UNBOUNDED NORM TOPOLOGY IN BANACH LATTICES

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Abstract. A net $(x_\alpha)$ in a Banach lattice $X$ is said to un-converge to a vector $x$ if $\| |x_\alpha - x| \wedge u \| \to 0$ for every $u \in X_+$. In this paper, we investigate un-topology, i.e., the topology that corresponds to un-convergence. We show that un-topology agrees with the norm topology iff $X$ has a strong unit. Un-topology is metrizable iff $X$ has a quasi-interior point. Suppose that $X$ is order continuous, then un-topology is locally convex iff $X$ is atomic. An order continuous Banach lattice $X$ is a KB-space iff its closed unit ball $B_X$ is un-complete. For a Banach lattice $X$, $B_X$ is un-compact iff $X$ is an atomic KB-space. We also study un-compact operators and the relationship between un-convergence and weak*-convergence.

1. Introduction and preliminaries

For a net $(x_\alpha)$ in a vector lattice $X$, we write $x_\alpha \xrightarrow{o} x$ if $(x_\alpha)$ converges to $x$ in order. That is, there is a net $(u_\gamma)$, possibly over a different index set, such that $u_\gamma \downarrow 0$ and for every $\gamma$ there exists $\alpha_0$ such that $|x_\alpha - x| \leq u_\gamma$ whenever $\alpha \geq \alpha_0$. We write $x_\alpha \xrightarrow{uo} x$ and say that $(x_\alpha)$ uo-converges to $x$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in X_+$; “uo” stands for “unbounded order”. For a net $(x_\alpha)$ in a normed lattice $X$, we write $x_\alpha \xrightarrow{un} x$ if $(x_\alpha)$ converges to $x$ in norm. We write $x_\alpha \xrightarrow{un} x$

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and say that \((x_\alpha)\) **un-converges** to \(x\) if \(|x_\alpha - x| \wedge u \rightarrow 0\) for every \(u \in X_+\); “un” stands for “unbounded norm”.

A variant of uo-convergence was originally introduced in [Nak48], while the term “uo-convergence” was first coined in [DeM64]. Relationships between uo, weak, and weak\(^*\) convergences were investigated in [Wic77, GX14, Gao14]. Relationships between uo-convergence and almost everywhere convergence were investigated and applied in [GX14, EM16, GTX]. We refer the reader to [GTX] for a further review of properties of uo-convergence. Un-convergence was introduced in [Tro04] and further investigated in [DOT]. For unexplained terminology on vector and Banach lattices we refer the reader to [AA02, AB06]. All vector lattices are assumed to be Archimedean.

Let us start by briefly going over some of the known properties of these modes of convergence; we refer the reader to [GTX, DOT] for details. Both uo-convergence and un-convergence respect linear and lattice operations; limits are unique. In particular, \(x_\alpha \stackrel{\text{uo}}{\rightarrow} x\) iff \(|x_\alpha - x| \stackrel{\text{uo}}{\rightarrow} 0\); similarly, \(x_\alpha \stackrel{\text{un}}{\rightarrow} x\) iff \(|x_\alpha - x| \stackrel{\text{un}}{\rightarrow} 0\). For order bounded nets, uo-convergence agrees with order convergence while un-convergence agrees with norm convergence. It follows that order intervals are uo- and un-closed. For sequences in \(L_p(\mu)\), where \(1 \leq p < \infty\) and \(\mu\) is a finite measure, it is easy to see that uo-convergence agrees with convergence almost everywhere, see, e.g., [DeM64] Example 2]. Under the same assumptions, un-convergence agrees with convergence in measure, see [Tro04, Example 23]. We write \(L_p\) for \(L_p[0,1]\).

Suppose that \(X\) is a vector lattice. By [GTX] Corollary 3.6, every disjoint sequence in \(X\) is uo-null. Recall that a sublattice \(Y\) of \(X\) is **regular** if the inclusion map preserves suprema and infima of arbitrary subsets. It was shown in [GTX, Theorem 3.2] that uo-convergence is stable under passing to and from regular sublattices. That is, if \((y_\alpha)\) is a net in a regular sublattice \(Y\) of \(X\) then \(y_\alpha \stackrel{\text{uo}}{\rightarrow} 0\) in \(Y\) iff \(y_\alpha \stackrel{\text{uo}}{\rightarrow} 0\) in \(X\) (in fact, this property characterizes regular sublattices).

It is clear that if \(X\) is an order continuous normed lattice then uo-convergence implies un-convergence. Let \(X\) be a Banach lattice and \((x_n)\) a un-null sequence in \(X\). Then \((x_n)\) has a uo-null subsequence by
Proposition 4.1 of [DOT]. A disjoint sequence need not be un-null. For example, the standard unit sequence \((e_n)\) in \(\ell_\infty\) is not un-null. However, a un-null sequence has an asymptotically disjoint subsequence. More precisely, we have the following.

**Theorem 1.1.** ([DOT, Theorem 3.2]) Let \((x_\alpha)\) be a un-null net. There is an increasing sequence of indices \((\alpha_k)\) and a disjoint sequence \((d_k)\) such that \(x_{\alpha_k} - d_k \to 0\).

While uo-convergence need not be given by a topology, it was observed in [DOT] that un-convergence is topological. For every \(\varepsilon > 0\) and non-zero \(u \in X_+\), put

\[V_{\varepsilon,u} = \{x \in X : \|x\chi u\| < \varepsilon\}.\]

The collection of all sets of this form is a base of zero neighborhoods for a topology, and the convergence in this topology agrees with un-convergence. We will refer to this topology as un-topology.

Every time a new linear topology is discovered, one is expected to ask several natural questions: is this topology metrizable? Is it locally-convex? Complete? Can one characterize (relatively) compact sets? Is this topology stronger or weaker than other known topologies? In this paper, we study these and similar questions for un-topology. In other words, our motivation for this paper is to investigate topological properties of un-topology.

Throughout this paper, \(X\) will be assumed to be a Banach lattice, unless specified otherwise. We write \(B_X\) for the closed unit ball of \(X\). It was observed in [DOT] that \(x_\alpha \rightharpoonup x\) implies \(\|x\| \leq \liminf \|x_\alpha\|\). This yields that \(B_X\) is un-closed.

The following facts will be used throughout the paper.

**Lemma 1.2.**

(i) If \((x_\alpha)\) is an increasing net in a vector lattice \(X\) and \(x_\alpha \rightharpoonup x\) then \(x_\alpha \uparrow x\);

(ii) If \((x_\alpha)\) is an increasing net in a normed lattice \(X\) and \(x_\alpha \rightharpoonup x\) then \(x_\alpha \uparrow x\) and \(x_\alpha \rightharpoonup x\).

**Proof.** Without loss of generality, \(x_\alpha \geq 0\) for all \(\alpha\); otherwise, pick any index \(\alpha_0\) and consider the net \((x_\alpha - x_{\alpha_0})_{\alpha \geq \alpha_0}\), which converges to
\(x - x_{a_0}\). Since lattice operations are uo- and un-continuous, we have \(x \geq 0\).

(i) Take any \(z \in X_+\). It follows from uo-continuity of lattice operations that \(x_\alpha \land z \xrightarrow{uo} x \land z\). Since the net \((x_\alpha \land z)\) is order bounded and increasing, this yields \(x_\alpha \land z \xrightarrow{o} x \land z\) and, therefore \(x_\alpha \land z \uparrow x \land z\). It follows that \(x_\alpha \land z \leq x\) for every \(\alpha\) and every \(z \in X_+\). Applying this with \(z = x_\alpha\) we get \(x_\alpha \leq x\). Thus, the net \((x_\alpha)\) is order bounded and, therefore, \(x_\alpha \xrightarrow{o} x\), hence \(x_\alpha \uparrow x\).

(ii) The proof is similar and uses the fact that every monotone norm convergent net converges in order to the same limit. We note that \(x_\alpha \land z \xrightarrow{\|\cdot\|} x \land z\) and, therefore, \(x_\alpha \land z \uparrow x \land z\) for every \(z \in X_+\). It follows that the net \((x_\alpha)\) is order bounded, which yields \(x_\alpha \xrightarrow{\|\cdot\|} x\) and, therefore, \(x_\alpha \uparrow x\).

Recall that [DOT] Question 2.14 asks whether \(x_\alpha \xrightarrow{un} 0\) implies that there exists an increasing sequence of indices \((\alpha_k)\) such that \(x_{\alpha_k} \xrightarrow{un} 0\).

The following counterexample was kindly provided to us by E. Emelyanov.

**Example 1.3.** Let \(\Omega\) be an uncountable set; let \(X\) be the closed sublattice of \(\ell_\infty(\Omega)\) consisting of all the functions with countable support. For \(\omega \in \Omega\), we write \(e_\omega\) for the characteristic function of \(\{\omega\}\).

Let \(\Lambda\) be the set of all countable subsets of \(\Omega\), ordered by inclusion. For each \(\alpha \in \Lambda\), pick any \(\omega \notin \alpha\) and put \(x_\alpha = e_\omega\). We claim that \(x_\alpha \xrightarrow{un} 0\). Indeed, let \(u \in X_+\); let \(\alpha_0\) be the support of \(u\). Then \(x_\alpha \land u = 0\) whenever \(\alpha \geq \alpha_0\).

On the other hand, let \((\omega_k)\) be any sequence in \(\Omega\); we claim that the sequence \((e_{\omega_k})\) is not un-null. Indeed, put \(\beta = \{\omega_k : k \in \mathbb{N}\}\) and let \(u\) be the characteristic function of \(\beta\). Then \(e_{\omega_k} \land u = e_{\omega_k}\) for every \(k\); hence it does not converge in norm to zero.

In particular, if \((\alpha_k)\) is an increasing sequence of indices in \(\Lambda\) then \((x_{\alpha_k})\) is not un-null.

Let \(e \in X_+\). Recall that the band \(B_e\) generated by \(e\) is norm closed and contains the principal ideal \(I_e\); hence \(I_e \subseteq T_e \subseteq B_e\). Recall also that
• *e* is a **strong unit** when \( I_e = X \); equivalently, for every \( x \geq 0 \) there exists \( n \in \mathbb{N} \) such that \( x \leq ne \);

• *e* is a **quasi-interior point** if \( \bar{T}_e = X \); equivalently, \( x \wedge ne \xrightarrow{\|\cdot\|} x \) for every \( x \in X_+ \);

• *e* is a **weak unit** if \( B_e = X \); equivalently, \( x \wedge ne \uparrow x \) for every \( x \in X_+ \).

In particular, strong unit \( \Rightarrow \) quasi-interior point \( \Rightarrow \) weak unit.

## 2. Strong units

It is easy to see that each \( V_{\varepsilon,u} \) is solid. It is also absorbing, that is, for every \( x \in X \) there exists \( \lambda > 0 \) such that \( \lambda x \in V_{\varepsilon,u} \). The following lemma is a dichotomy: it says that \( V_{\varepsilon,u} \) is either “very small” or “very large”.

**Lemma 2.1.** Let \( \varepsilon > 0 \), and \( 0 \neq u \in X_+ \). Then \( V_{\varepsilon,u} \) is either contained in \([-u,u]\) or contains a non-trivial ideal.

