf{N}\}$ is b-order bounded in the Banach lattice c_0 but A is not order bounded in c_0, where e_n is the sequence of reals numbers with all terms zero except for the n'th which is 1.

Définition 2

An operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact if it maps each b-order bounded subset of E into a relatively weakly compact subset in X.

The class of b-weakly compact operators

The class of b-weakly compact operator is introduced in ($[Alpay\text{-}Altin\text{-}Tonyali]$, **On property (b) of vector lattices. Positivity, (2003)**).

Recall that

A subset A of a Banach lattice E is called b-order bounded in E if it is order bounded in the topological bidual E''.

Every order bounded subset of E is b-order bounded. The converse is not true in general. In fact, the subset $A = \{e_n : n \in \mathbf{N}\}$ is b-order bounded in the Banach lattice c_0 but A is not order bounded in c_0, where e_n is the sequence of reals numbers with all terms zero except for the n'th which is 1.

Définition 2

An operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact if it maps each b-order bounded subset of E into a relatively weakly compact subset in X.

The class of b-weakly compact operators

Each weakly compact operator is b-weakly compact and each b-weakly compact operator is order weakly compact.
But, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]$ is b-weakly compact but it is not weakly compact and the identity operator $Id_{c_0} : c_0 \to c_0$ is order weakly compact but it is not b-weakly compact.
There is a b-weakly compact operator whose adjoint is not b-weakly compact. In fact, the identity operator $Id_{\ell^1} : \ell^1 \longrightarrow \ell^1$ is b-weakly compact but its adjoint $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is not one.
And conversely, there is an operator which is not b-weakly compact while its adjoint is one. In fact, the identity operator $Id_{c_0} : c_0 \longrightarrow c_0$ is not b-weakly compact but its adjoint $Id_{\ell^1} : l^1 \longrightarrow \ell^1$ is b-weakly compact.

The class of b-weakly compact operators

Each weakly compact operator is b-weakly compact and each b-weakly compact operator is order weakly compact.
But, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]$ is b-weakly compact but it is not weakly compact and the identity operator $Id_{c_0} : c_0 \to c_0$ is order weakly compact but it is not b-weakly compact.
There is a b-weakly compact operator whose adjoint is not b-weakly compact. In fact, the identity operator $Id_{\ell^1} : \ell^1 \longrightarrow \ell^1$ is b-weakly compact but its adjoint $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is not one.
And conversely, there is an operator which is not b-weakly compact while its adjoint is one. In fact, the identity operator $Id_{c_0} : c_0 \longrightarrow c_0$ is not b-weakly compact but its adjoint $Id_{\ell^1} : l^1 \longrightarrow \ell^1$ is b-weakly compact.

The class of b-weakly compact operators

A Banach lattice E is said to be a KB-space, whenever every increasing norm bounded sequence of E^+ is norm convergent.
For example, each reflexive Banach lattice (resp. AL-space) is a KB-space.
Each KB-space has an order continuous norm, but a Banach lattice with an order continuous norm is not necessary a KB-space.
In fact, the Banach lattice c_0 has an order continuous norm but it is not a KB-space. However, if E is a Banach lattice, the topological dual E' is a KB-space if and only if its norm is order continuous.

The class of b-weakly compact operators

We need the following result, which is a consequence of Theorem 4.60 of ([$Aliprantis$-$Burkinshaw$], **Positive operators, 2006**) and Corollary of ([$Alpay$-$Altin$], **A note on b-weakly compact operators, Positivity (2007), p. 577**)

Proposition 1

Let $T : E \to X$ be an operator from a Banach lattice E into a Banach space X. If T factors through a KB-space, then T is b-weakly compact.

The class of b-weakly compact operators

We need the following result, which is a consequence of Theorem 4.60 of ($[Aliprantis\text{-}Burkinshaw]$, **Positive operators, 2006**) and Corollary of ($[Alpay\text{-}Altin]$, **A note on b-weakly compact operators, Positivity (2007), p. 577**)

Proposition 1

Let $T : E \to X$ be an operator from a Banach lattice E into a Banach space X. If T factors through a KB-space, then T is b-weakly compact.

The class of b-weakly compact operators

Our first result, on the duality of b-weakly compact operators, gives a sufficient and necessary condition for which the b-weak compactness of an operator implies the b-weak compactness of its adjoint.

Théorème 11

Let E and F be two Banach lattices. Then the following conditions are equivalent :

1. *Each operator from F' into E' is b-weakly compact.*
2. *If $T : E \to F$ is a b-weakly compact operator, then its adjoint T' is b-weakly compact.*
3. *One of the following assertions is valid :*
   a) *E' is a KB-space.*
   b) *F' is a KB-space.*

The class of b-weakly compact operators

Our first result, on the duality of b-weakly compact operators, gives a sufficient and necessary condition for which the b-weak compactness of an operator implies the b-weak compactness of its adjoint.

Théorème 11

Let E and F be two Banach lattices. Then the following conditions are equivalent :

1. *Each operator from F' into E' is b-weakly compact.*
2. *If $T : E \to F$ is a b-weakly compact operator, then its adjoint T' is b-weakly compact.*
3. *One of the following assertions is valid :*
   a) *E' is a KB-space.*
   b) *F' is a KB-space.*

