MINIMAL VECTORS IN ARBITRARY BANACH SPACES

VLADIMIR G. TROITSKY

ABSTRACT. We extend the method of minimal vectors to arbitrary Banach spaces. It is proved, by a variant of the method, that certain quasinilpotent operators on arbitrary Banach spaces have hyperinvariant subspaces.

The method of *minimal vectors* was introduced by Ansari and Enflo in [AE98] in order to prove the existence of invariant subspaces for certain classes of operators on a Hilbert space. Pearcy used it in [P] to prove a version of Lomonosov's theorem. Androulakis in [A] adapted the technique to super-reflexive Banach spaces. In [CPS] the method was independently generalized to reflexive Banach spaces. There has been hope that this technique could provide a positive solution to the invariant subspace problem for these spaces. In this note we present a version of the method of minimal vectors (based on [A]) that works for arbitrary Banach spaces. In particular, it applies in the spaces where there are known examples of operators without invariant subspaces, e.g., [Enf76, Enf87, Rea84, Rea85]. This shows that the method of minimal vectors alone cannot solve the invariant subspace problem for "good" spaces.

Suppose that X is a Banach space. For simplicity, we assume that X is a real Banach space, though the results can be adapted to the complex case in a straightforward manner. In the following, $B(x_0, \varepsilon)$ stands for the closed ball of radius ε centered at x_0 while $B^\circ(x_0, \varepsilon)$ stands for the open ball, and $S(x_0, \varepsilon)$ stands for the corresponding sphere.

2000 *Mathematics Subject Classification.* Primary: 47A15.
Key words and phrases. Invariant subspace, minimal vector.

Proc. Amer. Math. Soc., **132** (2004), 1177–1180.

Let Q be a bounded operator on X. Since we will be interested in the hyperinvariant subspaces of Q, we can assume, without loss of generality, that Q is one-to-one and has dense range, since otherwise $\ker Q$ or $\overline{\operatorname{Range} Q}$ would be hyperinvariant for Q. By $\{Q\}'$ we denote the commutant of Q.

Fix a point $x_0 \neq 0$ in X and a positive real $\varepsilon < \|x_0\|$. Let $K = Q^{-1}B(x_0,\varepsilon)$. Clearly, K is a convex closed set. Note that $0 \notin K$ and $K \neq \varnothing$ because Q has dense range. Let $d = \inf_K \|z\|$. Then $d > 0$. It is observed in [AE98, A] that if X is reflexive, then there exists $z \in K$ with $\|z\| = d$. Such a vector is called a ***minimal vector*** for x_0, ε and Q. Even without the reflexivity condition, however, one can always find $y \in K$ with $\|y\| \leqslant 2d$; such a y will be referred to as a ***2-minimal vector*** for x_0, ε and Q.

The set $K \cap B(0,d)$ is the set of all minimal vectors; in general, this set may be empty. If z is a minimal vector, since $z \in K = Q^{-1}B(x_0,\varepsilon)$ then $Qz \in B(x_0,\varepsilon)$. Since z is an element of minimal norm in K, then, in fact, $Qz \in S(x_0,\varepsilon)$. Since Q is one-to-one, we have

$$QB(0,d) \cap B(x_0,\varepsilon) = Q\big(B(0,d) \cap K\big) \subseteq S(x_0,\varepsilon).$$

It follows that $QB(0,d)$ and $B^\circ(x_0,\varepsilon)$ are two disjoint convex sets. Since one of them has nonempty interior, they can be separated by a continuous linear functional (see, e.g., [AB99, Theorem 5.5]). That is, there exists a functional f with $\|f\| = 1$ and a positive real c such that $f_{|QB(0,d)} \leqslant c$ and $f_{|B^\circ(x_0,\varepsilon)} \geqslant c$. By continuity, $f_{|B(x_0,\varepsilon)} \geqslant c$. We say that f is a ***minimal functional*** for x_0, ε, and Q.

