Krivine’s Function Calculus and Bochner Integration

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Abstract. We prove that Krivine’s Function Calculus is compatible with integration. Let \((\Omega, \Sigma, \mu)\) be a finite measure space, \(X\) a Banach lattice, \(x \in X^n\), and \(f: \mathbb{R}^n \times \Omega \to \mathbb{R}\) a function such that \(f(\cdot, \omega)\) is continuous and positively homogeneous for every \(\omega \in \Omega\), and \(f(s, \cdot)\) is integrable for every \(s \in \mathbb{R}^n\). Put \(F(s) = \int f(s, \omega) d\mu(\omega)\) and define \(F(x)\) and \(f(x, \omega)\) via Krivine’s Function Calculus. We prove that under certain natural assumptions \(F(x) = \int f(x, \omega) d\mu(\omega)\), where the right hand side is a Bochner integral.

1. Motivation

In [Kal12], the author defines a real-valued function of two real or complex variable via \(F(s, t) = \int_0^{2\pi} |s + e^{i\theta}t| d\theta\). This is a positively homogeneous continuous function. Therefore, given two vectors \(u\) and \(v\) in a Banach lattice \(X\), one may apply Krivine’s Function Calculus to \(F\) and consider \(F(u, v)\) as an element of \(X\). The author then claims that

\[
F(u, v) = \int_0^{2\pi} |u + e^{i\theta}v| d\theta,
\]

where the right hand side here is understood as a Bochner integral; this is used later in [Kal12] to conclude that \(\|F(u, v)\| \leq \int_0^{2\pi} \|u + e^{i\theta}v\| d\theta\) because Bochner integrals have this property: \(\|\int f\| \leq \int \|f\|\). A similar exposition is also found in [DGTJ84, p. 146]. Unfortunately, neither [Kal12] nor [DGTJ84] includes a proof of (1). In this note, we prove a general theorem which implies (1) as a special case.

2. Preliminaries

We start by reviewing the construction of Krivine’s Function Calculus on Banach lattices; see [LT79, Theorem 1.d.1] for details. For Banach lattice terminology, we refer the reader to [AA02, AB06].
Fix $n \in \mathbb{N}$. A function $F: \mathbb{R}^n \to \mathbb{R}$ is said to be **positively homogeneous** if

$$F(\lambda t_1, \ldots, \lambda t_n) = \lambda F(t_1, \ldots, t_n)$$

for all $t_1, \ldots, t_n \in \mathbb{R}$ and $\lambda \geq 0$.

Let $H_n$ be the set of all continuous positively homogeneous functions from $\mathbb{R}^n$ to $\mathbb{R}$. Let $S^\infty_n$ be the unit sphere of $\ell^n_\infty$, that is,

$$S^\infty_n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : \max_{i=1,\ldots,n} |t_i| = 1\}.$$

It can be easily verified that the restriction map $F \mapsto F|_{S^\infty_n}$ is a lattice isomorphism from $H_n$ onto $C(S^\infty_n)$. Hence, we can identify $H_n$ with $C(S^\infty_n)$. For each $i = 1, \ldots, n$, the $i$-th coordinate projection $\pi_i: \mathbb{R}^n \to \mathbb{R}$ clearly belongs to $H_n$.

Let $X$ be a (real) Banach lattice and $x = (x_1, \ldots, x_n) \in X^n$. There exists a unique lattice homomorphism $\Phi: H_n \to X$ such that $\Phi(\pi_i) = x_i$. The map $\Phi$ will be referred to as **Krivine’s function calculus**. We often write $F(x)$ or $\tilde{F}$ instead of $\Phi(F)$. This construction allows one to define expressions like $\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for $0 < p < \infty$ in every Banach lattice $X$; this expression is understood as $\Phi(F)$ where $F(t_1, \ldots, t_n) = \left(\sum_{i=1}^n |t_i|^p\right)^{1/p}$. It is also known that

$$\|F(x)\| \leq \|F\|_{C(S^\infty_n)} \cdot \left\|\bigvee_{i=1}^n |x_i|\right\|.$$