The class of b-weakly compact operators

Proof. $(1) \Rightarrow (2)$ Obvious.
$(2) \Rightarrow (3)$ If neither E' nor F' is a KB-space, we have to construct an operator $T : E \to F$ such that T is b-weakly compact and its adjoint T' is not b-weakly compact.
A- Since E' is not a KB-space (i.e. the norm of E' is not order continuous) then, by Theorem 2.4.2 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**), there is a positive order bounded disjoint sequence (f_n) of E' satisfying $\|f_n\| = 1$ for all n. Let $f = \vee_{n=1}^{\infty} f_n$ in E'.
We use this sequence (f_n) to define the positive operator $T_1 : E \to \ell^1$ by

$$T_1(x) = (f_n(x))_{n=1}^{\infty} \text{ for all } x \in E.$$

The class of b-weakly compact operators

B- Since F' is not a KB-space, there is a positive order bounded disjoint sequence (g_n) of F' satisfying $\|g_n\| = 1$ for all n. Since $\|g_n\| = \sup\{g_n(y) : 0 \leq y \in F \text{ and } \|y\| = 1\}$ holds for all n, then for each n we choose $y_n \in F^+$ with $\|y_n\| = 1$ and $g_n(y_n) \geq \frac{1}{2}\|g_n\| = \frac{1}{2}$. We use this sequence (y_n) to define the positive operator $T_2 : \ell^1 \to F$ defined by

$$T_2((a_n)) = \sum_{n=1}^{\infty} a_n.y_n \text{ for all } (a_n) \in \ell^1.$$

C- We consider the operator product $T = T_2 \circ T_1 : E \to \ell^1 \to F$ defined by

$$T(x) = \sum_{n=1}^{\infty} f_n(x).y_n \text{ for all } x \in E.$$

Since ℓ^1 is a KB-space, it follows from Proposition 1 that T is b-weakly compact. But its adjoint $T' : F' \longrightarrow E'$ defined by

$$T'(h) = \sum_{n=1}^{\infty} h(y_n) f_n \text{ for all } h \in F',$$

is not b-weakly compact.

$(3) \Rightarrow (1)$ Follows immediately from Proposition 1.

The class of b-weakly compact operators

Now, we give a sufficient and necessary condition for which an operator becomes b-weakly compact whenever its adjoint is b-weakly compact.

Théorème 12

Let E and F be two Banach lattices such that the norm of E is order continuous. Then the following conditions are equivalent :

1. *Each operator from E into F is b-weakly compact.*
2. *Each operator $T : E \to F$ is b-weakly compact whenever its adjoint T' is b-weakly compact.*
3. *One of the following assertions is valid :*
   a) *E is a KB-space.*
   a) *F is a KB-space.*

The class of b-weakly compact operators

Now, we give a sufficient and necessary condition for which an operator becomes b-weakly compact whenever its adjoint is b-weakly compact.

Théorème 12

Let E and F be two Banach lattices such that the norm of E is order continuous. Then the following conditions are equivalent :

1. *Each operator from E into F is b-weakly compact.*
2. *Each operator $T : E \to F$ is b-weakly compact whenever its adjoint T' is b-weakly compact.*
3. *One of the following assertions is valid :*
   a) *E is a KB-space.*
   a) *F is a KB-space.*

The class of b-weakly compact operators

Proof $(1) \Rightarrow (2)$ Obvious.
$(2) \Rightarrow (3)$ If neither E nor F is a KB-space, we have to construct an operator $T : E \to F$ such that T is not b-weakly compact but its adjoint T' is b-weakly compact.
A- Since E is not a KB-space, the identity operator Id_E is not b-weakly compact by Proposition 2.10 of ([Alpay-Altin-Tonyali]), **On property (b) of vector lattices. Positivity (2003)**).
Hence it follows from Proposition 2.8 of ([Alpay-Altin-Tonyali]), that E^+ contains a b-order bounded disjoint sequence (x_n) satisfying $\|x_n\| = 1$ for all n.
So, by Proposition 2.8 and Proposition 2.10 of ([Alpay-Altin-Tonyali]), there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| \le 1$ such that

$$g_n(x_n) = 1 \text{ for all } n \text{ and } g_n(x_m) = 0 \text{ for } n \neq m. \qquad (*)$$

The class of b-weakly compact operators

B- Since the norm of E is order continuous, it follows from Corollary 2.4.3 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) that $g_n \to 0$ for $\sigma(E', E)$. Hence the positive operator $T_1 : E \to c_0$ defined by

$$T_1(x) = (g_n(x))_{n=1}^{\infty} \text{ for each } x \in E,$$

is well defined.
C- Since F is not a KB-space, it follows from Theorem 4.61 of ($[Aliprantis\text{-}Burkinshaw]$, **Positive operators, 2006**) that c_0 is lattice embeddable in F.
If $T_2 : c_0 \to F$ is a lattice embedding, then the sequence (y_n) defined by $y_n = T_2(e_n)$ for all n, is bounded away from zero, i.e., there exists some $K > 0$ satisfying $\|y_n\| \geq K$ for all n.
D- Consider the positive operator $T = T_2 \circ T_1 : E \longrightarrow c_0 \longrightarrow F$.
It's adjoint $T' : F' \longrightarrow l^1 \longrightarrow E'$ is b-weakly compact. But T is not b-weakly compact.
$(3) \Rightarrow (1)$ Follows from Proposition 2.1 of ($[Altin]$, **Some properties of b-weakly compact operators. G. U. J. Sci. (2005)**).

The class of b-weakly compact operators

Remarque 4

The assumption "the norm of E is order continuous" is essential in Theorem 12.
For instance, each operator $T : \ell^\infty \to c_0$ is weakly compact (because ℓ^∞ is a Grothendieck space (i.e. if $x'_n \overset{w^*}{\to} 0$ in $(\ell^\infty)'$ then $x'_n \overset{w}{\to} 0$ in $(\ell^\infty)'$)) and hence each operator from ℓ^∞ into c_0 is b-weakly compact, but neither ℓ^∞ nor c_0 is a KB-space.

The class of almost Dunford-Pettis operators

A linear operator from a Banach lattice E into a Banach space F is almost Dunford-Pettis if $\|T(x_n)\| \to 0$ for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E.
There is no automatic duality result for the class of almost Dunford-Pettis operators.
In fact, the identity operator $Id_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]$ is almost Dunford-Pettis but its adjoint $Id_{L^\infty[0,1]} : L^\infty[0,1] \to L^\infty[0,1]$ is not almost Dunford-Pettis.
Conversely, the identity operator $Id_{c_0} : c_0 \to c_0$, is not almost Dunford-Pettis but its adjoint, which is the identity operator $Id_{\ell^1} : \ell^1 \to \ell^1$, is almost Dunford-Pettis.

The class of almost Dunford-Pettis operators

To establish a necessary and sufficient condition on the pair of Banach lattices E and F which guarantees that if $T : E \to F$ is almost Dunford-Pettis then so is $T' : F' \to E'$, we need to recall positive Schur property.