**Proof.** Suppose that \( V_{\varepsilon,u} \) is not contained in \([-u,u]\). Then there exists \( x \in V_{\varepsilon,u} \) such that \( x \notin [-u,u] \). Replacing \( x \) with \( |x| \), we may assume that \( x > 0 \). Let \( y = (x-u)^+ \); then \( y > 0 \). It is an easy exercise to show that \( (\lambda y) \wedge u \leq x \wedge u \) for every \( \lambda \geq 0 \); it follows that \( \lambda y \in V_{\varepsilon,u} \). Since \( V_{\varepsilon,u} \) is solid, it contains the principal ideal \( I_y \). □

**Lemma 2.2.** If \( V_{\varepsilon,u} \) is contained in \([-u,u]\) then \( u \) is a strong unit.

**Proof.** Let \( x \in X_+ \). There exists \( \lambda > 0 \) such that \( \lambda x \in V_{\varepsilon,u} \), hence \( \lambda x \in [-u,u] \). It follows that \( u \) is a strong unit. □

Recall that if \( e \) is a positive vector in \( X \) then the principal ideal \( I_e \) equipped with the norm

\[
\|x\|_e = \inf \{ \lambda > 0 : |x| \leq \lambda e \}
\]

is lattice isometric to \( C(K) \) for some compact Hausdorff space \( K \), with \( e \) corresponding to the constant one function \( 1 \); see, e.g., Theorems 3.4 and 3.6 in [AA02]. If \( e \) is a strong unit in \( X \) then \( I_e = X \); it is easy to see that in this case \( \|\cdot\|_e \) is equivalent to the original norm; it follows that \( X \) is lattice and norm isomorphic to \( C(K) \).
It is easy to see that if $x_\alpha \xrightarrow{\|\cdot\|} x$ then $x_\alpha \xrightarrow{\text{un}} x$, so norm topology generally is stronger than un-topology.

**Theorem 2.3.** Let $X$ be a Banach lattice. The following are equivalent.

(i) Un-topology agrees with norm topology;

(ii) $X$ has a strong unit.

*Proof.* Suppose that un-topology and norm topology agree. It follows that $V_{\varepsilon,u}$ is contained in $B_X$ for some $\varepsilon > 0$ and $u > 0$. By Lemma 2.1, we conclude that $V_{\varepsilon,u}$ is contained in $[-u,u]$; hence $u$ is a strong unit by Lemma 2.2.

Suppose now that $X$ has a strong unit. Then $X$ is lattice and norm isomorphic to $C(K)$ for some compact Hausdorff space $K$. Without loss of generality, $X = C(K)$. It follows from $x_\alpha \xrightarrow{\text{un}} 0$ that $|x_\alpha| \xrightarrow{\|\cdot\|} 0$. Since the norm in $C(K)$ is the sup-norm, it is easy to see that $x_\alpha \xrightarrow{\|\cdot\|} 0$. □

3. **QUASI-INTERIOR POINTS AND METRIZABILITY**

Given a net $(x_\alpha)$ in a vector lattice with a weak unit $e$, then $x_\alpha \xrightarrow{\text{un}} x$ iff $|x_\alpha - x| \wedge e \xrightarrow{0}$; see, e.g., [GTX, Corollary 3.5] (this was proved in [Kap97] in the special case when the lattice is order complete). That is, it suffices to test $\text{uo}$-convergence on a weak unit. Lemma 2.11 in [DOT] provides a similar statement for un-convergence and quasi-interior points. We now prove that this property actually characterizes quasi-interior points.

**Theorem 3.1.** Let $e \in X_+$. The following are equivalent.

(i) $e$ is a quasi-interior point;

(ii) For every net $(x_\alpha)$ in $X_+$, if $x_\alpha \wedge e \xrightarrow{\|\cdot\|} 0$ then $x_\alpha \xrightarrow{\text{un}} 0$;

(iii) For every sequence $(x_n)$ in $X_+$, if $x_n \wedge e \xrightarrow{\|\cdot\|} 0$ then $x_n \xrightarrow{\text{un}} 0$.

*Proof.* The implication $\text{(i)} \Rightarrow \text{(ii)}$ was proved in [DOT, Lemma 2.11].

$\text{(ii)} \Rightarrow \text{(iii)}$ is trivial. This leaves $\text{(iii)} \Rightarrow \text{(i)}$.

Suppose $\text{(iii)}$. Fix $x \in X_+$. We need to show that $x \wedge ne \xrightarrow{\|\cdot\|} x$ or, equivalently $(x - ne)^+ \xrightarrow{\|\cdot\|} 0$ as a sequence of $n$. Put $u = x \vee e$. The
ideal \( I_u \) is lattice isomorphic (as a vector lattice) to \( C(K) \) for some compact space \( K \), with \( u \) corresponding to \( \mathbb{1} \). Since \( x, e \in I_u \), we may consider \( x \) and \( e \) as elements of \( C(K) \). Note that \( x \vee e = \mathbb{1} \) implies that \( x \) and \( e \) never vanish simultaneously.

For each \( n \in \mathbb{N} \), we define

\[
F_n = \{ t \in K : x(t) \geq ne(t) \} \quad \text{and} \quad O_n = \{ t \in K : x(t) > ne(t) \}.
\]

Clearly, \( O_n \subseteq F_n \), \( O_n \) is open, and \( F_n \) is closed.

**Claim 1:** \( F_{n+1} \subseteq O_n \). Indeed, let \( t \in F_{n+1} \). Then \( x(t) \geq (n+1)e(t) \).

If \( e(t) > 0 \) then \( x(t) > ne(t) \), so that \( t \in O_n \). If \( e(t) = 0 \) then \( x(t) > 0 \), hence \( t \in O_n \).

By Urysohn’s Lemma, we find \( z_n \in C(K) \) such that \( 0 \leq z_n \leq x \), \( z_n \) agrees with \( x \) on \( F_{n+1} \) and vanishes outside of \( O_n \). We can also view \( z_n \) as an element of \( X \).

**Claim 2:** \( n(z_n \land e) \leq x \). Let \( t \in K \). If \( t \in O_n \) then \( n(z_n \land e)(t) \leq ne(t) < x(t) \). If \( t \notin O_n \) then \( z_n(t) = 0 \), so that the inequality is satisfied trivially.

**Claim 3:** \( (x - (n+1)e)^+ \leq z_n \). Again, let \( t \in K \). If \( t \in F_{n+1} \) then \( (x - (n+1)e)^+ \leq x(t) = z_n(t) \). If \( t \notin F_{n+1} \) then \( x(t) < (n+1)e(t) \), so that \( (x - (n+1)e)^+(t) = 0 \) and the inequality is satisfied trivially.

Now, Claim 2 yields \( 0 \leq z_n \land e \leq \frac{1}{n} \parallel x \parallel \rightarrow 0 \), so that \( z_n \land e \parallel \parallel 0 \). By assumption, this yields \( z_n \rightharpoonup 0 \). Since \( 0 \leq z_n \leq x \) for every \( n \), the sequence \( (z_n) \) is order bounded and, therefore, \( z_n \rightharpoonup 0 \). Now Claim 3 yields \( (x - (n+1)e)^+ \rightharpoonup 0 \), which concludes the proof.

**Theorem 3.2.** Un-topology is metrizable iff \( X \) has a quasi-interior point. If \( e \) is a quasi-interior point then \( d(x, y) = \parallel |x - y| \land e \parallel \) is a metric for \( un \)-topology.

**Proof.** Suppose that \( e \in X_+ \) is a quasi-interior point and put \( d(x, y) = \parallel |x - y| \land e \parallel \) for \( x, y \in X \). It can be easily verified that this defines a metric on \( X \). Indeed, \( d(x, x) = 0 \) and \( d(x, y) = d(y, x) \) for every \( x, y \in X \). If \( d(x, y) = 0 \) then \( |x - y| \land e = 0 \), hence \( |x - y| = 0 \) because \( e \) is a weak unit, so that \( x = y \). The triangle inequality follows from the fact that

\[
|x - z| \land e \leq |x - y| \land e + |y - z| \land e.
\]
Note also that $x_\alpha \xrightarrow{un} x$ iff $d(x_\alpha, x) \to 0$ for every net $(x_\alpha)$ in $X$.

Conversely, suppose that un-topology is metrizable; let $d$ be a metric for it. For each $n$, let $B_{\frac{1}{n}}$ be the ball of radius $\frac{1}{n}$ centred at zero for the metric, that is,

$$B_{\frac{1}{n}} = \{x \in X : d(x, 0) \leq \frac{1}{n}\}.$$ 

Since $B_{\frac{1}{n}}$ is a neighborhood of zero for the un-topology, it contains $V_{\varepsilon_n, u_n}$ for some $\varepsilon_n > 0$ and $u_n > 0$. Let $M_n = 2^n\|u_n\| + 1$; then the series $e = \sum_{n=1}^{\infty} \frac{u_n}{M_n}$ converges. Note that $M_n > 1$ and $u_n \leq M_n e$ for every $n$. We claim that $e$ is a quasi-interior point.

It suffices that Theorem 3.1(ii) is satisfied. Suppose that $x_\alpha \wedge e \xrightarrow{||\cdot||} 0$ for some net $(x_\alpha)$ in $X_+$. Fix $n$. It follows from

$$x_\alpha \wedge u_n \leq (M_n x_\alpha) \wedge (M_n e) = M_n(x_\alpha \wedge e) \xrightarrow{||\cdot||} 0$$

that $x_\alpha \wedge u_n \xrightarrow{||\cdot||} 0$. Then there exists $\alpha_0$ such that $\|x_\alpha \wedge u_n\| < \varepsilon_n$ whenever $\alpha \geq \alpha_0$. Consequently, $x_\alpha$ is in $V_{\varepsilon_n, u_n}$ and, therefore, in $B_{\frac{1}{n}}$. It follows that $x_\alpha \to 0$ in the metric, hence $x_\alpha \xrightarrow{un} 0$. 

Note that a linear Hausdorff topological space is metrizable iff it is first countable, i.e., has a countable base of neighborhoods of zero, see, e.g., [KN63, pp. 49]. Therefore, Theorem 3.2 implies, in particular, that un-topology is first countable iff $X$ has a quasi-interior point. This should be compared with Corollary 2.13 and Question 2.14 in [DOT] (we now know from Example 1.3 that Question 2.14 has a negative answer).

**Proposition 3.3.** Un-topology is stronger than or equal to a metric topology iff $X$ has a weak unit.

**Proof.** Suppose that un-topology is stronger than or equal to a topology given by a metric. Construct $e$ as in the second part of the proof of Theorem 3.2. We claim that $e$ is a weak unit. Suppose that $x \wedge e = 0$. It follows that $x \wedge u_n = 0$ for every $n$ and, therefore, $x \in V_{\varepsilon_n, u_n}$, hence $x \in B_{\frac{1}{n}}$. It follows that $x = 0$.

Conversely, let $e \in X_+$ be a weak unit. For $x, y \in X$, define $d(x, y) = ||x - y \wedge e||$. As in the first part of the proof of Theorem 3.2, this is a metric and $x_\alpha \xrightarrow{un} x$ implies $d(x_\alpha, x) \to 0$. 

