MARTINGALES IN BANACH LATTICES

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Dedicated to the memory of Yuri Abramovich, my friend and advisor.

Abstract. In this article we present a version of martingale theory in terms of Banach lattices. A sequence of contractive positive projections (E_n) on a Banach lattice F is said to be a filtration if $E_nE_m = E_{n\wedge m}$. A sequence (x_n) in F is a martingale if $E_nx_m = x_n$ whenever $n \leqslant m$. Denote by $M = M\bigl(F, (E_n)\bigr)$ the Banach space of all norm uniformly bounded martingales. It is shown that if F doesn't contain a copy of c_0 or if every E_n is of finite rank then M is itself a Banach lattice. Convergence of martingales is investigated and a generalization of Doob Convergence Theorem is established. It is proved that under certain conditions one has isometric embeddings $F \hookrightarrow M \hookrightarrow F^{**}$. Finally, it is shown that every martingale difference sequence is a monotone basic sequence.

In this paper we define a martingale in terms of Banach lattices. Rephrasing the title of [Wil91], this paper could be entitled *Martingales without probability.* We start with a brief review of classical martingale theory and Banach lattices.

1. Classical martingale theory

In the following, we fix a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_n)_{n=1}^\infty$, i.e., an increasing sequence of sub-sigma-algebras of $\mathcal{F}$. It is often convenient to assume that $\mathcal{F} = \bigvee_{n=1}^\infty \mathcal{F}_n$, as otherwise one can replace $\mathcal{F}$ with $\bigvee_{n=1}^\infty \mathcal{F}_n$. We will write $L_p(P)$ for $L_p(\Omega, \mathcal{F}, P)$.

A sequence $X = (x_n)_{n=1}^\infty$ of functions in $L_1(P)$ is called a ***martingale*** relative to $(\mathcal{F}_n)$ and P if $E(x_m \mid \mathcal{F}_n) = x_n$ whenever $m \geqslant n$, and a ***submartingale*** if $E(x_m \mid \mathcal{F}_n) \geqslant x_n$ whenever $m \geqslant n$. A martingale X is ***L_p-bounded*** if its ***L_p-martingale norm***, given by $\lVert X\rVert_p = \sup_n \lVert x_n\rVert_p$, is finite. Let the symbol $M_p = M_p\bigl(\Omega, \mathcal{F}, (\mathcal{F}_n), P\bigr)$ denote the space of all L_p-bounded martingales.

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A martingale X is called ***uniformly integrable*** if for every positive ε there exists a number K such that

$$\int_{|x_n|>K} |x_n|\, dP < \varepsilon$$

for every n. Doob Convergence Theorem [Doob53] asserts that a martingale X is uniformly integrable if and only if converges in L_1-norm to some function $x \in L_1(P)$. In this case $x_n = E(x \mid \mathcal{F}_n)$ and $\|x\|_1 = \lim_n \|x_n\|_1 = \|X\|$. Thus, the set of all uniformly integrable martingales is a proper subspace of M_1, isometrically isomorphic to $L_1(P)$, the isomorphism maps X to x. For example, the so called ***double-or-nothing*** martingale $x_n = 2^n \chi_{[0,2^{-n})}$ defined on the unit interval endowed with Lebesgue measure and dyadic filtration is L_1-bounded but not uniformly integrable. However, for $p > 1$ every L_p-bounded martingale is uniformly integrable, so that M_p is isometrically isomorphic to $L_p(P)$. For further details on classical theory of martingales see [Wil91, Doob53, Doob94].

2. Banach lattices

A ***vector lattice*** is a vector space equipped with a lattice order relation, which is compatible with the linear structure. A ***Banach lattice*** is a vector lattice with a Banach norm which is monotone, i.e., $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \|y\|$, and satisfies $\big\||x|\big\| = \|x\|$ for every two vectors x and y, where $|x| = x \vee (-x)$. The spaces $C(K)$ and $L_p(\mu)$ for $1 \leqslant p \leqslant +\infty$ are important examples of Banach lattices. A Banach lattice where $\|x + y\| = \|x\| + \|y\|$ holds for every two nonnegative vectors x and y is called an ***abstract L_1-space*** or ***AL-space***. A vector lattice is said to be ***Dedekind complete*** if every nonempty subset that is bounded above has a supremum. We say that a Banach lattice has ***order continuous norm*** if $\|x_\alpha\| \to 0$ for every decreasing net (x_α) with $\inf_\alpha x_\alpha = 0$. A Banach lattice with order continuous norm is Dedekind complete. A Banach lattice is called a ***Kantorovič-Banach space*** or a ***KB-space*** whenever every increasing norm bounded sequence (x_n) is norm convergent. In this case norm continuity of lattice operations implies that $\lim_n x_n = \sup_n x_n$. It is known that a Banach lattice F is a KB-space iff c_0 is not embeddable in F iff F is weakly sequentially complete. In particular, reflexive Banach lattices and AL-spaces are KB-spaces. The norm in a KB-space is order continuous. A sublattice E of a vector lattice is called an (order) ***ideal*** if $y \in E$ and $|x| \leqslant |y|$ imply $x \in E$. An ideal E is called a ***band*** if $x = \sup_\alpha x_\alpha$ implies $x \in E$ for every positive increasing net (x_α) in E. Two elements x and y in a vector lattice are said to be ***disjoint*** (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. If A is a nonempty subset of a vector lattice, then its ***disjoint complement*** A^d is the set of all elements of the lattice, disjoint to every element of A. A band E in a vector lattice F that satisfies $F = E \oplus E^d$ is reffered to as a ***projection band***. Every band in a Dedekind complete vector lattice is a projection band. An operator T on a Banach lattice F is ***positive*** if it preserves the cone F_+ of positive elements. It is easy to see that $T \geqslant 0$ iff $x \leqslant y$ implies $Tx \leqslant Ty$; furthermore, if $T \geqslant 0$ then $T(x \vee y) \geqslant (Tx) \vee (Ty)$ and $|Tx| \leqslant T|x|$ for any $x, y \in F$. See [AB85, AA02, LZ71, Zaan83, MN91] for more details on Banach lattices. Throughout the rest of the paper F will be a fixed Banach lattice.