A Banach space E is said to have the Schur property if every weakly convergent sequence to 0 in E is norm convergent to zero. For example, the Banach space ℓ^1 has the Schur property.

The Banach lattice E has the positive Schur property if each weakly null sequence with positive terms in E converges to zero in norm. For example, the Banach lattice $L^1([0,1])$ has the positive Schur property but does not have the Schur property.

The class of almost Dunford-Pettis operators

To establish a necessary and sufficient condition on the pair of Banach lattices E and F which guarantees that if $T : E \to F$ is almost Dunford-Pettis then so is $T' : F' \to E'$, we need to recall positive Schur property.

A Banach space E is said to have the Schur property if every weakly convergent sequence to 0 in E is norm convergent to zero. For example, the Banach space ℓ^1 has the Schur property.

The Banach lattice E has the positive Schur property if each weakly null sequence with positive terms in E converges to zero in norm. For example, the Banach lattice $L^1([0,1])$ has the positive Schur property but does not have the Schur property.

The class of almost Dunford-Pettis operators

To establish a necessary and sufficient condition on the pair of Banach lattices E and F which guarantees that if $T : E \to F$ is almost Dunford-Pettis then so is $T' : F' \to E'$, we need to recall positive Schur property.

A Banach space E is said to have the Schur property if every weakly convergent sequence to 0 in E is norm convergent to zero. For example, the Banach space ℓ^1 has the Schur property.

The Banach lattice E has the positive Schur property if each weakly null sequence with positive terms in E converges to zero in norm. For example, the Banach lattice $L^1([0,1])$ has the positive Schur property but does not have the Schur property.

The class of almost Dunford-Pettis operators

Proposition 2

A Banach lattice E does not have the positive Schur property if and only if there exists a disjoint weakly null sequence (x_n) in E_+ with $\|x_n\| = 1$ for all n.

Proof. If E does not have the positive Schur property we know that there exists a disjoint weakly null sequence (y_n) of E_+ such that (y_n) is not norm convergent to 0. By passing to a subsequence if necessarily, we may assume that there exists some $\alpha > 0$ with $\|y_n\| \geq \alpha$ for all n. Put $x_n = y_n/\|y_n\|$ for all n and it is easy to see that our requirements are satisfied. The converse is easy.

The class of almost Dunford-Pettis operators

Proposition 2

A Banach lattice E does not have the positive Schur property if and only if there exists a disjoint weakly null sequence (x_n) in E_+ with $\|x_n\| = 1$ for all n.

Proof. If E does not have the positive Schur property we know that there exists a disjoint weakly null sequence (y_n) of E_+ such that (y_n) is not norm convergent to 0. By passing to a subsequence if necessarily, we may assume that there exists some $\alpha > 0$ with $\|y_n\| \geq \alpha$ for all n. Put $x_n = y_n/\|y_n\|$ for all n and it is easy to see that our requirements are satisfied. The converse is easy.

The class of almost Dunford-Pettis operators

Our first result is the following

Théorème 13

Let E and F be two Banach lattices. The following conditions are equivalent :

1. *The adjoint of each positive almost Dunford-Pettis operator $T : E \longrightarrow F$ is almost Dunford-Pettis.*
2. *At least one of the following assertions is valid :*
   - a) *E' has an order continuous norm.*
   - b) *F' has the positive Schur property.*

Proof.

(2)(a)$\Longrightarrow$(1)

A- Let $T : E \to F$ be a positive and almost Dunford-Pettis. Let (f_n) be a disjoint sequence in F'_+ such that $f_n \to 0$ for $\sigma(F', F'')$. We have to prove that $\|T'(f_n)\| \to 0$.

It is clear that $0 \leq T'(f_n) \to 0$ for $\sigma(E', E)$.

The class of almost Dunford-Pettis operators

Our first result is the following

Théorème 13

Let E and F be two Banach lattices. The following conditions are equivalent :

1. *The adjoint of each positive almost Dunford-Pettis operator $T : E \longrightarrow F$ is almost Dunford-Pettis.*
2. *At least one of the following assertions is valid :*
   a) *E' has an order continuous norm.*
   b) *F' has the positive Schur property.*

Proof.

(2)(a)$\Longrightarrow$(1)

A- Let $T : E \to F$ be a positive and almost Dunford-Pettis. Let (f_n) be a disjoint sequence in F'_+ such that $f_n \to 0$ for $\sigma(F', F'')$. We have to prove that $\|T'(f_n)\| \to 0$.

It is clear that $0 \leq T'(f_n) \to 0$ for $\sigma(E', E)$.

By Corollary 2.7 of ($[Dodds\text{-}Fremlin]$, **Compact operators in Banach lattices. Israel J. Math. (1979)**), it suffices to show that $T'(f_n)(x_n) \to 0$ for each disjoint norm bounded sequence (x_n) in E_+.
B- As the norm of E' is order continuous, it follows from Corollary 2.4.14 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**), that for such a sequence we have $x_n \to 0$ for $\sigma(E, E')$.
Since T is almost Dunford-Pettis, $\|T(x_n)\| \to 0$. As $f_n \to 0$ for $\sigma(F', F'')$, (f_n) is norm bounded. Hence $T'(f_n)(x_n) = f_n\,(T(x_n)) \to 0$ and we are done.
(2)(b)$\Longrightarrow$(1),
we will prove that in fact the adjoint of every operator $T : E \to F$ is almost Dunford-Pettis.
To see this, let (f_n) be a disjoint sequence in F'_+ such that $f_n \to 0$ for $\sigma(F', F'')$. Since F' has the positive Schur property, $\|f_n\| \to 0$ and hence $\|T'(f_n)\| \to 0$ from which the result follows.