Let $L_n$ be the sublattice of $H_n$ or, equivalently, of $C(S^\infty_n)$, generated by the coordinate projections $\pi_i$ as $i = 1, \ldots, n$. It follows from the Stone-Weierstrass Theorem that $L_n$ is dense in $C(S^\infty_n)$. It follows from $\Phi(\pi_i) = x_i$ that $\Phi(L_n)$ is the sublattice generated by $x_1, \ldots, x_n$ in $X$, hence Range $\Phi$ is contained in the closed sublattice of $X$ generated by $x_1, \ldots, x_n$. It follows from, e.g., Exercise 8 on [AB06, p.204] that this sublattice is separable.

Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $X$ a Banach space. A function $f: \Omega \to X$ is **measurable** if there is a sequence $(f_n)$ of simple functions from $\Omega$ to $X$ such that $\lim_n \|f_n(\omega) - f(\omega)\| = 0$ almost everywhere. If, in addition, $\int \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) \to 0$ then $f$ is **Bochner integrable** with $\int_A f \, d\mu = \lim_n \int_A f_n \, d\mu$ for every measurable set $A$. In the following theorem, we collect a few standard facts about Bochner integral for future reference; we refer the reader to [DU77, Chapter II] for proofs and further details.

**Theorem 2.1.** Let $f: \Omega \to X$.

(i) If $f$ is the almost everywhere limit of a sequence of measurable functions then $f$ is measurable.
(ii) If \( f \) is separable-valued and there is a norming set \( \Gamma \subseteq X^* \) such that \( x^* f \) is measurable for every \( x^* \in \Gamma \) then \( f \) is measurable.

(iii) A measurable function is Bochner integrable iff \( \| f \| \) is integrable.

(iv) If \( f(\omega) = u(\omega)x \) for some fixed \( x \in X \) and \( u \in L_1(\mu) \) then \( f \) is measurable and Bochner integrable.

(v) If \( f \) is Bochner integrable and \( T : X \to Y \) is a bounded operator from \( X \) to a Banach space \( Y \) then \( T(\int f \, d\mu) = \int T f \, d\mu \).

### 3. Uniformly continuous case

Throughout the rest of the paper, we assume that \( (\Omega, \Sigma, \mu) \) be a finite measure space, \( n \in \mathbb{N} \), and \( f : \mathbb{R}^n \times \Omega \to \mathbb{R} \) is such that \( f(\cdot, \omega) \) is in \( H_n \) for every \( \omega \in \Omega \) and \( f(s, \cdot) \) is integrable for every \( s \in \mathbb{R}^n \). For every \( s \in \mathbb{R}^n \), put \( F(s) = \int f(s, \omega) \, d\mu(\omega) \).

It is clear that \( F \) is positively homogeneous.

In general, \( F \) need not be continuous. For example, let \( n = 1 \), let \( \mu \) be a measure on \( \mathbb{N} \) given by \( \mu(\{k\}) = 2^{-k} \), and let \( f_k = f(\cdot, k) \) be defined as follows: it vanishes outside \([0, \frac{2}{2^k}]\), equals \( 2^k \) at \( \frac{1}{2^k} \), and is linear in between. It follows from \( F(s) = \sum_{k=1}^{\infty} 2^{-k} f_k(s) \) that \( F(0) = 0 \) and \( F(2^{-k}) \geq 2^{-k} f_k(2^{-k}) = 1 \), hence \( F \) is discontinuous at zero.