3. Main definitions

We will introduce a generalization of the concept of a martingale. By a martingale we will mean a sequence of elements in a Banach lattice satisfying certain properties. Douglas proved in [Doug65] (see also [AAB93]) that the conditional expectations are the only contractive projections on $L_1(P)$ preserving constant functions. In view of this result of Douglas it will be natural in our setting to replace conditional expectations with positive contractive projections. This argument justifies the following notation. A sequence of positive contractive projections (E_n) on a Banach lattice F is called a ***filtration*** if $E_n E_m = E_{n \wedge m}$. Note that the ranges of E_n's form a nested increasing sequence. Here E_n's play the role of conditional expectations.

A filtration is said to be ***dense*** if $E_n x \to x$ in norm for every $x \in F$, or, equivalently, if $\bigcup_{n=1}^{\infty} \operatorname{Range} E_n$ is dense in F. This is analogous to the condition $\mathcal{F} = \bigvee_{n=1}^{\infty} \mathcal{F}_n$ for classical filtrations. But, unlike in the case of classical filtration, generally we cannot simply replace F with $\overline{\bigcup_{n=1}^{\infty} \operatorname{Range} E_n}$ because the latter set need not be a Banach sublattice of F.

Note, however, that if a projection P is ***strictly positive***, i.e., $Px > 0$ whenever $x > 0$, then $\operatorname{Range} P$ is a sublattice of F (see [Sch74, dJ82]). Therefore, if E_m is strictly positive for some m then $\overline{\bigcup_{n=1}^{\infty} \operatorname{Range} E_n}$ is a Banach sublattice of F. Indeed, for every $x > 0$ and every $n \geqslant m$ we have $E_n x \neq 0$ because $0 < E_m x = E_m E_n x$. It follows that E_n is strictly positive for all $n \geqslant m$, so that $\operatorname{Range} E_n$ is a sublattice. Thus, when a filtration contains a strictly positive projection, one can assume that it is dense by replacing F with $\overline{\bigcup_{n=1}^{\infty} \operatorname{Range} E_n}$.

A sequence $X = (x_n)$ of elements of F is called a ***martingale*** relative to a filtration (E_n) if $E_n x_m = x_n$ whenever $n \leqslant m$. It follows, in particular, that $x_n \in \operatorname{Range} E_n$ for every n. A sequence $X = (x_n)$ is called a ***submartingale*** if $E_n x_m \geqslant x_n$ whenever $n \leqslant m$. A (sub)martingale is said to be ***bounded*** if its ***martingale norm*** given by

$\|X\| = \sup_n \|x_n\|$ is finite. Notice that if (x_n) is a martingale or a positive submartingale, then the sequence $\|x_n\|$ is increasing. Indeed, $\|x_n\| \leqslant \|E_n x_{n+1}\| \leqslant \|x_{n+1}\|$. Therefore, $\|X\| = \lim_n \|x_n\|$.

Denote by $M = M\bigl(F, (E_n)\bigr)$ the class of all bounded martingales. Note that M is a closed subspace of the space $\bigl(\bigoplus_{n=1}^{\infty} F\bigr)_{\infty}$, hence M is a Banach space.

4. Examples

In the case of classical martingales we have $F = L_1(P)$ and $E_n = E(\cdot \mid \mathcal{F}_n)$. It was mentioned in the very beginning that one can usually assume that the filtration is dense. Observe, that a classical filtration satisfies another important property: it ***preserves norms of positive vectors***, that is, $\|E_n x\| = \|x\|$ for every n and every $x \in F_+$.

Next, we present several examples of filtrations and martingales related to bases in Banach spaces. For the terminology related to bases in Banach spaces we refer the reader to [LT77].

Example 1. Suppose that (e_i) is a 1-unconditional basis in F such that $\sum_{n=1}^{\infty} \alpha_i e_i \geqslant 0$ iff $\alpha_i \geqslant 0$ for all $i \geqslant 1$. For every $n \geqslant 1$ let E_n be the n-th basis projection given by $E_n\bigl(\sum_{i=1}^{\infty} \alpha_i e_i\bigr) = \sum_{i=1}^{n} \alpha_i e_i$, then (E_n) is a dense filtration. Notice that (x_n) is a martingale iff there exists a sequence of scalars (α_i) such that $x_n = \sum_{i=1}^{n} \alpha_i e_i$ for every $n \geqslant 1$. A martingale (x_n) is convergent if and only there exists $z = \sum_{i=1}^{\infty} \alpha_i e_i$ such that $x_n = \sum_{i=1}^{n} \alpha_i e_i$, in this case x_n converges to z. The basis is boundedly complete if and only if every bounded martingale converges.

Example 2. Consider the special case of Example 1 when $F = c_0$. Again, $X = (x_n)$ is a martingale in c_0 iff there exists a sequence of scalars (α_i) such that $x_n = \sum_{i=1}^{n} \alpha_i e_i$ for every $n \geqslant 1$. Notice that X is bounded if and only if the sequence (α_i) is bounded, and in this case $\|X\| = \sup_i |\alpha_i|$. Thus, in this case M can be identified with ℓ_∞. Observe that here M is non-separable even though X is separable.

Example 3. Let $F = C[0,1]$ and (e_i) be the Schauder system in $C[0,1]$. Again, let E_n be the n-th basis projection. One can easily see that for $f \in C[0,1]$ its image $E_n f$ agrees with f on a set of dyadic points and is linear between those points. In particular, every E_n is a positive operator. Since the Schauder system is a monotone basis, each E_n is a contraction. Hence, (E_n) is a dense filtration on F. Clearly, not every martingale is convergent. For example, put $x_n(0)$ equal 0 at 0, equal 1 at all the other dyadic points corresponding to E_n, and linear in between. Then (x_n) is a non-convergent martingale.