□
When is every un-null sequence norm bounded? If $X$ has a strong unit then, by Theorem 2.3, un-topology agrees with norm topology, hence every un-null sequence is norm null and, in particular, norm bounded. This justifies the following question: *If every un-null sequence in $X$ is norm bounded (or even norm null), does this imply that $X$ has a strong unit?* The following example shows that, in general, the answer is negative.

**Example 3.4.** Let $X$ be as in Example 1.3. Clearly, $X$ does not have a strong unit; it does not even have a weak unit. Yet, every un-null sequence in $X$ is norm null. Indeed, suppose that $x_n \xrightarrow{\text{un}} 0$. Let $u$ be the characteristic function of $\bigcup_{n=1}^{\infty} \text{supp} x_n$. By assumption, $|x_n| \wedge u \xrightarrow{\|\cdot\|} 0$. It follows that for every $\varepsilon \in (0, 1)$ there exists $n_0$ such that for every $n \geq n_0$ we have $\| |x_n| \wedge u \| < \varepsilon$. It follows that $\|x_n\| < \varepsilon$.

However, we will see that the answer is affirmative under certain additional assumptions.

Recall that every disjoint sequence is uo-null. Thus, if $\dim X = \infty$, one can take any non-zero disjoint sequence, scale it to make it norm unbounded, and thus produce a uo-null sequence which is not norm bounded. However, this trick does not work for un-topology because a disjoint sequence need not be un-null. Moreover, we have the following.

**Proposition 3.5.** The following are equivalent.

(i) $X$ is order continuous;

(ii) Every disjoint sequence in $X$ is un-null;

(iii) Every disjoint net in $X$ is un-null.

*Proof.* (i)$\Rightarrow$(ii) because every disjoint sequence is uo-null and, therefore, un-null. To show that (ii)$\Rightarrow$(i), note that every order bounded disjoint sequence is norm null and apply [AB06, Theorem 4.14].

(iii)$\Rightarrow$(ii) is trivial. To show that (ii)$\Rightarrow$(iii), suppose that there exists a disjoint net $(x_\alpha)$ which is not un-null. Then there exist $\varepsilon > 0$ and $u \in X_+$ such that for every $\alpha$ there exists $\beta > \alpha$ with $\| |x_\beta| \wedge u \| > \varepsilon$. Inductively, we find an increasing sequence $(\alpha_k)$ of indices such that $\| |x_{\alpha_k}| \wedge u \| > \varepsilon$. Hence, the sequence $(x_{\alpha_k})$ is disjoint but not un-null. \qed
Corollary 3.6. If $X$ is order continuous and every un-null sequence in $X$ is norm bounded then $\text{dim} \ X < \infty$ (and, therefore, $X$ has a strong unit).

Proof. Suppose $\text{dim} \ X = \infty$. Then there exists a non-zero disjoint sequence in $X$. Scaling it if necessary, we may assume that it is not norm bounded. Yet it is un-null. A contradiction. \qed

Note that Example 2.7 in [DOT] is an example of a disjoint but non un-null sequence in an infinite-dimensional Banach lattice which is not order continuous and lacks a strong unit.

Proposition 3.7. If $X$ has a quasi-interior point and every un-null sequence is norm bounded then $X$ has a strong unit.

Proof. By Theorem 3.2, the un-topology on $X$ is metrizable. Fix such a metric. As before, for each $n$, let $B_{\frac{1}{n}}$ be the ball of radius $\frac{1}{n}$ centred at zero for the metric. For each $n$, $B_{\frac{1}{n}}$ contains $V_{\varepsilon_n, u_n}$ for some $\varepsilon_n > 0$ and $u_n > 0$. If $V_{\varepsilon_n, u_n} \subseteq [-u_n, u_n]$ for some $n$ then $u_n$ is a strong unit by Lemma 2.2. Otherwise, by Lemma 2.1, each $V_{\varepsilon_n, u_n}$ contains a non-trivial ideal. Pick any $x_n$ in this ideal with $\|x_n\| = n$. Then the sequence $(x_n)$ is norm unbounded; yet $x_n \in B_{\frac{1}{n}}$ for every $n$, so that $x_n \xrightarrow{\text{un}} 0$; a contradiction. \qed

4. Un-convergence in a sublattice

Recall that if $(y_n)$ is a net in a regular sublattice $Y$ of a vector lattice $X$ then $y_n \xrightarrow{\text{un}} 0$ in $Y$ iff $y_n \xrightarrow{\text{un}} 0$ in $X$. The situation is very different for un-convergence. Let $Y$ be a sublattice of a normed lattice $X$ and $(y_n)$ a net in $Y$. If $y_n \xrightarrow{\text{un}} 0$ in $X$ then, clearly, $y_n \xrightarrow{\text{un}} 0$ in $Y$. However, the following examples show that the converse fails even for closed ideals or bands.

Example 4.1. The sequence of the standard unit vectors $(e_n)$ is un-null in $c_0$ but not in $\ell_\infty$, even though $c_0$ is a closed ideal in $\ell_\infty$.

Example 4.2. Let $X = C[-1, 1]$ and $Y$ be the set of all $f \in X$ which vanish on $[-1, 0]$. It is easy to see that $Y$ is a band (though it is not a projection band). Let $(f_n)$ be a sequence in $Y_+$ such that $\|f_n\| = 1$
and \( \text{supp } f_n \subseteq \left[ \frac{1}{n+1}, \frac{1}{n} \right] \). Since \( X \) has a strong unit, the un-topology on \( X \) agrees with the norm topology, hence \((f_n)\) is not un-null in \( X \). However, it is easy to see that \((f_n)\) is un-null in \( Y \).

Nevertheless, there are some good news. Recall that a sublattice \( Y \) of a vector lattice \( X \) is **majorizing** if for every \( x \in X_+ \) there exists \( y \in Y_+ \) with \( x \leq y \).

**Theorem 4.3.** Let \( Y \) be a sublattice of a normed lattice \( X \) and \((y_\alpha)\) a net in \( Y \) such that \( y_\alpha \overset{\text{un}}{\rightarrow} 0 \) in \( Y \). Each of the following conditions implies that \( y_\alpha \overset{\text{un}}{\rightarrow} 0 \) in \( X \).

(i) \( Y \) is majorizing in \( X \);

(ii) \( Y \) is norm dense in \( X \);

(iii) \( Y \) is a projection band in \( X \).

**Proof.** Without loss of generality, \( y_\alpha \geq 0 \) for every \( \alpha \). (i) is straightforward. To prove (ii), take \( u \in X_+ \) and fix \( \varepsilon > 0 \). Find \( v \in Y_+ \) with \( \|u - v\| < \varepsilon \). By assumption, \( y_\alpha \wedge v \overset{\|\|}{\rightarrow} 0 \). We can find \( \alpha_0 \) such that \( \|y_\alpha \wedge v\| < \varepsilon \) whenever \( \alpha \geq \alpha_0 \). It follows from \( u \leq v + |u - v| \) that \( y_\alpha \wedge u \leq y_\alpha \wedge v + |u - v| \), so that

\[
\|y_\alpha \wedge u\| \leq \|y_\alpha \wedge v\| + \|u - v\| < 2\varepsilon.
\]

It follows that \( y_\alpha \wedge u \overset{\|\|}{\rightarrow} 0 \). Hence \( y_\alpha \overset{\text{un}}{\rightarrow} 0 \) in \( X \).

To prove (iii), let \( u \in X_+ \). Then \( u = v + w \) for some positive \( v \in Y \) and \( w \in Y^d \). It follows from \( y_\alpha \perp w \) that \( y_\alpha \wedge u = y_\alpha \wedge v \overset{\|\|}{\rightarrow} 0 \). \( \square \)

Recall that every (Archimedean) vector lattice \( X \) is majorizing in its **order (or Dedekind) completion** \( X^\delta \); see, e.g., [AB06, p. 101].

**Corollary 4.4.** If \( X \) is a normed lattice and \( x_\alpha \overset{\text{un}}{\rightarrow} x \) in \( X \) then \( x_\alpha \overset{\text{un}}{\rightarrow} x \) in the order completion \( X^\delta \) of \( X \).

**Corollary 4.5.** If \( X \) is a KB-space and \( x_\alpha \overset{\text{un}}{\rightarrow} 0 \) in \( X \) then \( x_\alpha \overset{\text{un}}{\rightarrow} 0 \) in \( X^{**} \).

**Proof.** By [AB06, Theorem 4.60], \( X \) is a projection band in \( X^{**} \). The conclusion now follows from Theorem 4.3[iii]. \( \square \)

Example 4.1 shows that the assumption that \( X \) is a KB-space cannot be removed.
Corollary 4.6. Let $Y$ be a sublattice of an order continuous Banach lattice $X$. If $y_\alpha \xrightarrow{\text{un}} 0$ in $Y$ then $y_\alpha \xrightarrow{\text{un}} 0$ in $X$.

Proof. Suppose that $y_\alpha \xrightarrow{\text{un}} 0$ in $Y$. By Theorem 4.3(i), $y_\alpha \xrightarrow{\text{un}} 0$ in the ideal $I(Y)$ generated by $Y$ in $X$. By Theorem 4.3(ii), $y_\alpha \xrightarrow{\text{un}} 0$ in the closure $\overline{I(Y)}$ of the ideal. Since $X$ is order continuous, $I(Y)$ is a projection band in $X$. It now follows from Theorem 4.3(iii) that $y_\alpha \xrightarrow{\text{un}} 0$ in $X$. □

Question 4.7. Let $B$ be a band in $X$. Suppose that every net in $B$ which is un-null in $B$ is also un-null in $X$. Does this imply that $B$ is a projection band?

Proposition 4.8. Every band in a normed lattice is un-closed.

Proof. Let $B$ be a band and $(x_\alpha)$ a net in $B$ such that $x_\alpha \xrightarrow{\text{un}} x$. Fix $z \in B^d$. Then $|x_\alpha| \wedge z = 0$ for every $\alpha$. Since lattice operations are un-continuous, we have $|x| \wedge z = 0$. It follows that $x \in B^{dd} = B$. □

Remark 4.9. Let $B$ be a projection band a normed lattice $X$. We write $P_B$ for the corresponding band projection. It follows easily from $0 \leq P_B \leq I$ that if $x_\alpha \xrightarrow{\text{un}} x$ in $X$ then $P_B x_\alpha \xrightarrow{\text{un}} P_B x$ both in $X$ and in $B$.

Dense band decompositions. Let $X$ be a Banach lattice. By a dense band decomposition of $X$ we mean a family $\mathcal{B}$ of pairwise disjoint projection bands in $X$ such that the linear span of all of the bands in $\mathcal{B}$ is norm dense in $X$.

Lemma 4.10. Let $\mathcal{B}$ be a family of pairwise disjoint projection bands in a Banach lattice $X$. $\mathcal{B}$ is a dense band decomposition of $X$ iff for every $x \in X$ and every $\varepsilon > 0$ there exist $B_1, \ldots, B_n$ in $\mathcal{B}$ such that $\|x - \sum_{i=1}^{n} P_{B_i} x\| < \varepsilon$.