Suppose that \( f(\cdot, \omega) \) is continuous on \( S_n^\infty \) uniformly on \( \omega \), that is,

\[
(3) \quad \text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } |f(s, \omega) - f(t, \omega)| < \varepsilon \\
\text{for all } s, t \in S_n^\infty \text{ and all } \omega \in \Omega \text{ provided that } \|s - t\|_\infty < \delta.
\]

This condition forces \( F \) to be continuous. Indeed, fix \( \varepsilon > 0 \); let \( \delta \) be as in \( (3) \), then

\[
(4) \quad |F(s) - F(t)| \leq \int |f(s, \omega) - f(t, \omega)| \, d\mu(\omega) < \varepsilon \mu(\Omega)
\]

whenever \( s, t \in S_n^\infty \) with \( \|s - t\|_\infty < \delta \). Therefore, \( F \in H_n \).

Let \( X \) be a Banach lattice, \( x \in X^n \), and \( \Phi : H_n \to X \) the corresponding function calculus. Since \( F \in H_n \), \( \tilde{F} = F(x) = \Phi(F) \) is defined as an element of \( X \). On the other hand, for every \( \omega \), the function \( s \in \mathbb{R}^n \mapsto f(s, \omega) \) is in \( H_n \), hence we may apply \( \Phi \) to it. We denote the resulting vector by \( \tilde{f}(\omega) \) or \( f(x, \omega) \). This produces a function \( \omega \in \Omega \mapsto f(x, \omega) \in X \).

**Theorem 3.1.** Suppose that \( f(\cdot, \omega) \) is continuous on \( S_n^\infty \) uniformly on \( \omega \). Then \( f(x, \omega) \) is Bochner integrable as a function of \( \omega \) and \( F(x) = \int f(x, \omega) \, d\mu(\omega) \).

The integral in the right hand side is understood as a Bochner integral. The rest of this section is the proof of this theorem. Without loss of generality, by scaling \( \mu \) and \( x \), we may assume that \( \mu \) is a probability measure and \( \left\| \sum_{i=1}^{n} |x_i| \right\| = 1 \); this
will simplify computations. In particular, (2) becomes \( \|H(x)\| \leq \|H\|_{C(S^n_{\infty})} \) for every \( H \in C(S^n_{\infty}) \). Note also that \( x \) in the theorem is a "fake" variable as \( x \) is fixed. It may be more accurate to write \( \tilde{F} \) and \( \tilde{f}(\omega) \) instead of \( F(x) \) and \( f(x, \omega) \), respectively. Hence, we need to prove that \( \tilde{f} \) as a function from \( \Omega \) to \( X \) is Bochner integrable and its Bochner integral is \( \tilde{F} \).

Fix \( \varepsilon > 0 \). Let \( \delta \) be as in (3). It follows from (4) that

\[
|F(s) - F(t)| < \varepsilon \quad \text{whenever} \quad s, t \in S^n_{\infty} \quad \text{with} \quad \|s - t\|_{\infty} < \delta.
\]

Each of the \( 2n \) faces of \( S^n_{\infty} \) is a translate of the \((n - 1)\)-dimensional unit cube \( B^n_{\infty} \). Partition each of these faces into \((n - 1)\)-dimensional cubes of diameter less than \( \delta \), where the diameter is computed with respect to the \( \|\cdot\|_{\infty} \)-metric. Partition each of these cubes into simplices. Therefore, there exists a partition of the entire \( S^n_{\infty} \) into finitely many simplices of diameter less than \( \delta \). Denote the vertices of these simplices by \( s_1, \ldots, s_m \). Thus, we have produced a triangularization of \( S^n_{\infty} \) with nodes \( s_1, \ldots, s_m \).