Example 4. Let $F = L_p[0,+\infty)$ $(1 \leqslant p < +\infty)$ and put $E_n x = x \cdot \chi_{[0,n]}$, i.e., E_n "cuts-off" the tail of x after n. One can easily see that (E_n) is a dense filtration. A sequence (x_n) is a martingale if $x_n = \sum_{i=1}^n h_i$ where (h_i) is a sequence in F such that $\operatorname{supp} h_i \subseteq [i-1, i]$. It can be easily verified that the map $x \in F \to (x \cdot \chi_{[0,n]})$ is an isometry from F onto M.

5. When is M a Banach lattice?

Introduce an order on $\left(\bigoplus_{n=1}^\infty F\right)_\infty$ as follows: if $X = (x_n)$ and $Y = (y_n)$, we say that $X \leqslant Y$ if $x_n \leqslant y_n$ for each n. With this order M is an ordered Banach space. Clearly, the norm is monotone: if $0 \leqslant X \leqslant Y$ then $\|X\| \leqslant \|Y\|$.

Notice that if the filtration preserves the norms of positive vectors and $X = (x_n)$ is a positive martingale then $\|x_n\| = \|E_n x_m\| = \|x_m\|$ whenever $n \leqslant m$, so that $\|x_n\| = \|X\|$ for every n. Thus, in this case every positive martingale is bounded.

It is not immediately obvious if M is a lattice in the order we just introduced, and how one could compute $X \vee Y$, $X \wedge Y$, and $|X|$ for two martingales $X = (x_n)$ and $Y = (y_n)$ in M. The "natual guess" that $X \vee Y = (x_n \vee y_n)_{n=1}^\infty$, $X \wedge Y = (x_n \wedge y_n)_{n=1}^\infty$, and $|X| = \left(|x_n|\right)_{n=1}^\infty$ turns out to be wrong. Even when (x_n) is a martingale, $(|x_m|)$ doesn't have to be a martingale. For example, let $(\Omega, \mathcal{F}, P)$ be the unit segment endowed with Lebesgue measure and $(\mathcal{F}_n)$ be the dyadic filtration. Consider an L_1-bounded martingale defined in the following way: $x_1 \equiv 0$, $x_n = \chi_{[0,1/2)} - \chi_{[1/2,1]}$ for $n > 1$. It is easy to see that $(|x_n|)$ is not a martingale. Notice, however, that if (x_n) and (y_n) are two submartingales then $(x_n \vee y_n)$ is a submartingale. Indeed, if $n \geqslant m$ then $E_m(x_n \vee y_n) \geqslant (E_m x_n) \vee (E_m y_n) \geqslant (x_m \vee y_m)$. In particular, if (x_n) is a martingale, then $(|x_n|)$ is a submartingale.

Lemma 5. *Let $X = (x_n)$ and $Y = (y_n)$ be two bounded submartingales.*

(i) *For a fixed n, the sequence $\left(E_n(x_m \vee y_m)\right)_{m=n}^\infty$ is increasing, norm bounded by $\|X\| + \|Y\|$, and bounded below by $x_n \vee y_n$.*

(ii) *If, in addition, this sequence converges in norm to some (z_n) for each n, then $Z = (z_n)$ is a martingale, and it is the least martingale satisfying $X \leqslant Z$ and $Y \leqslant Z$.*

Proof. Let $X = (x_n)$ and $Y = (y_n)$ be two bounded submartingales and $n \leqslant m$, notice that $E_n(x_m \vee y_m) \geqslant (E_n x_m) \vee (E_n y_m) \geqslant x_n \vee y_n$. Furthermore,

$$E_n(x_{m+1} \vee y_{m+1}) = E_n E_m(x_{m+1} \vee y_{m+1}) \geqslant E_n(E_m x_{m+1} \vee E_m y_{m+1}) \geqslant E_n(x_m \vee y_m).$$

Finally,

$$\left\|E_n(x_m \vee y_m)\right\| \leqslant \|x_m \vee y_m\| \leqslant \left\||x_m| + |y_m|\right\| \leqslant \|X\| + \|Y\|.$$

Suppose that $\lim_m E_n(x_m \vee y_m) = z_n$ for each n, and set $Z = (z_n)$. First, observe that Z is a martingale. Indeed, for $k \leqslant n$ we have

$$E_k z_n = E_k\bigl(\lim_{m\to\infty} E_n(x_m \vee y_m)\bigr) = \lim_{m\to\infty} E_k E_n(x_m \vee y_m) = \lim_{m\to\infty} E_k(x_m \vee y_m) = z_k.$$

Since $E_n(x_m \vee y_m) \geqslant x_n \vee y_n$ whenever $m \geqslant n$, we have $z_n \geqslant x_n \vee y_n$ for all n. Thus, $Z \geqslant X$ and $Z \geqslant Y$. On the other hand, suppose that $\widetilde{Z} = (\tilde{z}_n)$ is a martingale such that $\widetilde{Z} \geqslant X$ and $\widetilde{Z} \geqslant Y$. Then $\tilde{z}_m \geqslant x_m \vee y_m$ for all m, so that $\tilde{z}_n = E_n \tilde{z}_m \geqslant E_n(x_m \vee y_m)$ for all $m \geqslant n$. As $\lim_m E_n(x_m \vee y_m) = z_n$, this yields $\tilde{z}_n \geqslant z_n$, so that $\widetilde{Z} \geqslant Z$. □

Lemma 6. *Let $X \in M$ such that the limit $\lim_m E_n|x_m|$ exists for each n, denote it z_n. Put $Z = (z_n)$, then Z is a martingale, $Z = |X|$, and $\|Z\| = \|X\|$.*

Proof. Apply Lemma 5 to X and $-X$, it follows that Z is indeed a bounded martingale and $Z = |X|$. Notice that

$$\|z_n\| = \lim_{m\to\infty} \bigl\|E_n|x_m|\bigr\| \leqslant \lim_{m\to\infty} \|x_m\| = \|X\|.$$