We claim that $f(x_0) \geqslant \varepsilon$. Indeed, for every x with $\|x\| \leqslant 1$ we have $x_0 - \varepsilon x \in B(x_0,\varepsilon)$. It follows that $f(x_0 - \varepsilon x) \geqslant c$, so that $f(x_0) \geqslant c + \varepsilon f(x)$. Taking sup over all x with $\|x\| \leqslant 1$ we get $f(x_0) \geqslant c + \varepsilon\|f\| \geqslant \varepsilon$.

Observe that the hyperplane $Q^*f = c$ separates K and $B(0,d)$. Indeed, if $z \in B(0,d)$, then $(Q^*f)(z) = f(Qz) \leqslant c$, and if $z \in K$, then $Qz \in B(x_0,\varepsilon)$ so that $(Q^*f)(z) = f(Qz) \geqslant c$. For every z with $\|z\| \leqslant 1$ we have $dz \in B(0,d)$, so that $(Q^*f)(dz) \leqslant c$. It follows that $\|Q^*f\| \leqslant \frac{c}{d}$. On the other hand, for every $\delta > 0$ there exists $z \in K$ with $\|z\| \leqslant d + \delta$. Then $(Q^*f)(z) \geqslant c \geqslant \frac{c}{d+\delta}\|z\|$, whence $\|Q^*f\| \geqslant \frac{c}{d+\delta}$. It follows that $\|Q^*f\| = \frac{c}{d}$. For every $z \in K$ we have $(Q^*f)(z) \geqslant c = d\|Q^*f\|$. In particular, if y is a 2-minimal vector, then

$$(Q^*f)(y) \geqslant \tfrac{1}{2}\|Q^*f\|\|y\|. \tag{1}$$

We proceed to the main theorem.

Theorem. *Let Q be a quasinilpotent operator on a Banach space X, and suppose that there exists a closed ball B such that $0 \notin B$ and for every sequence (x_n) in B there is a subsequence (x_{n_i}) and a sequence (K_i) in $\{Q\}'$ such that $\|K_i\| \leqslant 1$ and $(K_i x_{n_i})$ converges in norm to a nonzero vector. Then Q has a hyperinvariant subspace.*

Remark. The hypothesis of the theorem is slightly weaker than the condition $(*)$ in [A], where it is required that for every $\varepsilon \in (0,1)$, there exists x_0 of norm one such that the ball $B(x_0, \varepsilon)$ satisfies the rest of the condition.

Proof. Without loss of generality, Q is one-to-one and has dense range. Let $x_0 \neq 0$ and $\varepsilon \in \big(0, \|x_0\|\big)$ be such that $B = B(x_0, \varepsilon)$. For every $n \geqslant 1$ choose a 2-minimal vector y_n and a minimal functional f_n for x_0, ε, and Q^n.

Since Q is quasinilpotent, there is a subsequence (y_{n_i}) such that $\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0$. Indeed, otherwise there would exist $\delta > 0$ such that $\frac{\|y_{n-1}\|}{\|y_n\|} > \delta$ for all n, so that $\|y_1\| \geqslant \delta\|y_2\| \geqslant \ldots \geqslant \delta^n\|y_{n+1}\|$. Since $Q^n y_{n+1} \in Q^{-1}B$, we have

$$\|Q^n y_{n+1}\| \geqslant d \geqslant \frac{\|y_1\|}{2} \geqslant \frac{\delta^n}{2}\|y_{n+1}\|.$$

It follows that $\|Q^n\| \geqslant \delta^n/2$, which contradicts the quasinilpotence of Q.

Since $\|f_{n_i}\| = 1$ for all i, we can assume (by passing to a further subsequence), that (f_{n_i}) weak*-converges to some $g \in X^*$. Since $f_n(x_0) \geqslant \varepsilon$ for all n, it follows that $g(x_0) \geqslant \varepsilon$. In particular, $g \neq 0$.