Proof. Suppose that $\mathcal{B}$ is a dense band decomposition of $X$. Let $x \in X$ and $\varepsilon > 0$. By assumption, we can find distinct bands $B_1, \ldots, B_n$ and vectors $x_1 \in B_1, \ldots, x_n \in B_n$ such that $\|x - \sum_{i=1}^{n} x_i\| < \varepsilon$. Put $Q = I - \sum_{i=1}^{n} P_{B_i}$. Then $Q$ is also a band projection, hence it is a
lattice homomorphism and $0 \leq Q \leq I$. Note also that $Qx_i = 0$ for $i = 1, \ldots, n$. We have

$$|x - \sum_{i=1}^{n} x_i| \geq Q|\sum_{i=1}^{n} x_i| = |\sum_{i=1}^{n} Qx_i| = |x - \sum_{i=1}^{n} P_Bx|.$$ 

It follows that $\|x - \sum_{i=1}^{n} P_Bx\| < \varepsilon$.

The converse implication is trivial. \qed

Our definition of a disjoint band decomposition is partially motivated by following fact.

**Theorem 4.11.** ([LT79, Proposition 1.a.9]) Every order continuous Banach lattice admits a dense band decomposition $\mathcal{B}$ such that each band in $\mathcal{B}$ has a weak unit.

It is easy to see that if $X$ is an order continuous Banach lattice and $\mathcal{B}$ is a pairwise disjoint collection of bands such that $x = \sup \{P_Bx : B \in \mathcal{B}\}$ for every $x \in X_+$ then $\mathcal{B}$ is a dense band decomposition.

**Theorem 4.12.** Suppose that $\mathcal{B}$ is a dense band decomposition of a Banach lattice $X$. Then $x_\alpha \overset{\text{un}}{\rightarrow} x$ in $X$ iff $P_Bx_\alpha \overset{\text{un}}{\rightarrow} P_Bx$ in $B$ for each $B \in \mathcal{B}$.

**Proof.** Without loss of generality, $x = 0$ and $x_\alpha \geq 0$ for every $\alpha$. The forward implication follows immediately from Remark 4.9. To prove the converse, suppose that $P_Bx_\alpha \overset{\text{un}}{\rightarrow} 0$ in $B$ for each $B \in \mathcal{B}$. Let $u \in X_+$; it suffices to show that $x_\alpha \wedge u \overset{\text{un}}{\rightarrow} 0$. Fix $\varepsilon > 0$. Find $B_1, \ldots, B_n \in \mathcal{B}$ such that $\|u - \sum_{i=1}^{n} P_{B_i}u\| < \varepsilon$. Since $P_{B_i}x_\alpha \overset{\text{un}}{\rightarrow} 0$ in $B_i$ as $i = 1, \ldots, n$, we can find $\alpha_0$ such that $\|P_{B_i}x_\alpha \wedge P_{B_i}u\| < \frac{\varepsilon}{n}$ for every $\alpha \geq \alpha_0$ and every $i = 1, \ldots, n$. It follows from $x_\alpha \wedge P_{B_i}u \in B_i$ that $x_\alpha \wedge P_{B_i}u = P_{B_i}x_\alpha \wedge P_{B_i}u$. Therefore,

$$\|x_\alpha \wedge u\| \leq \left\|x_\alpha \wedge \sum_{i=1}^{n} P_{B_i}u\right\| + \left\|u - \sum_{i=1}^{n} P_{B_i}u\right\| \leq \left\|\sum_{i=1}^{n} P_{B_i}x_\alpha \wedge P_{B_i}u\right\| + \varepsilon$$

$$= \left\|\sum_{i=1}^{n} P_{B_i}x_\alpha \wedge P_{B_i}u\right\| + \varepsilon \leq n \cdot \frac{\varepsilon}{n} + \varepsilon \leq 2\varepsilon.$$ 

\qed
Remark 4.13. Recall that a positive non-zero vector $a$ in a vector lattice $X$ is an atom if the principal ideal $I_a$ generated by $a$ coincides with span $a$. In this case, $I_a$ is a projection band, and the corresponding band projection $P_a$ has form $f_a \otimes a$ for some positive functional $f_a$, that is, $P_a x = f_a(x)a$. We say that $X$ is non-atomic if it has no atoms. We say that $X$ is atomic if $X$ is the band generated by all the atoms. In the latter case, $x = \sup \{ f_a(x) : a \text{ is an atom} \}$ for every $x \in X_+$. See, e.g., [Sch74, p. 143].

It follows that if $X$ is an order continuous atomic Banach lattice, the family $\{ I_a : a \text{ is an atom} \}$ is a dense band decomposition of $X$. Applying Theorem 4.12, we conclude that in such spaces un-convergence is exactly the “coordinate-wise” convergence:

**Corollary 4.14.** Let $X$ be an atomic order continuous Banach lattice. Then $x_\alpha \overset{\text{un}}{\rightharpoonup} x$ iff $f_a(x_\alpha) \to f_a(x)$ for every atom $a$.

**Remark 4.15.** The order continuity assumption cannot be removed. Indeed, $\ell_\infty$ is atomic, the sequence $(e_n)$ converges to zero coordinate-wise, yet it is not un-null.

The following results extends [DOT, Proposition 6.2].

**Proposition 4.16.** The following are equivalent:

(i) $x_\alpha \overset{\text{w}}{\rightharpoonup} 0$ implies $x_\alpha \overset{\text{un}}{\rightharpoonup} 0$ for every net $(x_\alpha)$ in $X$;

(ii) $x_n \overset{\text{w}}{\rightharpoonup} 0$ implies $x_n \overset{\text{un}}{\rightharpoonup} 0$ for every sequence $(x_n)$ in $X$;

(iii) $X$ is atomic and order continuous.

**Proof.** (i) $\Rightarrow$ (ii) is trivial. The implication (ii) $\Rightarrow$ (iii) is a part of [DOT, Proposition 6.2]. The implication (iii) $\Rightarrow$ (i) follows from Corollary 4.14. □

5. **AL-representations and local convexity**

In this section, we will show that un-topology on an order continuous Banach lattice $X$ is locally convex iff $X$ is atomic. Our main tool is the relationship between un-convergence in $X$ and in an AL-representation of $X$. 
It was observed in [Tro04, Example 23] that for a net \( (x_\alpha) \) in \( L^p(\mu) \) where \( \mu \) is a finite measure and \( 1 \leq p < \infty \), one has \( x_\alpha \xrightarrow{\text{un}} 0 \) iff \( x_\alpha \xrightarrow{\mu} 0 \) (i.e., the net converges to zero in measure). Note that this does not extend to \( \sigma \)-finite measures. Indeed, let \( X = L^p(\mathbb{R}) \) and let \( x_n \) be the characteristic function of \([n, n+1]\). Then \( x_n \xrightarrow{\text{un}} 0 \) but \( (x_n) \) does not converge to zero in measure. On the other hand, let \( (x_\alpha) \) be a net in \( L^p(\mu) \) where \( \mu \) is a \( \sigma \)-finite measure, let \( (\Omega_n) \) be a countable partition of \( \Omega \) into sets of finite measure; it follows from Theorem 4.12 that \( x_\alpha \xrightarrow{\text{un}} 0 \) iff the restriction of \( x_\alpha \) to \( \Omega_n \) converges to zero in measure for every \( n \).

Suppose that \( X \) is an order continuous Banach lattice with a weak unit \( e \). By [LT79, Theorem 1.b.14], \( X \) can be represented as an ideal of \( L^1(\mu) \) for some probability measure \( \mu \). More precisely, there is a lattice isomorphism from \( X \) onto a norm-dense ideal of \( L^1(\mu) \); with a slight abuse of notation we will view \( X \) itself as an ideal of \( L^1(\mu) \)). Moreover, this representation may be chosen so that \( e \) corresponds to \( 1 \), \( L_\infty(\mu) \) is a norm-dense ideal in \( X \), and both inclusions in \( L_\infty(\mu) \subseteq X \subseteq L^1(\mu) \) are continuous. We call \( L_1(\mu) \) an \textbf{AL-representation} for \( X \) and \( e \).

Let \( (x_n) \) be a sequence in \( X \). It was shown in [GTX, Remark 4.6] that \( x_n \xrightarrow{\text{uo}} 0 \) in \( X \) iff \( x_n \xrightarrow{\text{a.e.}} 0 \) in \( L^1(\mu) \). It was shown in [DOT, Theorem 4.6] that \( x_n \xrightarrow{\text{un}} 0 \) in \( X \) iff \( x_n \xrightarrow{\mu} 0 \) in \( L^1(\mu) \). Since un-topology and the topology of convergence in measure are both metrizable on \( X \) because \( X \) has a weak unit, it follows that these two topologies coincide on \( X \). In particular, \( x_\alpha \xrightarrow{\text{un}} 0 \) in \( X \) iff \( x_\alpha \xrightarrow{\mu} 0 \) in \( L^1(\mu) \) for every net \( (x_\alpha) \) in \( X \). This may also be deduced from Amemiya’s Theorem (see, e.g., Theorem 2.4.8 in [MN91]) as follows:

\[
x_\alpha \xrightarrow{\text{un}} 0 \text{ in } X \iff \|x_\alpha \wedge e\|_X \to 0 \quad \text{Amemiya} \quad \|x_\alpha \wedge 1\|_{L^1} \to 0 \quad \xrightarrow{\text{un}} \quad x_\alpha \xrightarrow{\mu} 0 \text{ in } L^1(\mu)
\]

for every net \( (x_\alpha) \) in \( X_+ \).

**Proposition 5.1.** Let \( X \) be a non-atomic order continuous Banach lattice and \( W \) a neighborhood of zero for un-topology. If \( W \) is convex then \( W = X \).

**Proof.** Fix \( e \in X_+ \); we will show that \( e \in W \). We know that \( V_{\varepsilon,u} \subseteq W \) for some \( \varepsilon > 0 \) and \( u > 0 \). Consider the principal band \( B_e \). Since \( X \) is order continuous, \( B_e \) is a projection band in \( X \); let \( P_e \) be the
corresponding band projection. Furthermore, $B_e$ is a non-atomic order continuous Banach lattice with a weak unit. Let $L_1(\Omega, \mathcal{F}, \mu)$ be an AL-representation for $B_e$ with $e = \mathbb{1}$. Note that the measure $\mu$ is non-atomic because if a measurable set $A$ were an atom for $\mu$ then its characteristic function $\chi_A$ would be an atom in $X$. Fix $n \in \mathbb{N}$. Using the non-atomicity of $\mu$, we find a measurable partition $A_{n,1}, \ldots, A_{n,n}$ of $\Omega$ with $\mu(A_{n,i}) = \frac{1}{n}$ as $i = 1, \ldots, n$; see, e.g., Exercise 2 in [Hal70, p. 174]. Since $L_\infty(\mu) \subseteq B_e \subseteq L_1(\mu)$, we may view the characteristic functions $\chi_{A_{n,i}}$ as elements of $B_e$. Consider the vectors $(n\chi_{A_{n,i}}) \wedge u$ as $i = 1, \ldots, n$; they belong to $B_e$, so that we may view them as functions in $L_1(\mu)$. Let $g_n$ be the function in this list whose norm in $X$ is maximal; if there are more than one, pick any one. Repeating this construction for every $n \in \mathbb{N}$, we produce a sequence $(g_n)$ in $[0,u] \cap B_e$.