On the other hand, for $n \leqslant m$ we have $|x_n| = |E_n x_m| \leqslant E_n|x_m|$. It follows that

$$\bigl\|z_n\bigr\| = \lim_{m\to\infty} \bigl\|E_n|x_m|\bigr\| \geqslant \|x_n\|.$$

This yields $\|Z\| = \|X\|$. □

Theorem 7. *If F is a KB-space then $M = M\bigl(F, (E_n)\bigr)$ is a Banach lattice with lattice operations given by*

$$\begin{aligned} (X \vee Y)_n &= \lim_{m\to\infty} E_n(x_m \vee y_m) \\ (X \wedge Y)_n &= \lim_{m\to\infty} E_n(x_m \wedge y_m) \\ (X^+)_n &= \lim_{m\to\infty} E_n(x_m^+) \\ (X^-)_n &= \lim_{m\to\infty} E_n(x_m^-) \\ |X|_n &= \lim_{m\to\infty} E_n|x_m| \end{aligned} \tag{1}$$

Proof. Let $X = (x_n)$ and $Y = (y_n)$ be two bounded martingales in M. It follows from Lemma 5(i) that for every n the sequence $\bigl(E_n(x_m \vee y_m)\bigr)_{m=n}^{\infty}$ is increasing in m and norm bounded, hence it converges. It follows then by Lemma 5(ii) that $X \vee Y$ exists and is given by $(X \vee Y)_n = \lim\limits_{m\to\infty} E_n(x_m \vee y_m)$. The other formulae in (1) follow immediately. This proves that M is a lattice. Finally, $\bigl\||X|\bigr\| = \|X\|$ by Lemma 6, so that M is a Banach lattice. □

Corollary 8. *If F is an AL-space and (E_n) is a filtration on F, then $M = M\bigl(F, (E_n)\bigr)$ is an AL-space.*

Proof. Theorem 7 yeilds that M is a Banach lattice. Suppose that $X, Y \in M_+$, $X = (x_n)$, and $Y = (y_n)$. Then $x_n, y_n \in F_+$ for each n, so that $\|x_n + y_n\| = \|x_n\| + \|y_n\|$. It follows that

$$\|X + Y\| = \lim_{n\to\infty} \|x_n + y_n\| = \lim_{n\to\infty} \bigl(\|x_n\| + \|y_n\|\bigr) = \|X\| + \|Y\|.$$

□

In the case of classical martingales, it was shown by Krickeberg [Kr56] that the space M_1 of all L_1-bounded martingales in $L_1(P)$ is a Dedekind complete vector lattice. In particular, every L_1-bounded martingales can be written as a difference of two positive martingales. Corollary 8 yields the following refinement of Krickeberg's result.

Corollary 9. *The space M_1 of all L_1-bounded martingales in $L_1(P)$ is a Banach lattice and, moreover, an AL-space.*

Next, we show that if a filtration consists of finite rank operators, then the space of bounded martingales is a Banach lattice.

Lemma 10. *An increasing norm bounded sequence contained in a finite-dimensional subspace of a Banach lattice has a supremum and converges to it in norm.*

Proof. Suppose that (x_n) is an increasing sequence contained in the unit ball B_E of a finite-dimensional subspace E of a Banach lattice. By the continuity of lattice operations, $x_n \leqslant \lim_i x_{n_i}$ for every $n \geqslant 0$ and every convergent subsequence (x_{n_i}). Therefore, if x_{m_i} is another convergent subsequence then $\lim_i x_{m_i} \leqslant \lim_i x_{n_i}$. It follows that all convergent subsequences of (x_n) have the same limit. Compactness of B_E completes the proof. □

Proposition 11. *If F is a Banach lattice and (E_n) is a filtration on F such that E_n is of finite rank for each n, then $M = M\bigl(F, (E_n)\bigr)$ is a Banach lattice with lattice operations given by (1).*

Proof. Let $X = (x_n)$ and $Y = (y_n)$ be two martingales in M. It follows from Lemma 5(i) that for every n the sequence $\bigl(E_n(x_m \vee y_m)\bigr)_{m=n}^{\infty}$ is increasing in m. This sequence is contained in the range of E_n, so that it converges by Lemma 5(ii), and if we denote the limit by z_n then $Z = (z_n)$ is a martingale and $Z = X \vee Y$. This proves the first formula in (1). The other formulae follow easily, so that M is a lattice. Finally, Lemma 6 yields that M is a Banach lattice. □

Observe that the filtrations in Examples 1–3 consist of finite-rank projections. It follows that bounded martingales in those examples form Banach lattices.

6. Regular martingales

Given a martingale X in a Banach lattice F, we say that X is ***regular*** if there exists a positive martingale Y such that $X \leqslant Y$. It is easy to see that a martingale is regular if and only if it is a difference of two positive martingales. We denote by $M_r = M_r\bigl(F,(E_n)\bigr)$ the set of all regular bounded martingales. Clearly, M_r is a linear subspace of M. It follows from Theorem 7 that if F is a KB-space then for every bounded martingale X in F we have $X \leqslant |X|$, so that X is regular, hence $M_r = M$.