Let \( a \in \mathbb{R}^m \). Define a function \( L: S^n_{\infty} \to \mathbb{R} \) by setting \( L(s_j) = a_j \) as \( j = 1, \ldots, m \) and then extending it to each of the simplices linearly; this can be done because every point in a simplex can be written in a unique way as a convex combination of its vertices. We write \( L = Ta \). This gives rise to a linear operator \( T: \mathbb{R}^m \to C(S^n_{\infty}) \). For each \( j = 1, \ldots, m \), let \( e_j \) be the \( j \)-th unit vector in \( \mathbb{R}^m \); put \( d_j = Te_j \). Clearly,

\[
T a = \sum_{j=1}^{m} a_j d_j \quad \text{for every} \quad a \in \mathbb{R}^m.
\]

Let \( H \in C(S^n_{\infty}) \). Let \( L = Ta \) where \( a_j = H(s_j) \). Then \( L \) agrees with \( H \) at \( s_1, \ldots, s_m \). We write \( L = SH \); this defines a linear operator \( S: C(S^n_{\infty}) \to C(S^n_{\infty}) \). Clearly, this is a linear contraction.

Suppose that \( H \in C(S^n_{\infty}) \) is such that \( |H(s) - H(t)| < \varepsilon \) whenever \( \|s - t\|_{\infty} < \delta \). Let \( L = SH \). We claim that \( \|L - H\|_{C(S^n_{\infty})} < \varepsilon \). Indeed, fix \( s \in S^n_{\infty} \). Let \( s_1, \ldots, s_{jn} \) be the vertices of a simplex in the triangularization of \( S^n_{\infty} \) that contains \( s \). Then \( s \) can be written as a convex combination \( s = \sum_{k=1}^{n} \lambda_k s_{jk} \). Note that \( \|s - s_{jk}\|_{\infty} < \delta \) for all \( j = 1, \ldots, n \). It follows that

\[
|L(s) - H(s)| = \left| \sum_{k=1}^{n} \lambda_k L(s_{jk}) - \sum_{k=1}^{n} \lambda_k H(s) \right| \leq \sum_{j=1}^{n} \lambda_k |H(s_{jk}) - H(s)| < \varepsilon.
\]

This proves the claim.
Let $G = SF$. It follows from (5) and the preceding observation $\|G - F\|_{C(S^n_\infty)} < \varepsilon$, so that

(7) \[ \|G(x) - F(x)\| < \varepsilon. \]

Similarly, for every $\omega \in \Omega$, apply $S$ to $f(\cdot, \omega)$ and denote the resulting function $g(\cdot, \omega)$. In particular, $g(s_j, \omega) = f(s_j, \omega)$ for every $\omega \in \Omega$ and every $j = 1, \ldots, m$. It follows also that

(8) \[ \|f(\cdot, \omega) - g(\cdot, \omega)\|_{C(S^n_\infty)} < \varepsilon \]

for every $\omega$, and, therefore

(9) \[ \|\bar{f}(\omega) - \bar{g}(\omega)\| = \|f(x, \omega) - g(x, \omega)\| < \varepsilon, \]

where $\bar{g}(\omega) = g(x, \omega)$ is the image under $\Phi$ of the function $s \in S^n_\infty \mapsto g(s, \omega)$. Note that

(10) \[ G(s_j) = F(s_j) = \int f(s_j, \omega) d\mu(\omega) = \int g(s_j, \omega) d\mu(\omega) \]

for every $j = 1, \ldots, m$. Since $G = SF = Ta$ where $a_j = F(s_j) = G(s_j)$ as $j = 1, \ldots, m$, it follows from (6) that

(11) \[ G = \sum_{j=1}^{m} G(s_j) d_j. \]

Similarly, for every $\omega \in \Omega$, we have

(12) \[ g(\cdot, \omega) = \sum_{j=1}^{m} g(s_j, \omega) d_j. \]

Applying $\Phi$ to (11) and (12), we obtain $\tilde{G} = G(x) = \sum_{j=1}^{m} G(s_j) d_j(x)$ and

\[ \tilde{g}(\omega) = g(x, \omega) = \sum_{j=1}^{m} g(s_j, \omega) d_j(x) = \sum_{j=1}^{m} f(s_j, \omega) d_j(x). \]

Together with Theorem 2.1(iv), this yields that $\tilde{g}$ is measurable and Bochner integrable. It now follows from (10) and (11) that

