SPECTRUM OF A WEAKLY HYPERCYCLIC OPERATOR MEETS THE UNIT CIRCLE

S. J. DILWORTH AND VLADIMIR G. TROITSKY

Abstract. It is shown that every component of the spectrum of a weakly hypercyclic operator meets the unit circle. The proof is based on the lemma that a sequence of vectors in a Banach space whose norms grow at geometrical rate doesn’t have zero in its weak closure.

Suppose that $T$ is a bounded operator on a nonzero Banach space $X$. Given a vector $x \in X$, we say that $x$ is hypercyclic for $T$ if the orbit $\text{Orb}_T \{x\} = \{T^n x\}$ is dense in $X$. Similarly, $x$ is said to be weakly hypercyclic if $\text{Orb}_T \{x\}$ is weakly dense in $X$. A bounded operator is called hypercyclic or weakly hypercyclic if it has a hypercyclic or, respectively, a weakly hypercyclic vector. It is shown in [CS] that a weakly hypercyclic vector need not be hypercyclic, and there exist weakly hypercyclic operators which are not hypercyclic. C. Kitai showed in [K] that every component of the spectrum of a hypercyclic operator intersects the unit circle. K. Chan and R. Sanders asked in [CS] if the spectrum of a weakly hypercyclic operator meets the unit circle. In this note we show that every component of the spectrum of a weakly hypercyclic operator meets the unit circle.

Lemma 1. Let $X$ be a Banach space and let $c > 1$. Suppose that $x_n \in X$ satisfies $\|x_n\| \geq c^n$ for all $n \geq 1$. Then $0 \notin \{x_n\}_w$.

Proof. Let $N$ be the smallest positive integer such that $c^N > 2$. We shall prove that there exist $F_1, \ldots, F_N \in X^*$ such that

\[ \max_{1 \leq k \leq N} |F_k(x_n)| \geq 1 \quad (n \geq 1). \]

Since $\|x_n\| \geq c^n$, by replacing $x_n$ by $(c^n/\|x_n\|) x_n$, it suffices to prove (1) for the case in which $\|x_n\| = c^n$ for all $n \geq 1$. First suppose that $c > 2$, so that $N = 1$. We have to construct $F_1 \in X^*$ such that $|F_1(x_1)| \geq 1$ for all $n \geq 1$. First choose $f_1 \in X^*$ with $f_1(x_1) = 1$. Then either $|f_1(x_2)| < 1$ or $|f_1(x_2)| \geq 1$. In the former case the Hahn-Banach theorem guarantees the existence of $g_2 \in X^*$ such that $\|g_2\| \leq 1/\|x_2\| = c^{-2}$, $|g_2(x_2)| = 1 - |f_1(x_2)|$, and $|(f_1 + g_2)(x_2)| = 1$. In the latter case, set $g_2 = 0$. Note that

$|((f_1 + g_2)(x_1))| \geq 1 - \|g_2\||x_1| \geq 1 - c^{-1}$.

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Set \( f_2 = f_1 + g_2 \). Repeating this argument, we can find \( g_3 \in X^* \) such that \( \|g_3\| \leq 1/\|x_3\| = c^{-3} \) and \( \|(f_2 + g_3)(x_3)\| \geq 1 \). Note that

\[
\|(f_2 + g_3)(x_1)\| \geq \|f_2(x_1)\| - \|g_3\|\|x_1\| \geq 1 - c^{-1} - c^{-2}
\]

and also that

\[
\|(f_2 + g_3)(x_2)\| \geq 1 - \|g_3\|\|x_2\| \geq 1 - c^{-1}.
\]

Set \( f_3 = f_2 + g_3 \). Continuing in this way we obtain \( f_n \in X^* \) such that (setting \( g_n = f_n - f_{n-1} \)) \( \|g_n\| \leq c^{-n} \) and

\[
\|f_n(x_k)\| \geq 1 - \sum_{i=1}^{n-k} c^{-i} \quad (1 \leq k \leq n).
\]

Thus, \( \{f_n\}_n \) is norm-convergent in \( X^* \) to some \( f \in X^* \). From (2), we obtain (since \( c > 2 \))

\[
|f(x_k)| = \lim_{n} |f_n(x_k)| \geq 1 - \sum_{i=1}^{\infty} c^{-i} = \frac{c - 2}{c - 1} > 0.
\]

Set \( F_1 = (c - 1)(c - 2)^{-1}f \), to complete the proof in the case \( c > 2 \).

Now suppose that \( 1 < c < 2 \). Set \( \alpha = c^N > 2 \). For each \( 1 \leq k \leq N \), consider the sequence \( y_n = x_{k+(n-1)N} \) \( (n \geq 1) \). Then \( \|y_n\| = (c^k/\alpha)n^\alpha \). Since \( \alpha > 2 \) there exists \( F_k \in X^* \) such that \( \|F_k(y_n)\| \geq 1 \) for all \( n \geq 1 \), which proves (1). \( \square \)

**Remark 2.** Recall that a closed subspace \( Y \) of \( X^* \) is said to be *norming* if there exists \( C > 0 \) such that

\[
\|x\| \leq C \sup \{|f(x)| : f \in Y, \|f\| \leq 1\} \quad (x \in X).
\]

The argument of Lemma 1 easily generalizes to give the following result. Suppose that \( Y \) is norming for \( X \) and that \( \{x_n\}_n \) is a sequence in \( X \) satisfying \( \|x_n\| \geq c^n \), where \( c > 1 \). Then \( 0 \) does not belong to the \( \sigma(X,Y) \)-closure of \( \{x_n\}_n \). In particular, Lemma 1 is valid for the weak-star topology when \( X \) is a dual space.