Example 12. *A non-regular martingale.* Let $F = L_\infty[0,1]$. For every n define E_n as follows:

$$E_n x(t) = \begin{cases} \text{the average of } x \text{ on } [0,2^{-n}] \text{ when } t \leqslant 2^{-n}; \\ x(t) \text{ otherwise.} \end{cases}$$

It is easy to see that (E_n) is a filtration. Put $D_n = [1/2^n, 1/2^{n-1}]$, let A_n and B_n be the left and the right halves of D_n respectively. Set $x_m = \sum_{k=1}^m 2^k(\chi_{A_k} - \chi_{B_k})$, then $X = (x_m)$ is an unbounded martingale. Show that there is no positive martingale that dominates X. Indeed, suppose that $Y = (y_n)$ is a positive martingale such that $X \leqslant Y$. Then $y_n \geqslant x_n^+$ for each n. It follows that $y_1 = E_1 y_n \geqslant E_1 x_n^+$. Notice that $x_n^+ = \sum_{k=1}^n 2^k \chi_{A_k}$, so that the restriction of x_n^+ to $[0,1/2]$ is $\sum_{k=2}^n 2^k \chi_{A_k}$, which has the average value of $n-1$ because the length of A_k is $1/2^{k+1}$. It follows that $E_1 x_n^+$ equals $n-1$ on $[0,1/2]$, so that $\|y_1\| \geqslant \|E_1 x_n^+\| \geqslant n-1$. It follows that $y_1 \notin L_\infty[0,1]$, contradiction.

Theorem 13. *If F has order continuous norm then $M_r = M_r\bigl(F,(E_n)\bigr)$ is a Banach lattice with lattice operations given by (1).*

Proof. Let $X = (x_n)$ and $Y = (y_n)$ be two martingales in M_r. Then there exist two positive martingales $\widetilde{X} = (\tilde{x}_n)$ and $\widetilde{Y} = (\tilde{y}_n)$ such that $X \leqslant \widetilde{X}$ and $Y \leqslant \widetilde{Y}$. It follows from Lemma 5(i) that for every n the sequence $\bigl(E_n(x_m \vee y_m)\bigr)_{m=n}^\infty$ is increasing in m. Moreover,

$$E_n(x_m \vee y_m) \leqslant E_n(\tilde{x}_m \vee \tilde{y}_m) \leqslant E_n(\tilde{x}_m + \tilde{y}_m) = \tilde{x}_n + \tilde{y}_n,$$

so that the sequence is order bounded. Since F has order continuous norm (and, in particular, is Dedekind complete), the sequence has a supremum, to which it converges in norm. Call the supremum z_n, let $Z = (z_n)$. By Lemma 5(i) $\|z_n\| \leqslant \|X\| + \|Y\|$. It follows from Lemma 5(ii) that Z is a bounded martingale (and it is regular because $Z \leqslant \widetilde{X} + \widetilde{Y}$) and that $Z = X \vee Y$. This proves the first formula in (1). The other formulae follow easily. In particular, M_r is a lattice. Lemma 6 yields $\bigl\||X|\bigr\| = \|X\|$ for every $X \in M_r$, so that the norm on M_r is a lattice norm.

It is left to show that M_r is a Banach space. It suffices to prove that M_r is a closed subspace of M. Suppose that $(X^{(n)})_{n=1}^\infty$ is a sequence of regular martingales such that $X^{(n)}$ converges in norm to some $X \in M$, show that X is regular. Without loss of generality, $\|X^{(n+1)} - X^{(n)}\| < 2^{-n}$ for every n. Since $X^{(n+1)} - X^{(n)}$ is regular, its modulus exists and

$$\big\||X^{(n+1)} - X^{(n)}|\big\| = \|X^{(n+1)} - X^{(n)}\| < 2^{-n},$$

so that the series $\sum_{n=1}^\infty |X^{(n+1)} - X^{(n)}|$ converges in M. Put $Y = \sum_{n=1}^\infty |X^{(n+1)} - X^{(n)}|$. Clearly, $Y \geqslant 0$. Now

$$X^{(m)} = X^{(1)} + \sum_{n=1}^{m-1} (X^{(n+1)} - X^{(n)}) \leqslant X^{(1)} + \sum_{n=1}^{m-1} |X^{(n+1)} - X^{(n)}| \leqslant |X^{(1)}| + Y.$$

It follows that $X \leqslant |X^{(1)}| + Y$, hence $X \in M_r$. $\square$

7. Bounded below filtrations

We say that a filtration (E_n) is ***bounded below on*** F_+ if there exists $n \geqslant 1$ and a constant $c > 0$ such that $\|E_n x\| \geqslant c\|x\|$ for every $x \geqslant 0$. In particular, if (E_n) preserves the norms of positive vectors, then it is bounded below on F_+. Observe that if (E_n) is bounded below then every positive (and, therefore, every regular) martingale is bounded. Indeed, $\|x_m\| \leqslant \frac{1}{c}\|E_n x_m\| = \frac{1}{c}\|x_n\|$ for every $m \geqslant n$, so that $\|X\| \leqslant \frac{1}{c}\|x_n\| < +\infty$. Notice also that if (E_n) is bounded below then E_n is strictly positive for some n, so that we can assume without loss of generality that the filtration is dense.

Proposition 14. *If F is a KB-space and (E_n) is bounded below on F_+, then $M = M\big(F,(E_n)\big)$ is again a KB-space.*

Proof. It follows from Theorem 7 that M is a Banach lattice. Consider an increasing sequence of martingales $(X^{(k)})_{k=1}^\infty$ with uniformly bounded norms $\|X^{(k)}\| \leqslant K$. Let $X^{(k)} = (x_n^{(k)})$. Then for each n the sequence $(x_n^{(k)})_{k=1}^\infty$ is increasing in k and is norm bounded by K. Since F is a KB-space, this sequence converges to some x_n and $\|x_n\| \leqslant K$. Let $X = (x_n)$, then for $n \leqslant m$ we have

$$E_n x_m = E_n\big(\lim_{k\to\infty} x_m^{(k)}\big) = \lim_{k\to\infty} E_n x_m^{(k)} = \lim_{k\to\infty} x_n^{(k)} = x_n,$$

so that $X \in M$. Clearly, $X = \sup_k X^{(k)}$, hence Proposition 15 implies that $\big\|X - X^{(k)}\big\| \to 0$. $\square$

Proposition 15. *If F has order continuous norm and (E_n) is bounded below on F_+ then the martingale norm is again order continuous.*