(1) $\Longrightarrow$ (2)
If (2) fails, then the norm of E' is not order continuous and F' does not has the positive Schur property.
Since the norm of E' is not order continuous, it follows from Theorem 2.4.2 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) that there exists a positive order bounded disjoint sequence (φ_n) of E'_+ with $\|\varphi_n\| = 1$ for all n. Let $0 \leq \varphi \in E'$ be such that $0 \leq \varphi_n \leq \varphi$ for all n.
Define the operator $U : E \to \ell_1$ by $U(x) = (\varphi_n(x))_{n=1}^{\infty}$ for $x \in E$.
Since $\sum_{n=1}^{\infty} |\varphi_n(x)| \leq \sum_{n=1}^{\infty} \varphi_n(|x|) \leq \varphi(|x|)$ for each $x \in E$, the operator U does indeed take values in ℓ_1 and it is clearly positive.
Since F' does not have the positive Schur property, it follows from Proposition 4.1 that there exists a disjoint weakly null sequence (f_n) in F'_+ with $\|f_n\| = 1$ for all n.

As $\|f_n\| = \sup\{f_n(y) : y \in F_+, \|y\| = 1\}$, for each n there exists $y_n \in F_+$ with $\|y_n\| = 1$ and $f_n(y_n) \geq \frac{1}{2}$. Define a positive operator $V : \ell_1 \to F$ by $V((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n$, which series is certainly norm convergent.

Let $T = V \circ U : E \to \ell^1 \to F$ so that $T(x) = \sum_{n=1}^{\infty} \varphi_n(x) y_n$ for $x \in E$.

It is clear that T is Dunford-Pettis as if $x_n \to 0$ weakly in E then $V(x_n) \to 0$ weakly in ℓ^1 and therefore, as ℓ^1 has the Schur property, $\|V(x_n)\| \to 0$ and hence $\|T(x_n)\| = \|U(V(x_n))\| \to 0$. So certainly T is almost Dunford-Pettis.

But its adjoint $T' : F' \to E'$ is not almost Dunford-Pettis.

To see this, note that if $h \in F'$ then $T'(h) = \sum_{n=1}^{\infty} h(y_n)\varphi_n$.

In particular, for every k we have $T'(f_k) = \sum_{n=1}^{\infty} f_k(y_n)\varphi_n \geq f_k(y_k)\varphi_k \geq 0$ and hence $\|T'(f_k)\| \geq \|f_k(y_k)\varphi_k\| = f_k(y_k) \geq \frac{1}{2}$. As (f_k) is disjoint and weakly null, T' is not almost Dunford-Pettis.

The class of almost Dunford-Pettis operators

We say that a Banach lattice E has the bi-sequence property if for each weak null disjoint sequence (x_n) in E and each weak* null sequence (f_n) in E'_+, we have $f_n(x_n) \to 0$.
In the very important special case that E has an order continuous norm, then this reduces to a condition that has been previously studied.

Proposition 3

A Banach lattice E, which has an order continuous norm, has the bi-sequence property if and only if it has the positive Schur property.

The bi-sequence property may similarly be simplified if we assume that E' has an order continuous norm.
A Banach lattice E has the dual positive Schur property if every weak null sequence in E'_+ is norm null.

The class of almost Dunford-Pettis operators

We say that a Banach lattice E has the bi-sequence property if for each weak null disjoint sequence (x_n) in E and each weak* null sequence (f_n) in E'_+, we have $f_n(x_n) \to 0$.
In the very important special case that E has an order continuous norm, then this reduces to a condition that has been previously studied.

Proposition 3

A Banach lattice E, which has an order continuous norm, has the bi-sequence property if and only if it has the positive Schur property.

The bi-sequence property may similarly be simplified if we assume that E' has an order continuous norm.
A Banach lattice E has the dual positive Schur property if every weak null sequence in E'_+ is norm null.

The class of almost Dunford-Pettis operators

Proposition 4

A Banach lattice E, for which E' has an order continuous norm, has the bi-sequence property if and only if it has the dual positive Schur property.

Now, we study the converse situation. We begin by giving sufficient conditions.

Théorème 14

Let E and F be two Banach lattices. If at least one of the following three conditions holds then each positive operator $T : E \to F$, such that $T' : F' \to E'$ is almost Dunford-Pettis, must itself be almost Dunford-Pettis :

1. *E has the positive Schur property.*
2. *E has the bi-sequence property and F has an order continuous norm.*
3. *F may be written as an order direct sum $F = G \bigoplus H$ where the*

The class of almost Dunford-Pettis operators

Proposition 4

A Banach lattice E, for which E' has an order continuous norm, has the bi-sequence property if and only if it has the dual positive Schur property.

Now, we study the converse situation. We begin by giving sufficient conditions.

Théorème 14

Let E and F be two Banach lattices. If at least one of the following three conditions holds then each positive operator $T : E \to F$, such that $T' : F' \to E'$ is almost Dunford-Pettis, must itself be almost Dunford-Pettis :

1. *E has the positive Schur property.*
2. *E has the bi-sequence property and F has an order continuous norm.*
3. *F may be written as an order direct sum $F = G \bigoplus H$ where the*

The class of almost Dunford-Pettis operators

Proposition 4

A Banach lattice E, for which E' has an order continuous norm, has the bi-sequence property if and only if it has the dual positive Schur property.

Now, we study the converse situation. We begin by giving sufficient conditions.

Théorème 14

Let E and F be two Banach lattices. If at least one of the following three conditions holds then each positive operator $T : E \to F$, such that $T' : F' \to E'$ is almost Dunford-Pettis, must itself be almost Dunford-Pettis :

1. *E has the positive Schur property.*
2. *E has the bi-sequence property and F has an order continuous norm.*
3. *F may be written as an order direct sum $F = G \bigoplus H$ where the*

Proof.

In case (1), very weakly null sequence in E is norm null, so that every bounded operator $T : E \to F$ is almost Dunford-Pettis.
In case (2), again every positive operator $T : E \to F$ is almost Dunford-Pettis.
In fact, let (x_n) be a weakly null disjoint sequence in E_+. It is clear that $0 \leq T(x_n) \to 0$ for $\sigma(F, F')$. Thus by Theorem 2.6 of ([Dodds-Fremlin]), it suffices to show that $f_n\left(T(x_n)\right) \to 0$ for each disjoint norm bounded sequence (f_n) in F'_+. Since the norm in F is order continuous, it follows from Theorem 2.4.3 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) that (f_n) is weak* null, and hence $(T'(f_n))$ is weak* null in E'_+. Finally, by (2)(b), we have
$f_n\left(T(x_n)\right) = T'(f_n)(x_n) \to 0$ and we are done.