We also make use of the following simple numerical fact. If \( (t_n) \) is a sequence in \( \mathbb{R}^+ \cup \{\infty\} \), then

\[
\limsup_{n \to \infty} \sqrt[n]{t_n} = \inf \{\nu > 0 : \lim_{n \to \infty} \frac{t_n}{n^\nu} = 0\} = \inf \{\nu > 0 : \limsup_{n \to \infty} \frac{t_n}{n^\nu} < \infty\}.
\]

In particular, if \( T \) is a bounded operator with spectral radius \( r \), then the Gelfand formula \( \lim_n \sqrt[n]{\|T^n\|} = r \) yields that \( \frac{\|T^n\|}{n^\nu} \to 0 \) for every scalar \( \lambda \) with \( |\lambda| > r \).

**Theorem 3.** If \( T \) is weakly hypercyclic, then every component of \( \sigma(T) \) meets \( \{z : \|z\| = 1\} \).

**Proof.** Let \( x \) be a weakly hypercyclic vector for \( T \). Let \( \sigma' = \sigma(T) \setminus \sigma \). Denote by \( X_\sigma \) and \( X_{\sigma'} \) the corresponding spectral subspaces, then \( X_\sigma \) and \( X_{\sigma'} \) are closed, \( T \)-invariant, and \( X = X_\sigma \oplus X_{\sigma'} \). Also, \( \sigma(T_{|X_\sigma}) = \sigma \) and \( \sigma(T_{|X_{\sigma'}}) = \sigma' \). Note that \( \sigma' \) might be empty, in which case we have \( X_\sigma = X \) and \( X_{\sigma'} = \{0\} \).

Denote by \( P_\sigma \) the spectral projection corresponding to \( \sigma \), then \( X_\sigma = \text{Range}P_\sigma \). Denote \( y = P_{\sigma'}x \). Without loss of generality, \( \|y\| = 1 \). Since \( P_\sigma \) is bounded and, therefore, weakly continuous, and \( \text{Orb}_T y = P_\sigma(\text{Orb}_T x) \), we conclude that \( \text{Orb}_T y \) is weakly dense in \( X_\sigma \). Thus, \( y \) is weakly hypercyclic for \( T_{|X_{\sigma'}} \).
Observe that the inclusion \( \sigma \subseteq \{ z : |z| < 1 \} \) is impossible. Indeed, in this case the spectral radius of \( T |_{X_\sigma} \) would be less than 1, so that \( T^n y \to 0 \), which contradicts \( y \) being weakly hypercyclic for \( T |_{X_\sigma} \).

Finally, we show that the inclusion \( \sigma \subseteq \{ z : |z| > 1 \} \) is equally impossible. In this case \( 0 \notin \sigma = \sigma(T |_{X_\sigma}) \), so that \( T |_{X_\sigma} \) is invertible. Denote the inverse by \( S \).

Then \( S \) is a bounded operator on \( X_\sigma \) and by the Spectral Mapping Theorem

\[
\sigma(S) = \{ \lambda \mid \lambda^{-1} \in \sigma(T |_{X_\sigma}) \} \subset \{ z : |z| < 1 \}.
\]

Therefore, \( r(S) < a \) for some \( 0 < a < 1 \). This yields \( \lim_n \frac{\|S^n\|}{a^n} = 0 \), so that \( \|S^n\| \leq a^n \) for all sufficiently large \( n \). In particular,

\[
1 = \|y\| = \|S^n T^n y\| \leq a^n \|T^n y\|,
\]

so that \( \|T^n y\| \geq \frac{1}{a^n} \). Lemma 1 asserts that \( 0 \notin \{ T^n y \}_n \), which contradicts \( y \) being weakly hypercyclic for \( T |_{X_\sigma} \). □

**Proposition 4.** Suppose that \( Y \) is norming for \( X \). If \( T \) has a hypercyclic vector for the \( \sigma(X,Y) \) topology, then the spectrum of \( T \) intersects \( \{ z : |z| = 1 \} \).

**Proof.** Suppose, to derive a contradiction, that \( \sigma(T) \) does not intersect the unit circle. We use the notation introduced above with \( \sigma = \sigma(T) \cap \{ z : |z| < 1 \} \) and \( \sigma' = \sigma(T) \setminus \sigma \).

Let \( x \) be a hypercyclic vector for the \( \sigma(X,Y) \)-topology. Then \( x = y + z \), where \( y \in X_\sigma \) and \( z \in X_{\sigma'} \). Since \( \|T^n y\| \to 0 \), it follows easily that \( z \) is hypercyclic, and hence that \( z \neq 0 \). But then there exists \( c > 1 \) such that \( \|T^n z\| \geq c^n \) for all sufficiently large \( n \), which contradicts Remark 2. □

**References**


Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.

Current address: Department of Mathematics, The University of Texas at Austin, Austin, Texas 78712, USA.

Department of Mathematical and Statistical Sciences, 632 CAB, University of Alberta, Edmonton, AB T6G 2G1, Canada.

E-mail address: dilworth@math.sc.edu, vtroitsky@math.ualberta.ca