Proof. Suppose that $\big(X^{(\alpha)}\big)$ is a decreasing net of martingales, $X^{(\alpha)} = (x_n^{(\alpha)})$, such that $\inf_\alpha X^{(\alpha)} = 0$. For a fixed n, the net $(x_n^{(\alpha)})_\alpha$ is positive and decreasing in α, so that order continuity of norm in F yields $\lim_\alpha x_n^{(\alpha)}$ exists, call it x_n. It is easy to see that $x_n \geqslant 0$ and that $X = (x_n)$ is a martingale:

$$E_m x_n = E_m\big(\lim_\alpha x_n^{(\alpha)}\big) = \lim_\alpha E_m x_n^{(\alpha)} = \lim_\alpha x_m^{(\alpha)} = x_m.$$

Clearly, X is bounded and it follows from $0 \leqslant X \leqslant X^{(\alpha)}$ for all α and $\inf_\alpha X_{(\alpha)} = 0$ that $X = 0$. Thus, $\lim_\alpha x_n^{(\alpha)} = 0$ for each n.

There exists $n \geqslant 1$ and a constant $c > 0$ such that $\|E_n x\| \geqslant c\|x\|$ for every $x \geqslant 0$. For every $m \geqslant n$ we have $\|x_n^{(\alpha)}\| = \|E_n x_m^{(\alpha)}\| \geqslant c\|x_m^{(\alpha)}\|$, so that $\big\|X^{(\alpha)}\big\| \leqslant \frac{1}{c}\|x_n^{(\alpha)}\| \to 0$. ☐

8. A submartingale is dominated by a martingale

We claim that every bounded submartingale in a KB-space is dominated by a unique martingale (which justifies the term ***submartingale***). Suppose that $X = (x_n)$ is a bounded submartingale on a KB-space F. Applying Lemma 5(i) with $Y = X$ we see that for a fixed n the sequence $\big(E_n x_m\big)_{m=n}^\infty$ is increasing and norm bounded by $\|X\|$. Since F is a KB-space, it converges to some z_n, let $Z = (z_n)$. By Lemma 5(ii) Z is a martingale; it is the least martingale such that $X \leqslant Z$. Notice that

$$\|z_n\| = \lim_{m\to\infty} \|E_n x_m\| \leqslant \lim_{m\to\infty} \|x_m\| = \|X\|,$$

so that Z is bounded and $\|Z\| \leqslant \|X\|$. In general, $\|Z\|$ need not equal $\|X\|$. Even in the real-valued case, let X be a submartingale of constant functions $x_n = -\frac{1}{n}\chi_{[0,1]}$, then $\|X\| > 0$ but $Z = 0$. Nevertheless, if (E_n) preserves the norms of positive vectors and X is a positive submartingale then $\|z_n\| = \lim_m\|E_n x_m\| = \lim_m\|x_m\| = \|X\|$ for all n, so that $\|Z\| = \|X\|$. Finally, notice that the condition $\|X\| < \infty$ can be replaced by $\sup_n\|x_n^+\| < \infty$.

9. Convergent martingales

Let F be a Banach lattice and (E_n) a filtration on F, denote $M = M\big(F, (E_n)\big)$. One can easily verify that for every $x \in F$ the sequence $X = (E_n x)_{n=1}^\infty$ is a martingale. Martingales of this form will be called ***fixed***. Note that $\|X\| \leqslant \|x\|$, so that the map $x \mapsto (E_n x)$ is a linear contraction from F into M.

Observe that every convergent martingale is fixed (on its limit). Indeed, if $\lim_m x_m = x$ then $\lim_m E_n x_m = E_n x$, but $E_n x_m = x_n$ when $m \geqslant n$, so that $E_n x = x_n$.

Suppose that, in addition, the filtration is dense, that is, $E_n x \to x$ for every $x \in F$. Then a martingale is fixed iff it is convergent. In this case $\|X\| = \lim_n\|x_n\| =$

$\lim_n \|E_n x\| = \|x\|$, hence the map $x \mapsto (E_n x)$ is an isometry. Thus, F isometrically embeds in M. With a slight abuse of notation, we will consider F as a subspace of M.

Furthermore, suppose that F is a KB-space. Let (x_n) be an increasing sequence in F_+, order bounded in M. Then it is norm bounded and, since F is a KB-space, it has a supremum in F. Since M is a KB-space, hence Dedekind complete, it follows that F is a projection band in M:

Proposition 16. *If F is a KB-space and (E_n) is dense then F is a projection band in $M\big(F,(E_n)\big)$.*

In the case of classical martingales this yields that uniformly integrable martingales form a projection band in M_1. It would be interesting to describe the disjoint complement of $L_1(P)$ in M_1, and, more generally, the describe the disjoint complement of F in M under the hypotheses of Proposition 16.

On the other hand, Example 2 shows that if F is not KB, then it doesn't even have to be complemented in M.

10. Uniform integrability

Dunford-Pettis Theorem [DP40] asserts that a subset of $L_1(P)$ is uniformly integrable if and only if it is relatively weakly compact. From this point of view, the following theorem is a generalization of Doob Convergence Theorem to martingales in Banach lattices.

Theorem 17. *A relatively weakly compact martingale in a Banach lattice is fixed.*

Proof. Suppose that (x_n) is a relatively weakly compact martingale in a Banach lattice F. By Eberlein-Šmulian Theorem it has a subsequence (x_{n_k}) weakly convergent to some $x \in F$. Since E_m is bounded and hence weakly continuous for every m, we have $\text{w-}\lim_k E_m x_{n_k} = E_m x$, but $E_m x_{n_k} = x_m$ for all sufficiently large values of k, so that $x_m = E_m x$. □

Corollary 18. *If F is reflexive then every bounded martingale is fixed. In particular, if F is reflexive and (E_n) is dense then $M = F$.*

A subset A in a Banach lattice F is said to be ***order bounded*** if $A \subseteq [-x, x]$ for some $x \in F_+$, and ***almost order bounded*** if for every $\varepsilon > 0$ there exists $x \in F_+$ such that $A \subseteq [-x, x] + \varepsilon B_F$, where B_F stands for the unit ball of F. One can easily see that a subset of $L_1(P)$ is uniformly integrable if and only if it is almost order bounded. If F is a Banach lattice with an order continuous norm then it follows from Theorems 10.17

and 12.9 of [AB85] that every almost order bounded set in F is relatively weakly compact. This provides us with yet another generalization of Doob Convergence Theorem.