(13) \[ G(x) = \sum_{j=1}^{m} G(s_j) d_j(x) = \sum_{j=1}^{m} \left( \int g(s_j, \omega) d\mu(\omega) \right) d_j(x) \]

\[ = \int \left( \sum_{j=1}^{m} g(s_j, \omega) d_j(x) \right) d\mu(\omega) = \int g(x, \omega) d\mu(\omega). \]

We will show next that $\tilde{f}$ is Bochner integrable. It follows from (9) and the fact that $\varepsilon$ is arbitrary that $\tilde{f}$ can be approximated almost everywhere (actually, everywhere)
by measurable functions; hence \( \tilde{f} \) is measurable by Theorem 2.1(i). Next, we claim that there exists \( M \in \mathbb{R}_+ \) such that \( |f(s, \omega) - f(1, \omega)| \leq M \) for all \( s \in S^n_\infty \) and all \( \omega \in \Omega \). Here \( 1 = (1, \ldots, 1) \). Indeed, let \( s \in S^n_\infty \) and \( \omega \in \Omega \). Find \( j_1, \ldots, j_l \) such that \( s_{j_1} = 1, s_{j_k} \) and \( s_{j_k+1} \) belong to the same simplex as \( k = 1, \ldots, l - 1 \), and \( s_{j_l} \) is a vertex of a simplex containing \( s \). It follows that

\[
|f(s, \omega) - f(1, \omega)| \leq |f(s, \omega) - f(s_{j_l}, \omega)| + \sum_{k=1}^{l-1} |f(s_{j_{k+1}}, \omega) - f(s_{j_k}, \omega)| \leq l \varepsilon \leq m \varepsilon.
\]

This proves the claim with \( M = m \varepsilon \). It follows that

\[
\|\tilde{f}(\omega)\| \leq \|f(\cdot, \omega)\|_{C(S^n_\infty)} = \sup_{s \in S^n_\infty} |f(s, \omega)| \leq |f(1, \omega)| + M.
\]

Since \( |f(1, \omega)| + M \) is an integrable function of \( \omega \), we conclude that \( \|\tilde{f}\| \) is integrable, hence \( \tilde{f} \) is Bochner integrable by Theorem 2.1(iii). It now follows from (9) that

\[
(14) \quad \left\| \int f(x, \omega) d\mu(\omega) - \int g(x, \omega) d\mu(\omega) \right\| \leq \int \left\| f(x, \omega) - g(x, \omega) \right\| d\mu(\omega) < \varepsilon
\]

Finally, combining (7), (13), and (14), we get

\[
\left\| F(x) - \int f(x, \omega) d\mu(\omega) \right\| < 2 \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, this proves the theorem.

4. An alternative approach

In this section, we present a less direct approach to the problem which leads to a stronger result. We use the same notation as before except that we do not assume that \( f \) is uniformly continuous.

**Theorem 4.1.** Suppose that \( F \) is continuous and the function \( M(\omega) := \|f(\cdot, \omega)\|_{C(S^n_\infty)} \) is integrable. Then \( f(x, \omega) \) is Bochner integrable as a function of \( \omega \) and \( F(x) = \int f(x, \omega) d\mu(\omega) \).

**Proof.** Special case: \( X = C(K) \) for some compact Hausdorff \( K \). By uniqueness of function calculus, Krivine’s function calculus \( \Phi \) agrees with “point-wise” function calculus. In particular,

\[
\tilde{F}(k) = F(x_1(k), \ldots, x_n(k)) \quad \text{and} \quad (\tilde{f}(\omega))(k) = f(x_1(k), \ldots, x_n(k), \omega)
\]

for all \( k \in K \) and \( \omega \in \Omega \). We view \( \tilde{f} \) as a function from \( \Omega \) to \( C(K) \).
We are going to show that \( \tilde{f} \) is Bochner integrable. It follows from \( \tilde{f}(\omega) \in \text{Range } \Phi \) that \( \tilde{f} \) a separable-valued function. For every \( k \in K \), consider the point-evaluation functional \( \varphi_k \in C(K)^* \) given by \( \varphi_k(x) = x(k) \). Then

\[
\varphi_k \tilde{f}(\omega) = (\tilde{f}(\omega))(k) = f(x_1(k), \ldots, x_n(k), \omega).
\]

for every \( k \in K \). By assumptions, this function is integrable; in particular, it is measurable. Since the set \( \{ \varphi_k : k \in K \} \) is norming in \( C(K)^* \), Theorem 2.1(vi) yields that \( \tilde{f} \) is measurable.