In case (3), consider first operators into G where we know that G'' is a KB-space and therefore has an order continuous norm. Let $T : E \to G$ be a positive operator such that $T' : G' \to E'$ is almost Dunford-Pettis. Let (x_n) be a weakly null disjoint sequence in E_+ and note that $0 \leq T(x_n) \to 0$ for $\sigma(G, G')$. Let (f_n) be a disjoint norm bounded sequence in G'_+. Since the norm of G'' is order continuous, it follows from Theorem 2.4.14 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) that (f_n) is a weakly null sequence in G'. Now, as the adjoint $T' : G' \to E'$ is almost Dunford-Pettis, we conclude that $\|T'(f_n)\| \to 0$. Hence $f_n\,(T(x_n)) = T'(f_n)(x_n) \to 0$ and we are done. Again, we use Corollary 2.4 of ($[Dodds\text{-}Fremlin]$, **Compact operators in Banach lattices. Israel J. Math. (1979)**) to conclude that $\|T(x_n)\| \to 0$ showing that T is almost Dunford-Pettis.

The class of almost Dunford-Pettis operators

If H has the positive Schur property then every positive operator $T : E \to H$ is almost Dunford-Pettis. For if (x_n) is a weakly null disjoint sequence in E_+, note that $0 \leq T(x_n) \to 0$ for $\sigma(H, H')$ so that $\|T(x_n)\| \to 0$ and T is almost Dunford-Pettis.
To prove (3) in general, suppose that $T : E \to G \oplus H$ is positive and such that $T' : G' \oplus H' \to E'$ is almost Dunford-Pettis. We will let P_B denote the band projection onto a band B. It is clear that both $T'_{|G'}$ and $T'_{|H'}$ are almost Dunford-Pettis from which our previous work shows that both $P_G \circ T$ and $P_H \circ T$ are almost Dunford-Pettis hence so is T itself.

The class of almost Dunford-Pettis operators

If we assume that the Banach lattice F is Dedekind σ-complete, we obtain the following necessary conditions.

Théorème 15

Let E and F be two Banach lattices such that F is Dedekind σ-complete. If every positive operator T from E into F is almost Dunford-Pettis whenever its adjoint T' from F' into E' is almost Dunford-Pettis, then one of the following assertions is valid :

1. *E has the positive Schur property.*
2. *E has the bi-sequence property and F has an order continuous.*
3. *F is a KB-space.*

Proof.

It suffices to establish the following two separate claims.
(α) If the norm of F is not order continuous, then E has the positive Schur property.
(β) If F is not a KB-space, then $f_n(x_n) \to 0$ for each weakly null disjoint sequence (x_n) in E and each weak* null sequence (f_n) in E'_+.
Assume that E does not have the positive Schur property and that the norm of F is not order continuous. Since E does not have the positive Schur property, it follows from Proposition 2 that there is disjoint weakly null sequence (x_n) in E_+ with $\|x_n\| = 1$ for all n.
Hence, by Theorem 116.3 of ($[Zaanen]$, **Riesz spaces II, 1983**), there exists a positive disjoint sequence (g_n) in E' with $\|g_n\| = g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ if $n \neq m$. Consider the operator $U : E \to \ell_\infty$ defined by $U(x) = (g_n(x))_{n=1}^{\infty}$ for each $x \in E$ which is clearly a positive operator taking values in ℓ_∞.

On the other hand, since the norm of F is not order continuous, it follows from Theorem 2.4.2 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) that there exists an order bounded disjoint sequence (y_n) in F_+ which is not norm convergent to zero. We can assume, without loss of generality, that $\|y_n\| = 1$ and that there is $y \in F_+$ with $0 \leq y_n \leq y$ for all n. Thus, since F is Dedekind σ-complete, it results from the proof of Theorem 117.3 of ($[Zaanen]$, **Riesz spaces II, 1983**)*)*, that the positive operator $V : \ell_\infty \to F$ defined by the order convergent series $V((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n$, for $(\lambda_n) \in \ell_\infty$, is a lattice isomorphism of ℓ_∞ into F.

Now, we consider the positive operator $T = V \circ U : E \to \ell_\infty \to F$ defined by the order convergent series $T(x) = \sum_{n=1}^{\infty} g_n(x) y_n$ for $x \in E$. Its adjoint $T' : F' \to \ell'_\infty \to E'$ is almost Dunford-Pettis. In fact, if (f_n) is a weakly null disjoint sequence in F'_+, then $0 \leq V'(f_n) \to 0$ for $\sigma(\ell'_\infty, \ell''_\infty)$. Since ℓ'_∞ has the positive Schur property, it follows that $\|V'(f_n)\| \to 0$. Hence $\|T'(f_n)\| = \|U'\left(V'(f_n)\right)\| \to 0$ and T' is almost Dunford-Pettis.
However, note that (x_n) is a disjoint weakly null sequence in E_+ and that $U(x_n) = e_n$, where e_n is the n'th standard basis vector in ℓ_∞.
Thus $\|T(x_n)\| = \|V\left(U(x_n)\right)\| = \|V(e_n)\| = \|y_n\| = 1$
for each n so that T is not almost Dunford-Pettis. We have now established claim (α).

To prove claim (β), let us suppose that F is not a KB-space. Then it follows from, for example, Theorem 4.61 of ($[Aliprantis\text{-}Burkinshaw]$, **Positive operators, 2006**) that c_0 is lattice embeddable in F i.e. there exists a lattice homomorphism $S : c_0 \to F$ and two strictly positive constants K and M such that $K\|(\alpha_n)\|_\infty \leq \|S\left((\alpha_n)\right)\| \leq M\|(\alpha_n)\|_\infty$ for all $(\alpha_n) \in c_0$.

Let (x_n) be a weakly null disjoint sequence in E and (f_n) a weak* null sequence in E'_+. We need only prove that $f_n(x_n) \to 0$. It is clear that the operator $R : E \to c_0$, defined by $R(x) = (f_k(x))_{k=1}^\infty$ is positive and does indeed map E linearly into c_0.