It follows that $g_n \leq P_e u$ for every $n$. Since $P_e u$ may be viewed as an element of $L_1(\mu)$ and the measure of the support of $g_n$ tends to zero, it follows that $\|g_n\|_{L_1} \to 0$. Amemiya’s Theorem yields $\|g_n\|_X \to 0$. Fix $n$ such that $\|g_n\|_X < \varepsilon$. It follows from the definition of $g_n$ that $\|(n\chi_{A_{n,i}}) \wedge u\|_X < \varepsilon$ as $i = 1, \ldots, n$, so that $n\chi_{A_{n,i}}$ is in $V_{\varepsilon,u}$ and, therefore, in $W$. Since $W$ is convex and

$$e = \mathbb{1} = \frac{1}{n} \sum_{i=1}^n n\chi_{A_{n,i}},$$

we have $e \in W$. Therefore, $X_+ \subseteq W$. Furthermore, it follows from $n\chi_{A_{n,i}} \in V_{\varepsilon,u}$ that $-n\chi_{A_{n,i}} \in V_{\varepsilon,u}$ for all $i = 1, \ldots, n$ and, therefore, $-e \in W$. This yields $X_- \subseteq W$. Finally, for every $x \in X$ we have

$$x = \frac{1}{2}(2x^+ + 2(-x^-)),$$

so that $x \in W$.

\[ \square \]

**Theorem 5.2.** Let $X$ be an order continuous Banach lattice. Un-topology on $X$ is locally convex iff $X$ is atomic.

**Proof.** Suppose that $X$ is atomic. By Corollary [4.14] un-topology is determined by the family of seminorms $x \mapsto |f_a(x)|$ where $a$ is an atom of $X$; hence the topology is locally convex.

Suppose that un-topology is locally convex but $X$ is not atomic. It follows that there is $e \in X_+$ such that $B_e$ is non-atomic. By Theorem [4.3] un-topology on $B_e$ agrees with the relative topology induced
on $B_e$ by un-topology on $X$; in particular, it is locally convex. On the other hand, Proposition 5.1 asserts that this topology on $B_e$ has no proper convex neighborhoods; a contradiction. □

Un-continuous functionals. Theorem 5.2 allows us to describe un-continuous linear functionals. For a functional $\varphi \in X^*$, we say that $\varphi$ is un-continuous if it is continuous with respect to the un-topology on $X$ or, equivalently, if $x_\alpha \xrightarrow{un} 0$ implies $\varphi(x_\alpha) \to 0$.

Proposition 5.3. The set of all un-continuous functionals in $X^*$ is an ideal.

Proof. It is straightforward to verify that this set is a linear subspace. Suppose that $\varphi$ in $X^*$ is un-continuous; we will show that $|\varphi|$ is also un-continuous. Fix $\delta > 0$. One can find $\varepsilon > 0$ and $u > 0$ such that $|\varphi(x)| < \delta$ whenever $x \in V_{\varepsilon,u}$. Fix $x \in V_{\varepsilon,u}$. Since $V_{\varepsilon,u}$ is solid, $|y| \leq |x|$ implies $y \in V_{\varepsilon,u}$ and, therefore, $|\varphi(y)| < \delta$. By the Riesz-Kantorovich formula, we get

$$||\varphi||(x) \leq |\varphi|(|x|) = \sup\{|\varphi(y)| : |y| \leq |x|\} \leq \delta.$$  

It follows that $|\varphi|$ is un-continuous. Hence, the set of all un-continuous functionals in $X^*$ forms a sublattice. It is easy to see that if $\varphi \in X_+^*$ is un-continuous and $0 \leq \psi \leq \varphi$ then $\psi$ is also un-continuous; this completes the proof. □

Recall that if $a$ is an atom then $f_a$ stands for the corresponding “coordinate functional”.

Corollary 5.4. Suppose that $X$ is an order continuous Banach lattice and $\varphi \in X^*$ is un-continuous.

(i) If $X$ is atomic then $\varphi = \lambda_1 f_{a_1} + \cdots + \lambda_n f_{a_n}$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $a_1, \ldots, a_n$ are atoms;

(ii) If $X$ is non-atomic then $\varphi = 0$.

Proof. By Proposition 5.3 we may assume that $\varphi \geq 0$; otherwise we consider $\varphi^+$ and $\varphi^-$.

Suppose $X$ is atomic; let $A$ be a maximal disjoint family of atoms. We claim that the set $F := \{a \in A : \varphi(a) \neq 0\}$ is finite. Indeed, otherwise, take a sequence $(a_n)$ of distinct atoms in $F$ and put $x_n = \frac{1}{\varphi(a_n)} a_n$. 


Then \( x_n \rightarrow 0 \) by Corollary 4.14 yet \( \varphi(x_n) = 1 \); a contradiction. This proves the claim.

Since \( X \) is order continuous, it follows from Remark 4.13 that \( X \) has a disjoint band decomposition \( X = B_F \oplus B_{A \setminus F} \). Since \( \varphi(a) = 0 \) for all \( a \in A \setminus F \), \( \varphi \) vanishes on the ideal \( I_{A \setminus F} \) and, therefore, on \( B_{A \setminus F} \) because \( \varphi \) is order continuous. On the other hand, since \( F \) is finite, \( B_F = \text{span} F \) and, therefore, is finite-dimensional. It follows that \( \varphi \) is a linear combination of \( \{ f_a : a \in F \} \).

Suppose now that \( X \) is non-atomic. Let \( W = \varphi^{-1}(-1,1) \). Then \( W \) is a convex neighborhood of zero for the un-topology. By Proposition 5.1 \( W = X \). This easily implies \( \varphi = 0 \). □

Case [i] of the preceding corollary essentially says that every un-continuous functional on an atomic order continuous space has finite support.

**Example 5.5.** Let \( X = \ell_2 \). By Corollary 5.4 the set of all un-continuous functionals in \( X^* \) may be identified with \( c_{00} \), the linear subspace of all sequences with finite support. Clearly, it is neither norm closed nor order closed; it is not even \( \sigma \)-order closed in \( X^* \).

**Example 5.6.** Let \( X = C_0(\Omega) \) where \( \Omega \) is a locally compact Hausdorff topological space. It was observed in [Tro04, Example 20] that the un-topology in \( X \) agrees with the topology of uniform convergence on compact subsets of \( \Omega \).

Let \( \varphi \in X_1^* \). By the Riesz Representation Theorem, there exists a regular Borel measure \( \mu \) such that \( \varphi(f) = \int f \, d\mu \) for every \( f \in X \); see, e.g., [Con99, Theorem III.5.7]. An argument similar to the proof of [Con99, Proposition IV.4.1] shows that \( \varphi \) is un-continuous iff \( \mu \) has compact support.

### 6. Un-completeness

Throughout this section, \( X \) is assumed to be an order continuous Banach lattice. Since un-topology is linear, one can talk about un-Cauchy nets. That is, a net \( (x_\alpha) \) is un-Cauchy if for every un-neighborhood \( U \) of zero there exists \( \alpha_0 \) such that \( x_\alpha - x_\beta \in U \) whenever \( \alpha, \beta \geq \alpha_0 \). We
investigate whether $X$ itself or some “nice” subset of $X$ is un-complete. First, we observe that the entire space is un-complete only when $X$ is finite-dimensional.

**Lemma 6.1.** Let $(x_n)$ be a positive disjoint sequence in an order continuous Banach lattice $X$ such that $(x_n)$ is not norm null. Put $s_n = \sum_{i=1}^{n} x_i$. Then $(s_n)$ is un-Cauchy but not un-convergent.

**Proof.** The sequence $(s_n)$ is monotone increasing and does not converge in norm; hence it is not un-convergent by Lemma 1.2(ii). To show that $(s_n)$ is un-Cauchy, fix any $\varepsilon > 0$ and a non-zero $u \in X_+$. Since $x_i$’s are disjoint, we have $s_n \wedge u = \sum_{i=1}^{n} (x_i \wedge u)$. The sequence $(s_n \wedge u)$ is increasing and order bounded, hence is norm Cauchy by Nakano’s Theorem; see [AB06, Theorem 4.9]. We can find $n_0$ such that $\|s_m \wedge u - s_n \wedge u\| < \varepsilon$ whenever $m \geq n \geq n_0$. Observe that

$$s_m \wedge u - s_n \wedge u = \sum_{i=n+1}^{m} (x_i \wedge u) = (s_m - s_n) \wedge u = |s_m - s_n| \wedge u.$$  

It follows that $\|s_m - s_n| \wedge u\| < \varepsilon$, so that $s_m - s_n \in V_{\varepsilon, u}$. □

**Proposition 6.2.** Let $X$ be an order continuous Banach lattice. $X$ is un-complete iff $X$ is finite-dimensional.

**Proof.** If $X$ is finite-dimensional then it has a strong unit, so that un-topology agrees with norm topology and is, therefore, un-complete. Suppose now that $\dim X = \infty$. Then $X$ contains a disjoint normalized positive sequence. By Lemma 6.1, $X$ is not un-complete. □

**Example 6.3.** Let $X = L_p$ with $1 < p < \infty$. Pick $0 \leq x \in L_1 \setminus L_p$ and put $x_n = x \wedge (n1)$. It is easy to see that $(x_n)$ is un-Cauchy in $L_p$, yet it does not un-converge in $L_p$.

Even when the entire space is not un-complete, the closed unit ball $B_X$ may still be un-complete; that is, complete in the topology induced by un-topology on $X$. Since $B_X$ is un-closed, it is un-complete iff every norm bounded un-Cauchy net in $X$ is un-convergent. The following theorem should be compared with [GX14, Theorem 4.7], where a similar statement was proved for uo-convergence.
Theorem 6.4. Let $X$ be an order continuous Banach lattice. Then $B_X$ is un-complete iff $X$ is a KB-space.

Proof. Suppose $X$ is not KB. Then $X$ contains a lattice copy of $c_0$. Let $(x_n)$ be the sequence in $X$ corresponding to the unit basis of $c_0$. Let $s_n = \sum_{i=1}^{n} x_i$. Clearly, $(s_n)$ is norm bounded. However, by Lemma 6.1, $(s_n)$ is un-Cauchy but not un-convergent.

Suppose now that $X$ is a KB-space. First, we consider the case when $X$ has a weak unit. In this case, un-topology on $X$ and, therefore, on $B_X$, is metrizable by Theorem 3.2. Hence, it suffices to prove that $B_X$ is sequentially un-complete. Let $(x_n)$ be a sequence in $B_X$ which is un-Cauchy in $X$. Let $L_1(\mu)$ be an AL-representation for $X$. It follows that $(x_n)$ is Cauchy with respect to convergence in measure in $L_1(\mu)$. By [Fol99, Theorem 2.30], there is a subsequence $(x_{n_k})$ which converges a.e. It follows that $(x_{n_k})$ is uo-Cauchy in $X$ by [GTX, Remark 4.6]. Then [GX14, Theorem 4.7] yields that $x_{n_k} \quad \text{un} \quad \to x$ for some $x \in X$. It follows that $x_{n_k} \quad \to x$. Since $(x_n)$ is un-Cauchy, this yields that $x_n \quad \to x$.