Corollary 19. *Every almost order bounded martingale in a Banach lattice with order continuous norm is fixed.*

Proposition 20. *A weakly convergent martingale is fixed.*

Proof. Suppose that (x_n) is a martingale such that $x_n \xrightarrow{w} x$. Then $\text{w-lim}_n E_m x_n = E_m x$ for all m, but $E_m x_n = x_m$ when $n \geqslant m$, so that $x_m = E_m x$. □

11. Dual filtration

Suppose that (E_n) is a filtration on a Banach lattice F. It is easy to see that the sequence of the adjoint operators (E_n^*) is a filtration on the dual Banach lattice F^*. Indeed, E_n^* is contractive and positive for every n, and $E_n^* E_m^* = (E_m E_n)^* = E_{n\wedge m}^*$ for all n and m. In particular, E_n^* is a projection.

Proposition 21. *If (E_n) is dense then $E_n^* f \xrightarrow{w^*} f$ and $\|E_n^* f\| \to \|f\|$ for every $f \in F^*$. In particular, the map $f \mapsto (E_n^* f)$ is an isometric embedding of F^* into $M\bigl(F^*, (E_n^*)\bigr)$.*

Proof. Suppose that (E_n) is dense. Then $E_n x \to x$ for each $x \in X$ so that $(E_n^* f)(x) = f(E_n x) \to f(x)$ for every $f \in F^*$, hence $E_n^* f \xrightarrow{w} f$. Show that $\|E_n^* f\| \to \|f\|$. Without loss of generality, $\|f\| = 1$. Clearly, $\|E_n^* f\| \leqslant 1$ for all n. There exists $x \in F$ such that $f(x) > 1 - \varepsilon$. Since $E_n x \to x$, there exists n such that $\|E_n x - x\| < \varepsilon$. It follows that $\bigl|f(E_n x) - f(x)\bigr| < \varepsilon$ so that

$$\bigl|E_n^* f(x) - 1\bigr| \leqslant \bigl|f(E_n x) - f(x)\bigr| + \bigl|f(x) - 1\bigr| < 2\varepsilon,$$

so that $\|E_n^* f\| > 1 - 2\varepsilon$. It follows that $\|E_n^* f\| \to 1$. □

We also have a dual version of this result.

Proposition 22. *If (E_n^*) is dense then $E_n x \xrightarrow{w} x$ and $\|E_n x\| \to \|x\|$ for every $x \in F$. In particular, the map $x \mapsto (E_n x)$ is an isometric embedding of F into $M\bigl(F, (E_n)\bigr)$.*

Proof. Suppose that (E_n^*) is dense. Then $E_n^* f \to f$ for every $f \in F^*$, so that $f(E_n x) = (E_n^* f)(x) \to f(x)$, for every $x \in F$, so that $E_n x \xrightarrow{w} x$. Fix $x \in F$ and consider it as an element of F^{**}, then it follows from Proposition 21 that $\|E_n x\| = \|E_n^{**} x\| \to \|x\|$. □

The following result is the dual version of Proposition 20.

Proposition 23. *Every weak*-convergent martingale in F^* relative to (E_n^*) is fixed.*

Proof. Suppose that (f_n) is a martingale in F^* relative to (E_n^*) such that $f_n \xrightarrow{w^*} f$. Since E_m^* is adjoint, hence weak*-continuous for each m, then $E_m^* f_n \xrightarrow{w^*} E_m^* f$. But $E_m^* f_n = f_m$ when $n \geqslant m$, so that $f_m = E_m^* f$ for each m. □

Theorem 24. *Given a Banach lattice F with filtration (E_n), denote $M = M\bigl(F,(E_n)\bigr)$ and $Y = \bigcup_{n=1}^{\infty} \operatorname{Range} E_n^*$. Then*

(i) *The sequence $(f(x_m))$ stabilizes for every martingale (x_m) and every $f \in Y$. Namely, if $f \in \operatorname{Range} E_n^*$ then $f(x_m) = f(x_n)$ for all $m \geqslant n$;*
(ii) *Every martingale in F is $\sigma(F,Y)$-Cauchy;*
(iii) *M embeds isometrically into Y^* via $X = (x_n) \mapsto \theta_X \in Y^*$ where $\theta_X(f) = f(x_n)$ for $f \in \operatorname{Range} E_n^*$;*
(iv) *If $X \in M$ then X is fixed as a martingale in F^{**}.*

Proof. To prove (i), consider a martingale $X = (x_m)$. Let $f \in \operatorname{Range} E_n^*$ for some n. Then $E_n^* f = f$, and for every $m \geqslant n$ we have $f(x_m) = (E_n^* f)(x_m) = f(E_n x_m) = f(x_n)$. It follows also that $\lim_m f(x_m)$ exists and equals $f(x_n)$. Thus, $f(x_m)$ converges for every $f \in Y$, this proves (ii).

Fix a bounded martingale $X = (x_n)$, for every $f \in \operatorname{Range} E_n^*$ define $\theta_X(f) = f(x_n) = \lim_m f(x_m)$. Then θ_X is defined for every $f \in Y$. Clearly, θ_X is linear. Since $|f(x_m)| \leqslant \|f\|\|x_m\| \leqslant \|f\|\|X\|$ for all m, it follows that $\|\theta_X\| \leqslant \|X\|$, so that $\theta_X \in Y^*$. Show that $\theta_X = \|X\|$, this will show that the map $X \mapsto \theta_X$ is an isometry and prove (iii). Fix n, By Hahn-Banach Theorem there exists $f \in F^*$ such that $\|f\| = 1$ and $f(x_n) = \|x_n\|$. Since $E_n^* f \in \operatorname{Range} E_n^*$ we have

$$\theta_X(E_n^* f) = E_n^* f(x_n) = f(E_n x_n) = f(x_n) = \|x_n\| \geqslant \|E_n^* f\|\|x_n\|$$

because $\|E_n^* f\| \leqslant 1$. Thus, $\|\theta_X\| \geqslant \|x_n\|$ for each n. It follows that $\|\theta_X\| = \|X\|$.