Consider the sequence $(Q^{n_i-1}y_{n_i-1})_{i=1}^{\infty}$. It is contained in B, so that by passing to yet a further subsequence, if necessary, we find a sequence (K_i) in $\{Q\}'$ such that $\|K_i\|\leqslant 1$ and $K_iQ^{n_i-1}y_{n_i-1}$ converges in norm to some $w\neq 0$. Put

$$Y=\{Q\}'Qw=\bigl\{TQw\mid T\in\{Q\}'\bigr\}.$$

One can easily verify that Y is a linear subspace of X invariant under $\{Q\}'$. Notice that Y is nontrivial because Q is one-to-one and $0\neq Qw\in Y$. We will show that $Y\subseteq\ker g$, so that $\overline{Y}$ is a proper Q-hyper-invariant subspace.

Take $T\in\{Q\}'$; we will show that $g(TQw)=0$. It follows from (1) that $\bigl(Q^{*n_i}f_{n_i}\bigr)(y_{n_i})\neq 0$ for every i, so that $X=\operatorname{span}\{y_{n_i}\}\oplus\ker\bigl(Q^{*n_i}f_{n_i}\bigr)$. Then one can write $TK_iy_{n_i-1}=\alpha_iy_{n_i}+r_i$, where α_i is a scalar and $r_i\in\ker\bigl(Q^{*n_i}f_{n_i}\bigr)$. We claim that $\alpha_i\to 0$. Indeed,

$$\bigl(Q^{*n_i}f_{n_i}\bigr)\bigl(TK_iy_{n_i-1}\bigr)=\alpha_i\bigl(Q^{*n_i}f_{n_i}\bigr)(y_{n_i}),\tag{2}$$

and, combining this with (1), we get

$$\bigl|\bigl(Q^{*n_i}f_{n_i}\bigr)\bigl(TK_iy_{n_i-1}\bigr)\bigr|\geqslant\frac{|\alpha_i|}{2}\bigl\|Q^{*n_i}f_{n_i}\bigr\|\|y_{n_i}\|.\tag{3}$$

On the other hand,

$$\bigl|\bigl(Q^{*n_i}f_{n_i}\bigr)\bigl(TK_iy_{n_i-1}\bigr)\bigr|\leqslant\bigl\|Q^{*n_i}f_{n_i}\bigr\|\cdot\|T\|\cdot\|y_{n_i-1}\|.\tag{4}$$

It follows from (3) and (4) that

$$|\alpha_i|\leqslant 2\|T\|\frac{\|y_{n_i-1}\|}{\|y_{n_i}\|}\to 0.$$

Then (2) yields that

$$\begin{aligned}\Bigl|f_{n_i}\bigl(Q^{n_i}TK_iy_{n_i-1}\bigr)\Bigr|&=\Bigl|\alpha_if_{n_i}\bigl(Q^{n_i}y_{n_i}\bigr)\Bigr|\\&\leqslant|\alpha_i|\cdot\|f_{n_i}\|\cdot\|Q^{n_i}y_{n_i}\|\leqslant|\alpha_i|\cdot 1\cdot\bigl(\|x_0\|+\varepsilon\bigr)\to 0,\end{aligned}$$

so that $f_{n_i}\bigl(Q^{n_i}TK_iy_{n_i-1}\bigr)\to 0$. On the other hand, since $T,K_i\in\{Q\}'$ we have

$$Q^{n_i}TK_iy_{n_i-1}=TQK_iQ^{n_i-1}y_{n_i-1}\to TQw$$

in norm, while $f_{n_i}\xrightarrow{w^*}g$, so that $f_{n_i}\bigl(Q^{n_i}TK_iy_{n_i-1}\bigr)\to g(TQw)$. Hence, $g(TQw)=0$. $\square$

Clearly, the argument will work as well for λ-minimal vectors for any $\lambda > 1$.