Now consider the general case. Let $X$ be a KB-space and $(x_\alpha)$ a net in $B_X$ such that $(x_\alpha)$ is un-Cauchy in $X$; we need to prove that the net is un-convergent. We may assume without loss of generality that $x_\alpha \geq 0$ for every $\alpha$; otherwise, consider $(x_\alpha^+) \text{ and } (x_\alpha^-)$, which are also un-Cauchy because $|x_\alpha^+ - x_\beta^+| \leq |x_\alpha - x_\beta|$ and $|x_\alpha^- - x_\beta^-| \leq |x_\alpha - x_\beta|$. By Theorem 4.11, there exists a dense band decomposition $B$ of $X$ such that each $B$ in $B$ has a weak unit. Put

$$C = \{B_1 \oplus \cdots \oplus B_n : B_1, \ldots, B_n \in B\}.$$

Note that $C$ is a family of bands with weak units. Furthermore, $C$ is a directed set when ordered by inclusion, so the family of band projections $(P_C)_{C \in C}$ may be viewed as a net.

For every $C \in C$, the net $(P_C x_\alpha)$ is un-Cauchy by Remark 4.9. Since $C$ has a weak unit, the first part of the proof yields that $(P_C x_\alpha)$ un-converges to some positive vector $x_C$ in $C$. This produces a net $(x_C)_{C \in C}$. It is easy to verify that $x_C = x_{B_1} + \cdots + x_{B_n}$ whenever $C = B_1 \oplus \cdots \oplus B_n$ for some $B_1, \ldots, B_n \in B$. It follows that the net $(x_C)_{C \in C}$ is increasing. On the other hand, $\|x_C\| \leq \liminf_{\alpha} \|P_C x_\alpha\| \leq 1$, so that this net is
norm bounded. Since $X$ is a KB-space, the net $(x_C)_{C \in C}$ converges in norm to some $x \in X$.

Fix $B \in B$. On one hand, norm continuity of $P_B$ yields $\lim_{C \in C} P_B x_C = P_B x$. On the other hand, for every $C \in C$ with $B \subseteq C$ we have $P_B x_C = x_B$, so that $\lim_{C \in C} P_B x_C = x_B$. It follows that $P_B x_C \xrightarrow{un} P_B x$ for every $B \in B$. Now Theorem 4.12 yields $\lim_{\alpha} x_\alpha \xrightarrow{un} x$.

The assumption that $X$ is order continuous cannot be removed: for example, $\ell_\infty$ is not a KB-space, yet its closed unit ball is un-complete (because the un and the norm topologies on $\ell_\infty$ agree).

Example 6.5. The following examples show that in general $B_X$ in Theorem 6.4 cannot be replaced with an arbitrary convex closed bounded set. Let $X = \ell_1$; let $C$ be the set of all vectors in $B_X$ whose coordinates sum up to zero. Clearly, $C$ is convex, closed, and bounded. Let $x_n = \frac{1}{2}(e_1 - e_n)$. Then $(x_n)$ is a sequence in $C$ which un-converges to $\frac{1}{2}e_1$ which is not in $C$. Thus, $C$ is not un-closed in $X$; in particular, $C$ is not un-complete.

It is easy to construct a similar example in $X = L_1$; take $C = \{ x \in B_X : \int x = 0 \}$ and put $x_n = \chi_{[0,\frac{1}{2}]} - \frac{n}{2} \chi_{[\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]}$, $n \geq 2$.

Proposition 6.6. Suppose that $X^*$ is order continuous and $C$ is a norm closed convex norm bounded subset of $X$. Then $C$ is un-closed.

Proof. Suppose that $x_\alpha \xrightarrow{un} x$ for a net $(x_\alpha)$ in $C$ and a vector $x$ in $X$. Since $(x_\alpha)$ is norm bounded and $X^*$ is order continuous, [DOT] Theorem 6.4] guarantees that $(x_\alpha)$ converges to $x$ weakly. Since $C$ is convex and closed, it is weakly closed, hence $x \in C$.

Corollary 6.7. Let $X$ be a reflexive Banach lattice and $C$ a closed convex norm bounded subset of $X$. Then $C$ is un-complete.

Proof. Since $X$ is reflexive, $X$ is a KB-space and $X^*$ is order continuous. Let $(x_\alpha)$ be a un-Cauchy net in $C$. Theorem 6.4 yields that $x_\alpha \xrightarrow{un} x$ for some $x \in X$, while Proposition 6.6 implies that $x \in C$. □
7. Un-compact sets

The main result of this section is Theorem 7.5 which asserts that \( B_X \) is (sequentially) un-compact iff \( X \) is an atomic KB-space. We start with some auxiliary results. The following theorem shows that, under certain assumptions, un-compactness is a “local” property.

**Theorem 7.1.** Let \( X \) be a KB-space, \( \mathcal{B} \) a dense band decomposition of \( X \), and \( A \) a un-closed norm bounded subset of \( X \). Then \( A \) is un-compact iff \( P_B(A) \) is un-compact in \( B \) for every \( B \in \mathcal{B} \).

**Proof.** If \( A \) is un-compact then \( P_B(A) \) is un-compact in \( B \) for every \( B \in \mathcal{B} \) because \( P_B \) is un-continuous by Remark 4.9. To prove the converse, suppose that \( P_B(A) \) is un-compact in \( B \) for every \( B \in \mathcal{B} \). Let \( H = \prod_{B \in \mathcal{B}} B \), the formal product of all the bands in \( \mathcal{B} \). That is, \( H \) consists of families \((x_B)_{B \in \mathcal{B}}\) indexed by \( \mathcal{B} \), where \( x_B \in B \). We equip \( H \) with the topology of coordinate-wise un-convergence; this is the product of un-topologies on the bands that make up \( H \). This makes \( H \) a topological vector space. Define \( \Phi : X \to H \) via \( \Phi(x) = (P_Bx)_{B \in \mathcal{B}} \). Clearly, \( \Phi \) is linear. Since \( \mathcal{B} \) is a dense band decomposition, \( \Phi \) is one-to-one. By Theorem 4.12, \( \Phi \) is a homeomorphism from \( X \) equipped with un-topology onto its range in \( H \).

Let \( K \) be the subset of \( H \) defined by \( K = \prod_{B \in \mathcal{B}} P_B(A) \). By Tikhonov’s Theorem, \( K \) is compact in \( H \). It is easy to see that \( \Phi(A) \subseteq K \).

We claim that \( \Phi(A) \) is closed in \( H \). Indeed, suppose that \( \Phi(x_\alpha) \to h \) in \( H \) for some net \((x_\alpha)\) in \( A \). In particular, the net \((\Phi(x_\alpha))\) is Cauchy in \( H \). Since \( \Phi \) is a homeomorphism, the net \((x_\alpha)\) is un-Cauchy in \( A \). Since \((x_\alpha)\) is bounded and \( X \) is a KB-space, \((x_\alpha)\) un-converges to some \( x \in X \) by Theorem 6.4. Since \( A \) is un-closed, we have \( x \in A \). It follows that \( h = \Phi(x) \), so that \( h \in \Phi(A) \).

Being a closed subset of a compact set, \( \Phi(A) \) is itself compact. Since \( \Phi \) is a homeomorphism, we conclude that \( A \) is un-compact. \( \square \)

Next, we discuss relationships between the sequential and the general variants of un-closedness and un-compactness. Recall that for a set \( A \) in a topological space, we write \( \overline{A} \) for the closure of \( A \); we write \( \overline{A}' \) for the **sequential closure** of \( A \), i.e., \( a \in \overline{A}' \) iff \( a \) is the limit of a
sequence in $A$. We say that $A$ is **sequentially closed** if $\overline{A}^\sigma = A$. It is well known that for a metrizable topology, we always have $\overline{A}^\sigma = \overline{A}$.

For a set $A$ in a Banach lattice, we write $\overline{A}^{\text{un}}$ and $\overline{A}^{\sigma\text{-un}}$ for the un-closure and the sequential un-closure of $A$, respectively. Obviously, $\overline{A}^{\sigma\text{-un}} \subseteq \overline{A}^{\text{un}}$.

**Example 7.2.** In general, $\overline{A}^{\text{un}} \neq \overline{A}^{\sigma\text{-un}}$. Indeed, in the notation of Example 1.3, let $A = \{e_\omega : \omega \in \Omega\}$. It follows from Example 1.3 that zero is in $\overline{A}^{\text{un}}$ but not in $\overline{A}^{\sigma\text{-un}}$.

**Proposition 7.3.** Let $A$ be a subset of a Banach lattice $X$. If $X$ has a quasi-interior point or $X$ is order continuous then $\overline{A}^{\text{un}} = \overline{A}^{\sigma\text{-un}}$.

**Proof.** If $X$ has a quasi-interior point then its un-topology is metrizable by Theorem 3.2, hence $\overline{A}^{\text{un}} = \overline{A}^{\sigma\text{-un}}$.

Suppose that $X$ is order continuous. Suppose that $x \in \overline{A}^{\text{un}}$; we need to show that $x \in \overline{A}^{\sigma\text{-un}}$. Without loss of generality, $x = 0$. This means that $A$ contains a un-null net $(x_\alpha)$. By Theorem 1.1, there exists an increasing sequence of indices $(\alpha_k)$ and a disjoint sequence $(d_k)$ such that $x_{\alpha_k} - d_k \xrightarrow{\text{un}} 0$. It follows that $x_{\alpha_k} - d_k \xrightarrow{\text{un}} 0$. Since $(d_k)$ is disjoint, it is uo-null and, since $X$ is order continuous, un-null. It follows that $x_{\alpha_k} \xrightarrow{\text{un}} 0$ and, therefore, $0 \in \overline{A}^{\sigma\text{-un}}$. \[\square\]

Recall that a topological space is said to be **sequentially compact** if every sequence has a convergent subsequence. In a Hausdorff topological vector space which is metrizable (or, equivalently, first countable), sequential compactness is equivalent to compactness, see, e.g., [Roy88, Theorem 7.21]. We do not know whether un-compactness and sequential un-compactness are equivalent in general, yet we have the following partial result.

**Proposition 7.4.** Let $A$ be a subset of a Banach lattice $X$.

(i) If $X$ has a quasi-interior point, then $A$ is sequentially un-compact iff $A$ is un-compact.

(ii) Suppose that $X$ is order continuous. If $A$ is un-compact then $A$ is sequentially un-compact.

(iii) Suppose that $X$ is a KB-space. If $A$ is norm bounded and sequentially un-compact then $A$ is un-compact.
Proof. (i) follows immediately from Theorem 3.2.

(ii) Let \((x_n)\) be a sequence in \(A\). Find \(e \in X_+\) such that \((x_n)\) is contained in \(B_e\) (e.g., take \(e = \sum_{n=1}^{\infty} \frac{x_n}{\|x_n\|+1}\)). Since \(B_e\) is un-closed, the set \(A \cap B_e\) is un-compact in \(B_e\). Since \(e\) is a quasi-interior point for \(B_e\), the un-topology on \(B_e\) is metrizable, hence \(A \cap B_e\) is sequentially un-compact. It follows that there is a subsequence \((x_{n_k})\) which un-converges in \(B_e\) to some \(x \in A \cap B_e\). By Theorem 4.3(iii), \(x_{n_k} \to x\) in \(X\).