Clearly, \( |(\tilde{f}(\omega))(k)| \leq M(\omega) \) for every \( k \in K \) and \( \omega \in \Omega \), so that \( \| \tilde{f}(\omega) \|_{C(K)} \leq M(\omega) \) for every \( \omega \). It follows that \( \int \| \tilde{f}(\omega) \|_{C(K)} d\mu(\omega) \) exists and, therefore, \( \tilde{f} \) is Bochner integrable by Theorem 2.1(vii).

Put \( h := \int \tilde{f}(\omega) d\mu(\omega) \), where the right hand side is a Bochner integral. Applying Theorem 2.1(v), we get

\[
h(k) = \varphi_k(h) = \int \varphi_k(\tilde{f}(\omega)) d\mu(\omega) = \int f(x_1(k), \ldots, x_n(k), \omega) d\mu(\omega)
\]

\[
= F(x_1(k), \ldots, x_n(k)) = \tilde{F}(k).
\]

for every \( k \in K \). It follows that \( \int \tilde{f}(\omega) d\omega = \tilde{F} \).

**General case.** Let \( e = |x_1| \vee \ldots |x_n| \). The principal order ideal \( I_e \) in \( X \) equipped with the norm

\[
\| x \|_e = \inf \{ \lambda > 0 : |x| \leq \lambda e \}
\]

is lattice isometric to \( C(K) \) for some compact Hausdorff \( K \). Note also that \( |x| \leq \|x\|_e e \) for every \( x \in I_e \); this yields \( \|x\| \leq \|x\|_e e \), hence the inclusion map \( T : (I_e, \| \cdot \|_e) \to X \) is bounded. Identifying \( I_e \) with \( C(K) \), we may view \( T \) as a bounded lattice embedding from \( C(K) \) into \( X \).

By the construction on Krivine’s Function Calculus, \( \Phi \) actually acts into \( I_e \), i.e., \( \Phi = T\Phi_0 \), where \( \Phi_0 \) is the \( C(K) \)-valued function calculus. By the special case, we know that \( \int \tilde{f}(\omega) d\mu(\omega) = \tilde{F} \) in \( C(K) \). Applying \( T \), we obtain the same identity in \( X \) by Theorem 2.1(vi).

Finally, we analyze whether any of the assumptions may be removed. Clearly, one cannot remove the assumption that \( F \) is continuous; otherwise, \( \tilde{F} \) would make no sense. We claim that the assumption that \( M \) is integrable cannot be removed as well. Indeed, consider the special case when \( X = C(S^*_\infty) \) and \( x_i = \pi_i \) as \( i = 1, \ldots, n \). In this case, \( \Phi \) is the identity map and \( \tilde{f}(\omega) = f(\cdot, \omega) \). It follows from Theorem 2.1(vii) that \( \tilde{f} \) is Bochner integrable iff \( \| \tilde{f} \| \) is integrable iff \( M \) is integrable.
It is easy to see that if $f$ is continuous uniformly on $\omega$ then $M$ is integrable; we implicitly proved it inside the proof of Theorem 3.1. Thus, Theorem 4.1 is formally stronger than Theorem 3.1. Even though Theorem 3.1 is weaker, we included it because its proof is direct.

Finally, the assumption that $f(\cdot, \omega)$ is in $H_n$ for every $\omega$ may clearly be relaxed to “for almost every $\omega$”.

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