Suppose that Q is a quasinilpotent operator commuting with a compact operator K. Then Q satisfies the hypothesis of the theorem. Indeed, without loss of generality, $\|K\| = 1$. Fixing $\varepsilon = \frac{1}{3}$, there exists x_0 with $\|x_0\| = 1$ such that $\|Kx_0\| \geqslant \frac{2}{3}$ and $0 \notin KB(x_0, \varepsilon)$. For every sequence (x_n) in $B(x_0, \varepsilon)$, the sequence (Kx_n) has a convergent subsequence (Kx_{n_i}). Take $K_i = K$ for all i; since $0 \notin KB(x_0, \varepsilon)$ then $\lim_i K_i x_{n_i} \neq 0$. It follows from the theorem that *if Q is a quasinilpotent operator on a real or complex Banach space commuting with a nonzero compact operator, then Q has a hyperinvariant subspace.* This fact is not new though: for complex Banach spaces it is a special case of the celebrated Lomonosov's theorem [Lom73], and for real Banach spaces it follows from Theorem 2 of [Hoo81].

Acknowledgements. Thanks are due to A. Litvak, N. Tomczak-Jaegermann, and R. Vershynin for their interest in the work and helpful suggestions. The author would also like to thank the Department of Mathematics of the University of Alberta for its support and hospitality.

References

[AB99] Charalambos D. Aliprantis and Kim C. Border, *Infinite-dimensional analysis, a hitchhiker's guide*, second ed., Springer-Verlag, Berlin, 1999. MR **2000k:**46001

[AE98] Shamim Ansari and Per Enflo, *Extremal vectors and invariant subspaces*, Trans. Amer. Math. Soc. **350** (1998), no. 2, 539–558. MR **98d:**47019

[A] George Androulakis, *A note on the method of minimal vectors*, Trends in Banach spaces and operator theory (Memphis, TN, 2001), Contemp. Math., vol. 321, Amer. Math. Soc., Providence, RI, 2003, pp. 29–36. MR 1 978 805

[CPS] I. Chalendar, J.R. Partington, and M. Smith, *Approximation in reflexive Banach spaces and applications to the invariant subspace problem*, Proc. American Math. Soc., to appear.

[Enf76] P. Enflo, *On the invariant subspace problem in Banach spaces*, Séminaire Maurey–Schwartz (1975–1976) Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. 14-15, Centre Math., École Polytech., Palaiseau, 1976, pp. 1–7. MR **MR57:**13530

[Enf87] Per Enflo, *On the invariant subspace problem for Banach spaces*, Acta Math. **158** (1987), no. 3-4, 213–313. MR **88j:**47006

[Hoo81] N. D. Hooker, *Lomonosov's hyperinvariant subspace theorem for real spaces*, Math. Proc. Cambridge Philos. Soc. **89** (1981), no. 1, 129–133. MR **84a:**47009

[Lom73] V. I. Lomonosov, *Invariant subspaces of the family of operators that commute with a completely continuous operator*, Funkcional. Anal. i Priložen. **7** (1973), no. 3, 55–56. MR **MR54:**8319

[P] C. Pearcy, *On technique of Enflo*, Proc. American Math. Soc., to appear.

[Rea84] C. J. Read, *A solution to the invariant subspace problem*, Bull. London Math. Soc. **16** (1984), no. 4, 337–401. MR **MR86f:**47005

[Rea85] ———, *A solution to the invariant subspace problem on the space* l_1, Bull. London Math. Soc. **17** (1985), no. 4, 305–317. MR **MR87e:**47013

[RR73] Heydar Radjavi and Peter Rosenthal, *Invariant subspaces*, Springer-Verlag, New York, 1973, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77. MR **MR51:**3924

Department of Mathematics, University of Alberta, Edmonton, AB, T6G 2G1. Canada.

E-mail address: `vtroitsky@math.ualberta.ca`