Let $T = S \circ R : E \to c_0 \to F$. It is clear that $T' : F' \to c_0' = \ell_1 \to E'$ is Dunford-Pettis and hence almost Dunford-Pettis. By our assumption T is almost Dunford-Pettis. Thus, since (x_n) is a weakly null disjoint sequence in E, we have $\|T(x_n)\| \to 0$. But, for all n,

$\|T(x_n)\| = \|S \circ R(x_n)\| = \|S\left((f_k(x_n))_{k=1}^\infty\right)\| \geq K \left\|(f_k(x_n))_{k=1}^\infty\right\|_\infty \geq K|f_n(x_n)|$

so that $f_n(x_n) \to 0$.

The class of almost Dunford-Pettis operators

Remarque 5

The assumption that F is Dedekind σ-complete is essential for Theorem 4.4 to hold. For instance, if we take $E = \ell_\infty$ and $F = c$, the Banach lattice of all convergent sequences then it is clear that every operator $T : \ell_\infty \to c$ is weakly compact, see the proof of Proposition 1 in ($[Wnuk]$, **Remarks on J. R. Holubs paper concerning Dunford-Pettis operators. Math. Japon., (1993)***), hence is Dunford-Pettis, as ℓ_∞ has the Dunford-Pettis property and therefore is almost Dunford-Pettis. Yet none of the three possible conditions listed in Theorem 4.4 holds.*

The class of almost Dunford-Pettis operators

How do Theorems 4.4 and 4.3 match up to each other ? Apart from the assumption of Dedekind σ-completeness in the latter case, the gap between the two results is that one of the necessary conditions is that F be a KB-space whilst a sufficient condition is that F be the sum $G \oplus H$ where G'' is a KB-space and H has the positive Schur property. We have no example of a KB-space which cannot be written as such a sum.

Now, if in Theorem 4.4, instead of assuming that the Banach lattice F is Dedekind σ-complete, we assume that the Banach lattice E has an order continuous norm, we obtain a slightly different set of necessary conditions.

The class of almost Dunford-Pettis operators

How do Theorems 4.4 and 4.3 match up to each other ? Apart from the assumption of Dedekind σ-completeness in the latter case, the gap between the two results is that one of the necessary conditions is that F be a KB-space whilst a sufficient condition is that F be the sum $G \oplus H$ where G'' is a KB-space and H has the positive Schur property. We have no example of a KB-space which cannot be written as such a sum.
Now, if in Theorem 4.4, instead of assuming that the Banach lattice F is Dedekind σ-complete, we assume that the Banach lattice E has an order continuous norm, we obtain a slightly different set of necessary conditions.

The class of almost Dunford-Pettis operators

Théorème 16

Let E and F be Banach lattices such that E has an order continuous norm. If each positive operator $T : E \to F$, such that $T' : F' \to E'$ is almost Dunford-Pettis, must itself be almost Dunford-Pettis then one of the following assertions is valid :

1. *E has the positive Schur property.*
2. *F is a KB-space.*

Proof. Suppose that E does not have the positive Schur property and F is not a KB-space. Since E does not have the positive Schur property, it follows from Proposition 4.1 that there exists a disjoint weakly null sequence (x_n) in E_+ with $\|x_n\| = 1$ for all n. Hence, by Theorem 116.3 of ($[Zaanen]$, **Riesz spaces II, 1983**)) there exists a positive disjoint sequence (g_n) in E' with $\|g_n\| = g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.

The class of almost Dunford-Pettis operators

Théorème 16

Let E and F be Banach lattices such that E has an order continuous norm. If each positive operator $T : E \to F$, such that $T' : F' \to E'$ is almost Dunford-Pettis, must itself be almost Dunford-Pettis then one of the following assertions is valid :

1. *E has the positive Schur property.*
2. *F is a KB-space.*

Proof. Suppose that E does not have the positive Schur property and F is not a KB-space. Since E does not have the positive Schur property, it follows from Proposition 4.1 that there exists a disjoint weakly null sequence (x_n) in E_+ with $\|x_n\| = 1$ for all n. Hence, by Theorem 116.3 of ($[Zaanen]$, **Riesz spaces II, 1983**)*)* there exists a positive disjoint sequence (g_n) in E' with $\|g_n\| = g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.

As the norm in E is order continuous, it follows from Corollary 2.4.3 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) that $g_n \to 0$ for $\sigma(E', E)$. Hence we may define a positive operator $U : E \to c_0$ by $U(x) = (g_n(x))_{n=1}^{\infty}$ for $x \in E$. On the other hand, since F is not a KB-space, it follows from Theorem 4.61 of ($[Aliprantis\text{-}Burkinshaw]$, **Positive operators, 2006**) that c_0 is lattice embeddable in F so there exists a lattice embedding $V : c_0 \to F$ and strictly positive constants K and M such that
$K\|(\alpha_n)\|_\infty \leq \|V((\alpha_n))\| \leq M\|(\alpha_n)\|_\infty$ for all $(\alpha_n) \in c_0$.
Now consider the positive operator $T = V \circ U : E \to c_0 \to F$. It is clear that its adjoint $T' : F' \to c_0' = \ell_1 \to E'$ is Dunford-Pettis and hence almost Dunford-Pettis. But T is not almost Dunford-Pettis. To see this, note that (x_n) is a disjoint weakly null sequence in E_+, that as above we have $U(x_n) = e_n$ and that
$\|T(x_n)\| = \|V(U(x_n))\| = \|V(e_n)\| \geq K\|e_n\|_\infty = K > 0$
for each n. Thus $\|T(x_n)\| \nrightarrow 0$.
The example given after Theorem 4.4 shows also that the hypothesis of order continuity of the norm may not be omitted from the statement of Theorem 4.5.