(iii) Clearly, \(A\) is sequentially un-closed and, therefore, un-closed by Proposition 7.3. Let \(B\) be as in Theorem 4.11. For each \(B \in \mathcal{B}\), the band projection \(P_B\) is un-continuous by Remark 4.9, so that \(P_B(A)\) is sequentially un-compact in \(B\). Since \(B\) has a weak unit, the un-topology on \(B\) is metrizable, so that \(P_B(A)\) is un-compact in \(B\). The conclusion now follows from Theorem 7.1.

Theorem 7.5. For a Banach lattice \(X\), TFAE:

(i) \(B_X\) is un-compact;
(ii) \(B_X\) is sequentially un-compact;
(iii) \(X\) is an atomic KB-space.

Proof. First, observe that both (i) and (ii) imply that \(X\) is order continuous and atomic. Indeed, since order intervals are bounded and un-closed, they are (sequentially) un-compact. But on order intervals, the un-topology agrees with the norm topology, hence order intervals are norm compact. This implies that \(X\) is atomic and order continuous; see, e.g., [Wm99 Theorem 6.1].

Suppose (i). Since \(X\) is order continuous, Proposition 7.4(ii) yields (ii).

Suppose (iii). We already know that \(X\) is atomic. To show that \(X\) is a KB-space, let \((x_n)\) be an increasing norm bounded sequence in \(X_+\). By assumption, it has a un-convergent subsequence \((x_{n_k})\). By Lemma 1.2(ii), \((x_{n_k})\) converges in norm, hence \((x_n)\) converges in norm. This yields (iii).

Suppose (iii). Let \(A\) be a maximal disjoint family of atoms in \(X\). Then \(\{B_a : a \in A\}\) is a dense band decomposition of \(X\). For every \(a \in A\), \(P_a(B_X)\) is a closed bounded subset of the one-dimensional band
$B_a$, hence $P_a(B_X)$ is norm and un-compact in $B_a$. Theorem 7.1 now
implies that $B_X$ is un-compact, which yields (i). □

**Example 7.6.** Let $X = c_0$ and $x_n = e_1 + \cdots + e_n$. Then $(x_n)$ is a
sequence in $B_X$ with no un-convergent subsequences.

**Proposition 7.7.** Let $A$ be a subset of an order continuous Banach
lattice $X$. If $A$ is relatively un-compact then $A$ is relatively sequentially
un-compact.

**Proof.** Let $(x_n)$ be a sequence in $A$. Find $e \in X_+$ such that $(x_n)$ is
contained in $B_e$. Since $A^\text{un}$ is un-compact, the set $A^\text{un} \cap B_e$ is un-
compact in $B_e$ and, therefore, sequentially un-compact in $B_e$ because
the un-topology on $B_e$ is metrizable. Hence, there is a subsequence
$(x_{n_k})$ which un-converges in $B_e$ and, therefore, in $X$. □

8. **Un-convergence and weak*-convergence**

When does un-convergence imply weak*-convergence? It is
easy to see that, in general, un-convergence does not imply weak*-convergence. Indeed, let $X$ be an infinite-dimensional Banach lattice
with order continuous dual. Pick any unbounded disjoint sequence $(f_n)$
in $X^*$. Being unbounded, $(f_n)$ cannot be weak*-null. Yet it is un-null
by Proposition 3.3. However, if we restrict ourselves to norm bounded
nets, the situation is more interesting. The following result is analog-
ous to [Gao14, Theorem 2.1]. Recall that for a net $(f_\alpha)$ in $X^*$, we
write $f_\alpha \xrightarrow{[\sigma(X^*,X)]} 0$ if $|f_\alpha|(x) \to 0$ for every $x \in X_+$.

**Theorem 8.1.** Let $X$ be a Banach lattice such that $X^*$ is order con-
tinuous. The following are equivalent:

(i) $X$ is order continuous;

(ii) for any norm bounded net $(f_\alpha)$ in $X^*$, if $f_\alpha \xrightarrow{\text{un}} 0$, then $f_\alpha \xrightarrow{w^*} 0$;

(iii) for any norm bounded net $(f_\alpha)$ in $X^*$, if $f_\alpha \xrightarrow{\text{un}} 0$, then $f_\alpha \xrightarrow{[\sigma(X^*,X)]} 0$;

(iv) for any norm bounded sequence $(f_n)$ in $X^*$, if $f_n \xrightarrow{\text{un}} 0$, then $f_n \xrightarrow{w^*} 0$;
(v) for any norm bounded sequence \((f_n)\) in \(X^*\), if \(f_n \xrightarrow{\text{un}} 0\), then \(f_n \xrightarrow{|\sigma||X^*,X|} 0\).

The proof is similar to that of [Gao14, Theorem 2.1] except that in the proof of (iv) \(\Rightarrow\) (i) we use Proposition 3.5. Note that without the assumption that \(X^*\) is order continuous, we still get the following implications:

\[
\begin{align*}
(\bar{\text{i}}) & \Rightarrow (\bar{\text{ii}}) \iff (\bar{\text{iii}}) \Rightarrow (\bar{\text{iv}}) \iff (\bar{\text{v}}).
\end{align*}
\]

When does weak*-convergence imply un-convergence? Recall that for norm bounded nets, weak*-convergence implies uo-convergence in \(X^*\) iff \(X\) is atomic and order continuous by [Gao14, Theorem 3.4]. Furthermore, Proposition [4.16] immediately yields the following.

**Corollary 8.2.** If \(f_n \xrightarrow{w^*} 0\) implies \(f_n \xrightarrow{\text{un}} 0\) for every sequence in \(X^*\) then \(X^*\) is atomic and order continuous.

The following example shows that the converse is false in general.

**Example 8.3.** Let \(X = c\), the space of all convergent sequences. By [AB06a, Theorem 16.14], \(X^*\) may be identified with \(\ell_1 \oplus \mathbb{R}\) with the duality given by

\[
\langle (f, r), x \rangle = r \cdot \lim_{n} x_n + \sum_{n=1}^{\infty} f_n x_n,
\]

where \(x \in c\), \(f \in \ell_1\), and \(r \in \mathbb{R}\). It is easy to see that \(X^*\) is atomic and order continuous. Consider the sequence \(((e_n, 0))\) in \(X^*\), where \(e_n\) is the \(n\)-th standard unit vector in \(\ell_1\). It is easy to see that \((e_n, 0) \xrightarrow{w^*} (0, 1)\) in \(X^*\). On the other hand, this sequence is disjoint and, therefore, un-null. Take \(f_n = (e_n, -1)\); it follows that \((f_n)\) is weak*-null but not un-null. Note that in this example, \(X^*\) is order continuous while \(X\) is not.

Nevertheless, we will show that the converse implication is true under the additional assumption that \(X\) is order continuous.

**Theorem 8.4.** The following are equivalent:

(i) For every net \((f_\alpha)\) in \(X^*\), if \(f_\alpha \xrightarrow{w^*} 0\) then \(f_\alpha \xrightarrow{\text{un}} 0\);

(ii) \(X^*\) is atomic and both \(X\) and \(X^*\) are order continuous.
Proof. (i)⇒(ii) By Corollary 8.2, $X^*$ is atomic and order continuous. Suppose $X$ is not order continuous. By [MN91, Corollary 2.4.3] there exists a disjoint norm-bounded sequence $(f_n)$ in $X^*$ which is not weak*-null. One can then find a subsequence $(f_{n_k})$, a vector $x_0 \in X$ and a positive real $\varepsilon$ so that $|f_{n_k}(x_0)| > \varepsilon$ for every $k$. By the Alaoglu-Bourbaki Theorem, there is a subnet $(g_\alpha)$ of $(f_{n_k})$ such that $g_\alpha \wto g$ for some $g \in X^*$. Since $(f_{n_k})$ is disjoint and $X^*$ is order continuous, we have $f_{n_k} \uto 0$ and, therefore, $g_\alpha \uto 0$. By assumption, this yields $g = 0$, so that $g_\alpha \wto 0$. This contradicts $|g_\alpha(x_0)| > \varepsilon$ for every $\alpha$.

(ii)⇒(i) Let $f_\alpha \wto 0$ in $X$. Let $A$ be a maximal disjoint collection of atoms in $X^*$; for each atom $a \in A$ let $P_a$ and $\varphi_a$ be the corresponding band projection and the coordinate functional, respectively; $P_a$ and $\varphi_a$ are defined on $X^*$. By [MN91, Corollary 2.4.7], $P_a$ and, therefore, $\varphi_a$, is weak*-continuous. It follows that $\varphi_a(f_\alpha) \to 0$ in $\alpha$. Corollary 4.14 yields that $f_\alpha \uto 0$. □

Proposition 8.5. Suppose that $X^*$ is atomic. The following are equivalent.

(i) For every net $(f_\alpha)$ in $X^*$, if $f_\alpha \sigma(X^*,X) \to 0$ then $f_\alpha \uto 0$;
(ii) For every sequence $(f_n)$ in $X^*$, if $f_n \sigma(X^*,X) \to 0$ then $f_n \uto 0$;
(iii) $X^*$ is order continuous.

Proof. (i)⇒(ii) is trivial.

(ii)⇒(iii) The proof is similar to that of Proposition 4.16. To show that $X^*$ is order continuous, suppose that $(f_n)$ is an order bounded positive disjoint sequence in $X^*_+$. It follows that $f_n \sigma(X^*,X) \to 0$ and, by assumption, $f_n \uto 0$. Since the sequence is order bounded, this yields $f_n \to 0$. Therefore, $X^*$ is order continuous.

(iii)⇒(i) By [MN91] Proposition 2.4.5, band projections on $X^*$ are $\sigma(X^*,X)$-continuous. The proof is now analogous to the implication (ii)⇒(i) in Theorem 8.4. □

Simultaneous weak* and un-convergence. Section 4 of [Gao14] contains several results that assert that if a sequence or a net in $X^*$ converges in both weak* and uo-topology then it also converges in some other topology. Several of these results remain valid if uo-convergence
is replaced with un-convergence. In particular, this works for Proposition 4.1 in [Gao14]. Propositions 4.3, 4.4, and 4.6 in [Gao14] remain valid under the additional assumption that $X^*$ is order continuous (note that the dual positive Schur property already implies that $X^*$ is order continuous by [Wnuk13 Proposition 2.1]). The proofs are analogous to the corresponding proofs in [Gao14]. Alternatively, the un-versions of these may be deduced from the uo-versions using the following two facts: first, every un-convergent sequence has a uo-convergent subsequence and, second, a sequence $(x_n)$ converges to $x$ in a topology $\tau$ iff every subsequence $(x_{n_k})$ has a further subsequence $(x_{n_{k_i}})$ such that $x_{n_{k_i}} \overset{\tau}{\to} x$.

9. Un-compact operators

Throughout this section, let $E$ be a Banach space, $X$ a Banach lattice, and $T \in L(E, X)$. We say that $T$ is **(sequentially) un-compact** if $TB_E$ is relatively (sequentially) un-compact in $E$. Equivalently, for every bounded net $(x_\alpha)$ (respectively, every bounded sequence $(x_n)$) its image has a subnet (respectively, subsequence), which is un-convergent.