As the norm in E is order continuous, it follows from Corollary 2.4.3 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) that $g_n \to 0$ for $\sigma(E', E)$. Hence we may define a positive operator $U : E \to c_0$ by $U(x) = (g_n(x))_{n=1}^{\infty}$ for $x \in E$. On the other hand, since F is not a KB-space, it follows from Theorem 4.61 of ($[Aliprantis\text{-}Burkinshaw]$, **Positive operators, 2006**) that c_0 is lattice embeddable in F so there exists a lattice embedding $V : c_0 \to F$ and strictly positive constants K and M such that

$K\|(\alpha_n)\|_\infty \leq \|V((\alpha_n))\| \leq M\|(\alpha_n)\|_\infty$ for all $(\alpha_n) \in c_0$.

Now consider the positive operator $T = V \circ U : E \to c_0 \to F$. It is clear that its adjoint $T' : F' \to c_0' = \ell_1 \to E'$ is Dunford-Pettis and hence almost Dunford-Pettis. But T is not almost Dunford-Pettis. To see this, note that (x_n) is a disjoint weakly null sequence in E_+, that as above we have $U(x_n) = e_n$ and that

$\|T(x_n)\| = \|V(U(x_n))\| = \|V(e_n)\| \geq K\|e_n\|_\infty = K > 0$

for each n. Thus $\|T(x_n)\| \nrightarrow 0$.

The example given after Theorem 4.4 shows also that the hypothesis of order continuity of the norm may not be omitted from the statement of Theorem 4.5.

The class of order weakly compact operators

The class of order weakly compact operators was introduced in the paper ($[Dodds]$, **o-weakly compact mappings of Riesz spaces. Trans. Amer. Math. Soc. (1975)**). It contains the subspace of weakly compact operators and the subspace of AM-compact operators.

An operator T from a Banach lattice E into a Banach space F is said to be order weakly compact if for each $x \in E^+$, the subset $T([0, x])$ is a relatively weakly compact subset of F.

The class of order weakly compact operators

The class of order weakly compact operators was introduced in the paper ($[Dodds]$, **o-weakly compact mappings of Riesz spaces. Trans. Amer. Math. Soc. (1975)**). It contains the subspace of weakly compact operators and the subspace of AM-compact operators.

An operator T from a Banach lattice E into a Banach space F is said to be order weakly compact if for each $x \in E^{+}$, the subset $T\left([0, x]\right)$ is a relatively weakly compact subset of F.

The class of order weakly compact operators

The class of order weakly compact operators does not satisfy the duality property. In fact,

1. the identity operator $Id_{l^1} : l^1 \to l^1$ is order weakly compact but its adjoint operator, which is the identity operator $Id_{l^\infty} : l^\infty \to l^\infty$, is not order weakly compact.
2. the identity operator $Id_{l_\infty} : l^\infty \to l^\infty$, is not order weakly compact but its dual operator, which is the identity operator $Id_{(l^\infty)'} : (l^\infty)' \to (l^\infty)'$, is order weakly compact.

The class of order weakly compact operators

Our first result gives a sufficient and necessary conditions under which an order weakly compact operator has an adjoint which is order weakly compact :

Théorème 17

Let E and F be two Banach lattices. Then the following conditions are equivalent :

1. *Each regular order weakly compact operator $T : E \to F$ has an order weakly compact adjoint $T' : F' \longrightarrow E'$.*
2. *One of the following assertions is valid :*
   a) *the norm of E' is order continuous.*
   b) *the norm of F' is order continuous.*

Proof. $2 \Longrightarrow 1$. It is just a consequence of Theorem 22.1 of ($[Aliprantis\text{-}Burkinshaw]$, **Locally solid Riesz spaces. Pure and Applied Mathematics, 1978**.)
$1 \Longrightarrow 2$. Assume that the norm of E' (resp. F') is not order continuous. Then Theorem 2.4.14 and Proposition 2.3.11 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) imply that E (resp. F) contains a sublattice isomorphic to l^1 and there exists a positive projection $P_1 : E \longrightarrow l^1$ (resp. $P_2 : F \longrightarrow l^1$).
Since F' is Dedekind σ-complete, it follows from Corollary 2.4.3 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) that F' contains a sublattice isomorphic to l^∞.

We denote by $i_1 : l^1 \longrightarrow E$ (resp. $i_2 : l^1 \longrightarrow F$) the canonical injection of l^1 into E (resp. l^1 into F). We consider the operator product

$$i_2 \circ P_1 : E \longrightarrow l^1 \longrightarrow F.$$

It is an order weakly compact operator because $i_2 \circ P_1 = i_2 \circ Id_{l^1} \circ P_1$ and the identity operator Id_{l^1} is order weakly compact. But the operator $P_1' \circ i_2'$ is not order weakly compact. If not, i.e. if

$$P_1' \circ i_2' : F' \longrightarrow l^\infty \longrightarrow E'$$

is order weakly compact, then the operator product

$$i_1' \circ P_1' \circ i_2' : F \longrightarrow l^\infty$$

would be order weakly compact and hence its restriction to l^∞, which is just the identity operator Id_{l^∞}, would be order weakly compact. But this is impossible.

The class of order weakly compact operators

Conversely, whenever F is a Dedekind σ-complete Banach lattice, the following result gives a sufficient and necessary conditions under which an operator is order weakly compact if its adjoint is order weakly compact :

Théorème 18

Let E and F be two Banach lattices such that F is Dedekind σ-complete. Then the following conditions are equivalent :

1. *Each order bounded operator T from E into $F$is order weakly compact.*
2. *Each order bounded operator T from E into F is order weakly compact whenever its dual operator T' from F' into E' is order weakly compact.*
3. *One of the following assertions is valid :*
   a) *the norm of E is order continuous.*
   b) *the norm of F is order continuous.*

The class of order weakly compact operators

Conversely, whenever F is a Dedekind σ-complete Banach lattice, the following result gives a sufficient and necessary conditions under which an operator is order weakly compact if its adjoint is order weakly compact :

Théorème 18

Let E and F be two Banach lattices such that F is Dedekind σ-complete. Then the following conditions are equivalent :

1. *Each order bounded operator T from E into F is order weakly compact.*
2. *Each order bounded operator T from E into F is order weakly compact whenever its dual operator T' from F' into E' is order weakly compact.*
3. *One of the following assertions is valid :*
   a) *the norm of E is order continuous.*
   b) *the norm of F is order continuous.*

Proof. $1) \Longrightarrow 2)$. Obvious.
$2) \Longrightarrow 3)$. Assume that the norms of E and F are not order continuous.
As the norm of E is not order continuous, it follows from Theorem 2.4.2 of ($[Meyer\text{-}Nieberg]$, **Banach lattices, 1991**) and Lemma 2.2 the existence of a positive order bounded disjoint sequence (x_n) in E^+ with $\|x_n\| = 1$ for all n and there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| \leq 1$ for each n, such that

$$g_n(x_n) = 1 \text{ for all } n \text{ and } g_n(x_m) = 0 \text{ for } n \neq m \ (*).$$

We consider the operator S defined by the following :

$$S : E \longrightarrow l^\infty, x \longmapsto S(x) = (g_n(x))_{n=1}^\infty .$$

It is clear that S is positive.