Clearly, if $T$ is compact then it is un-compact and sequentially un-compact. Theorems 3.2 and 7.5 and Proposition 7.7 yield the following.

**Proposition 9.1.** Let $T \in L(E, X)$.

(i) If $X$ has a quasi-interior point then $T$ is un-compact iff it is sequentially un-compact;

(ii) If $X$ is order continuous and $T$ is un-compact then $T$ is sequentially un-compact;

(iii) If $X$ is an atomic KB-space then $T$ is un-compact and sequentially un-compact.

**Proposition 9.2.** The set of all un-compact operators is a linear subspace of $L(E, X)$. The set of all sequentially un-compact operators in $L(E, X)$ is a closed subspace.

**Proof.** Linearity is straightforward. To prove closedness, suppose that $(T_m)$ is a sequence of sequentially un-compact operators in $L(E, X)$ and $T_m \overset{\|\cdot\|}{\longrightarrow} T$. We will show that $T$ is sequentially un-compact.
Let \((x_n)\) be a sequence in \(B_E\). For every \(m\), the sequence \((T_m x_n)_n\) has a un-convergent subsequence. By a standard diagonal argument, we can find a common subsequence for all these sequences. Passing to this subsequence, we may assume without loss of generality that for every \(m\) we have \(T_m x_n \overset{\text{un}}{\longrightarrow} y_m\) for some \(y_m\). Note that 
\[ \|y_m - y_k\| \leq \liminf_n \|T_m x_n - T_k x_n\| \leq \|T_m - T_k\| \to 0, \]
so that the sequence \((y_m)\) is Cauchy and, therefore, \(y_m \overset{\|\cdot\|}{\longrightarrow} y\) for some \(y \in X\).

Fix \(u \in X^+\) and \(\varepsilon > 0\). Find \(m_0\) such that \(\|T_{m_0} - T\| < \varepsilon\) and \(\|y_{m_0} - y\| < \varepsilon\). Find \(n_0\) such that \(\|T_{m_0} x_n - y_{m_0} \wedge u\| < \varepsilon\) whenever \(n \geq n_0\). It follows from
\[ |T x_n - y| \wedge u \leq |T x_n - T_{m_0} x_n| + |T_{m_0} x_n - y_{m_0}| \wedge u + |y_{m_0} - y| \]
that \(\|T x_n - y| \wedge u\| < 3\varepsilon\), so that \(T x_n \overset{\text{un}}{\longrightarrow} y\). \(\square\)

We do not know whether the set of all un-compact operators is closed.

It is easy to see that if we multiply a (sequentially) un-compact operator by another bounded operator on the right, the product is again (sequentially) un-compact. The following example shows that this fails when we multiply on the left.

**Example 9.3.** The class of all (sequentially) un-compact operators is not a left ideal. Let \(T : \ell_1 \to L_1\) be defined via \(T e_n = r_n^+\), where \((e_n)\) is the standard unit basis of \(\ell_1\) and \((r_n)\) is the Rademacher sequence in \(L_1\). Note that \(T\) is neither un-compact nor sequentially un-compact because the sequence \((T e_n)\) has no un-convergent subsequences. On the other hand, \(T = T I_{\ell_1}\), where \(I_{\ell_1}\) is the identity operator on \(\ell_1\). Observe that \(I_{\ell_1}\) is un-compact by Proposition 9.1(iii).

**Proposition 9.4.** In the diagram \(E \overset{T}{\to} X \overset{S}{\to} Y\), suppose that \(T\) is (sequentially) un-compact and \(S\) is a lattice homomorphism. If the ideal generated by \(\text{Range } S\) is dense in \(Y\) then \(ST\) is (sequentially) un-compact.

**Proof.** We will prove the statement for the sequential case; the other case is analogous. Let \((h_n)\) be a norm bounded sequence in \(E\). By
assumption, there is a subsequence \((h_{n_k})\) such that \(Th_{n_k} \xrightarrow{un} x\) for some \(x \in X\). Let \(Z = \text{Range} S\); it is a sublattice of \(Y\). Fix \(u \in Z_+\). Then \(u = Sv\) for some \(v \in X_+\), and \(|Th_{n_k} - x| \wedge u \xrightarrow{\|\cdot\|} 0\). Applying \(S\), we get \(|STh_{n_k} - Sy| \wedge u \xrightarrow{\|\cdot\|} 0\). Therefore, \(STh_{n_k} \xrightarrow{un} Sx\) in \(Z\). It follows from Theorem 4.3(i) and (ii) that \(STh_{n_k} \xrightarrow{un} Sx\) in \(Y\). \(\square\)

**Example 9.5.** The set of all sequentially un-compact operators is not order closed. Let \(T\) be as in Example 9.3. Let \(T_n = TP_n\), where \(P_n\) is the \(n\)-th basis projection on \(\ell_1\), i.e., \(T_nh = \sum_{i=1}^n h_ir_i^+\) for \(h \in \ell_1\). It is easy to see that each \(T_n\) is finite rank and, therefore, sequentially un-compact. Note that \(T_n \uparrow T\), yet \(T\) is not sequentially un-compact.

**Proposition 9.6.** Suppose that for every sequence \((T_n)\) of sequentially un-compact operators in \(L(c_0, X)\), \(T_n \uparrow T\) implies that \(T\) is sequentially un-compact. Then \(X\) is a KB-space.

**Proof.** Suppose not. Then there is a lattice isomorphism \(T: c_0 \rightarrow X\). Put \(x_n = Te_n\), where \((e_n)\) is the standard unit basis of \(c_0\). Put \(T_n = TP_n\), where \(P_n\) is the \(n\)-th basis projection on \(c_0\), i.e., \(T_nh = \sum_{i=1}^n h_ir_i^+\) for \(h \in \ell_1\). It follows that \(T_nh \xrightarrow{\|\cdot\|} Th\), so that \(T_nh \uparrow Th\) for every \(h \geq 0\) and, therefore, \(T_n \uparrow T\). For each \(n\), \(T_n\) has finite rank and, therefore, is sequentially un-compact.

We claim that, nevertheless, \(T\) is not sequentially un-compact. Put \(w_n = e_1 + \cdots + e_n\) in \(c_0\). Note that \((w_n)\) is norm bounded and \(Tw_n = x_1 + \cdots + x_n\). Since \(T\) is an isomorphism, \((Tw_n)\) is not norm-convergent. Since \((Tw_n)\) is increasing, it is not un-convergent by Lemma 1.2(ii). Similarly, no subsequence of \((Tw_n)\) is un-convergent. \(\square\)

We do not know whether the converse is true.

Next, we study whether un-compactness is inherited under domination. The following example shows that, in general, the answer is negative.

**Example 9.7.** Let \(T\) be as in Example 9.3. Let \(S: \ell_1 \rightarrow L_1\) be defined via \(Se_n = 1\). Then \(S\) is a rank-one operator; hence it is compact and un-compact. Clearly, \(0 \leq T \leq S\). Yet \(T\) is not un-compact.
Proposition 9.8. Suppose that $S, T : E \to X$, $0 \leq S \leq T$, $X$ is a KB-space and $T$ is a lattice homomorphism. If $T$ is (sequentially) un-compact then so is $S$.

Proof. We will prove the sequential case; the other case is similar. Let $(h_n)$ be a bounded sequence in $E$. Passing to a subsequence, we may assume that $(Th_n)$ is un-convergent. In particular, it is un-Cauchy. Fix $u \in X_+$. Note that

$$|Sh_n - Sh_m| \land u \leq (S|h_n - h_m|) \land u \leq (T|h_n - h_m|) \land u = |Th_n - Th_m| \land u \|x\to 0$$

as $n, m \to \infty$. It follows that $(Sh_n)$ is un-Cauchy and, therefore, un-converges by Theorem 6.4.

We would like to mention that the class of un-compact operators is different from several other known classes of operators. We already mentioned that every compact operator is un-compact. The converse is false as the identity operator on any infinite-dimensional atomic KB-space is un-compact but not compact.

Recall that an operator between Banach lattices is AM-compact if it maps order intervals to relatively compact sets.

Proposition 9.9. Every order bounded un-compact operator is AM-compact.

Proof. Let $T : X \to Y$ be an order bounded un-compact operator between Banach lattices. Fix an order interval $[a, b]$ in $X$. Since $T$ is un-compact, $T[a, b] \subseteq C$ for some un-compact set $C$. Since $T$ is order bounded, $T[a, b] \subseteq [c, d]$ for some $c, d \in Y$. Note that $[c, d]$ is un-closed, hence $C \cap [c, d]$ is un-compact and, being order bounded, is compact. It follows that $T[a, b]$ is relatively compact.

Note that the converse is false: the identity operator on $c_0$ is AM-compact but not un-compact.

The identity operator on $\ell_1$ is un-compact, yet it is neither L-weakly compact nor M-weakly compact.

Finally, we note that if $T$ is sequentially un-compact and semi-compact then $T$ is compact. Indeed, let $(h_n)$ be a bounded sequence in $E$. There is a subsequence $(h_{n_k})$ such that $Th_{n_k} \xrightarrow{\text{un}} x$ for some $x \in X$. 
Since $T$ is semi-compact, the sequence $(Th_{n_k})$ is almost order bounded and, therefore, $Th_{n_k} \xrightarrow{\|\cdot\|} x$ by [DOT, Lemma 2.9].

Finally, we discuss when weakly compact operators are un-compact.

**Lemma 9.10.** If $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{un} y$ then $x = y$.

**Proof.** Without loss of generality, $y = 0$. By Theorem 1.1, there exist a subsequence $(x_{n_k})$ and a disjoint sequence $(d_k)$ such that $x_{n_k} - d_k \xrightarrow{\|\cdot\|} 0$. It follows that $x_{n_k} - d_k \xrightarrow{w} 0$, so that $d_k \xrightarrow{w} x$. Now [AB06, Theorem 4.34] yields $x = 0$. □

**Theorem 9.11.** A Banach lattice $X$ is atomic and order continuous iff $T$ is sequentially un-compact for every Banach space $E$ and every weakly compact operator $T : E \rightarrow X$.

**Proof.** The forward implication follows immediately from Proposition 4.16. To prove the converse, let $(x_n)$ be a weakly null sequence in $X$. By Proposition 4.16, it suffices to show that $x_n \xrightarrow{un} 0$. Define $T : \ell_1 \rightarrow X$ via $Te_n = x_n$. By [AB06, Theorem 5.26], $T$ is weakly compact. By assumption, $T$ is sequentially un-compact. It follows that $(Te_n)$ has a un-convergent subsequence, i.e., $x_{n_k} \xrightarrow{un} x$ for some $x \in X$ and a subsequence $(x_{n_k})$. Lemma 9.10 yields $x = 0$. By the same argument, every subsequence of $(x_n)$ has a further subsequence which is un-null; since un-convergence is topological, it follows that $x_n \xrightarrow{un} 0$. □

**Corollary 9.12.** Every operator from a reflexive Banach space to an atomic order continuous Banach lattice is sequentially un-compact.

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