On the other hand, since the norm of F is not order continuous, there exists $y \in F^+$ and an order bounded disjoint sequence (y_n) in F such that $0 \leq y_n \leq y$ and $\|y_n\| = 1$ for all n.
Now, as F is Dedekind σ-complete, it follows from the proof of Theorem 117.3 of ($[Zaanen]$, **Riesz spaces II, 1983**)*)* that the operator

$$\varphi : l^\infty \longrightarrow F,$$
$$(\lambda_1, \lambda_2, ...) \longmapsto \varphi\left((\lambda_1, \lambda_2, ...)\right) = (o) - i = \overset{\infty}{1} \sum \lambda_i y_i$$

defines a positive operator from l^∞ into F where the convergence is in the sens of the order.

We consider the operator product $T = \varphi \circ S : E \longrightarrow F$ defined by

$$T(x) = (o) - i = \overset{\infty}{1} \sum g_i(x) y_i \text{ for each } x \in E$$

It is well defined and positive, and its adjoint $T' = S' \circ \varphi'$ is order weakly compact. But, the operator T is not order weakly compact. In fact, since $(l^\infty)'$ has an order continuous norm, the positive operator $\varphi' : F' \longrightarrow (l^\infty)'$ is order weakly compact by Theorem 22.1 of ($[Aliprantis\text{-}Burkinshaw]$, **Locally solid Riesz spaces. Pure and Applied Mathematics, 1978**.) and hence $T' = S' \circ \varphi'$ is order weakly compact.

But, the operator T is not order weakly compact. To see this, note that $T(x_n) = y_n$ for all n (by $(*)$) and hence $\|T(x_n)\| = \|y_n\| = 1$ for all n. Now since (x_n) is an order bounded disjoint sequence in E^+, it follows from Dodds's Theorem 5.57 of ($[Aliprantis$-$Burkinshaw]$, **Positive operators, 2006**) that T is not order weakly compact. This completes the proof of $2) \Longrightarrow 3)$.

$3) \Longrightarrow 1)$. It is just a consequence of Theorem 22.1 of ($[Aliprantis$-$Burkinshaw]$, **Locally solid Riesz spaces. Pure and Applied Mathematics, 1978**).

The class of order weakly compact operators

Remarque 6

The condition "F is Dedekind σ-complete" is essential. In fact, each operator T from l^∞ into c (which is not Dedekind σ-complete) is weakly compact (see the proof of Proposition 1 of ($[Wnuk]$, **Remarks on J. R. Holub's paper concerning Dunford-Pettis operators. Math. Japon. (1993)***). Hence T is order weakly compact and its dual operator is also order weakly compact. But, the norm of the Banach lattice l^∞ and the norm of the Banach lattice c are not order continuous.*

Bibliographie

1. Aliprantis C.D. and Burkinshaw O., Locally solid Riesz spaces, Academic Press, 1978.
2. Aliprantis C. D. and Burkinshaw O., Positive operators. Reprint of the 1985 original. Springer, Dordrecht, 2006.
3. Alpay S., Altin B. and Tonyali C., On property (b) of vector lattices. Positivity 7, No. 1-2, (2003) 135-139.
4. Alpay, S. ; Altin, B. A note on b-weakly compact operators. Positivity 11 (2007), no. 4, 575–582.
5. B. Altin, Some properties of b-weakly compact operators. G. U. J. Sci. 18(3) (2005), 91-95.
6. Aqzzouz B., Nouira R. and Zraoula L., The duality problem for the class of AM-compact operators on Banach lattices. Canad. Math. Bull. Vol. 51 (1), (2008), 15-20.
7. Aqzzouz B. and Hmichane J., The duality problem for the class of order weakly compact operators. Glasg. Math. J. 51, No. 1, (2009) 101-108.
8. Aqzzouz B., Elbour A. and Hmichane J., The duality problem for the class of b-weakly compact operators. Positivity 13 (2009), no. 4, 683–692.

1. Aqzzouz B. and Elbour A., On the weak compactness of b-weakly compact operators. Positivity 14 (2010), no. 1, 75–81.
2. Aqzzouz B. and Elbour A., , Characterizations of the order weak compactness of semi-compact operators. J. Math. Anal. Appl. 355 (2009), no. 2, 541–547.
3. Chen Z. L. and Wickstead A. W., Some applications of Rademacher sequences in Banach lattices. Positivity 2 (1998), no. 2, 171–191.
4. Chen, Z. L. and Wickstead, A. W., L-weakly and M-weakly compact operators. Indag. Math. (N.S.) 10 (1999), no. 3, 321-336.
5. Dodds, P. G., o-weakly compact mappings of Riesz spaces. Trans. Amer. Math. Soc. 214 (1975), 389–402.
6. Dodds P.G. and Fremlin D.H., Compact operators on Banach lattices, Israel J. Math. 34 (1979) 287-320.
7. Fremlin D.H., Riesz spaces with the order continuity property I, Math. Proc. Cambr. Phil. Soc. 81 (1977) 31-42.
8. Meyer-Nieberg P., Banach lattices. Universitext. Springer-Verlag, Berlin, 1991.
9. Schaefer H.H., Banach lattices and positive operators, Springer-Verlag, Berlin and New York, 1974.

Thanks for your attentions