To prove (iv), extend θ_X to an element x^{**} of F^{**} by Hahn-Banach. Then for every $f \in F^*$ and every n we have

$$\langle E_n^{**} x^{**}, f\rangle = \langle x^{**}, E_n^* f\rangle = \theta_X(E_n^* f) = E_n^* f(x_n) = f(E_n x_n) = \langle f, x_n\rangle,$$

so that $E_n^{**} x^{**} = x_n$ for every n. □

Corollary 25. *If (E_n^*) is dense then $M = M\bigl(F,(E_n)\bigr)$ embeds isometrically into F^{**}. Namely, every martingale $X = (x_n)$ in M is w^*-convergent and the map $\Phi\colon X \mapsto$ $\text{w}^*\text{-}\lim_n x_n$ is an isometry from M into F^{**}. Also, $X \leqslant Y \Leftrightarrow \Phi(X) \leqslant \Phi(Y)$*

Proof. If (E_n^*) is dense than Y is dense in F^*, so that $Y^* = F^{**}$ and the first statement follows immediately from Theorem 24(iii). Given a bounded martingale $X = (x_n)$ in

F, let $\theta_X \in Y^*$ be as in Theorem 24(iii). Since Y is dense in F^*, we extend θ_X by continuity to all of F^*, so that we can write $\theta_X \in F^{**}$. Show that $x_n \xrightarrow{w^*} \theta_X$. Given $f \in F^*$, it follows from Theorem 24(i) that $\theta_X(E_m^* f) = (E_m^* f)(x_m) = f(E_m x_m) = f(x_m)$. Therefore,

$$\bigl|\langle x_m, f\rangle - \langle \theta_X, f\rangle\bigr| = \bigl|\langle \theta_X, E_m^* f\rangle - \langle \theta_X, f\rangle\bigr| \leqslant \|\theta_X\|\|E_m^* f - f\| \to 0.$$

□

Example 26. If (E_n) is a filtration given by a 1-unconditional basis as in Example 1, then (e_i) is shrinking iff (E_n^*) is dense, in this case $M\bigl(F,(E_n)\bigr)$ embeds isometrically in F^{**}. Notice that in Example 2 we actually have $M = F^{**}$.

The following can be viewd as a complement to Proposition 16.

Corollary 27. *If F is a KB-space and (E_n^*) is dense, then $M\bigl(F,(E_n)\bigr)$ is lattice isometric to F.*

Proof. Let Φ be as in Corollary 25. Since F is weakly sequentially complete, then $\Phi(X) = \mathrm{w}^*\text{-}\lim_n x_n$ belongs to F for every bounded martingale $X = (x_n)$. □

12. Martingales of martingales

Given a Banach lattice F and a filtration (E_n), let $M = M\bigl(F,(E_n)\bigr)$. One can view M as an enlargement or a completion of F. This enlargement operation is idempotent in the following sense. For every $k \leqslant 1$ consider the map $\mathcal{E}_k$ on M given as follows: if $X = (x_n)$ is an element of M then $\mathcal{E}_k X = Y$, where

$$y_n = \begin{cases} x_n & n \leqslant k, \\ x_k & n \geqslant k. \end{cases}$$

Clearly, $Y \in M$ and $\|Y\| \leqslant \|X\|$, so that $\mathcal{E}_k$ is a positive contractive projection. One can easily check that $\mathcal{E}_k\mathcal{E}_m = \mathcal{E}_{k\wedge m}$. Therefore, $(\mathcal{E}_k)$ is a filtration on M. Notice that if (E_n) preserves the norms of positive vectors, then so does $(\mathcal{E}_k)$. Indeed, if $X = (x_n) \in M_+$ then $\|X\| = \|x_n\|$ for all n. Let $Y = \mathcal{E}_k X$, then $Y = (y_n)$ is again a positive martingale, and $\|Y\| = \|y_1\| = \|x_1\| = \|X\|$.

We claim that $M\big(M,(\mathcal{E}_k)\big)=M$ up to the canonical isometry. Indeed, let $\big(X^{(k)}\big)$ be a martingale in $M\big(M,(\mathcal{E}_k)\big)$. It is easy to see that it has to be of the form

$$\begin{aligned} X^{(1)} &= (x_1,x_1,x_1,x_1,\dots) \\ X^{(2)} &= (x_1,x_2,x_2,x_2,\dots) \\ X^{(3)} &= (x_1,x_2,x_3,x_3,\dots) \\ &\vdots \end{aligned}$$

for some sequence $X=(x_n)$. Then for $m\leqslant n$ we have $E_m x_n = E_m X^{(n)}_n = X^{(n)}_m = x_m$, so that X is a martingale. Furthermore,

$$\|X\| = \sup_n \|x_n\| = \sup_k \|X^{(k)}\| = \big\|\big(X^{(k)}\big)\big\| < +\infty,$$

so that $X\in M$. Finally, $\mathcal{E}_k X = X^{(k)}$, so that the martingale $\big(X^{(k)}\big)$ is fixed on X, and, clearly, X is the only element of M with this property. It follows that the map $X\in M\mapsto(\mathcal{E}_k X)$ is a surjective isometry between M and $M\big(M,(\mathcal{E}_k)\big)$.

13. Martingale difference sequences

Given a filtration (E_n), if $m<n$, then it is can be easily verified that $E_n - E_m$ is a projection. Define the sequence (P_n) of ***difference projections*** via $P_1=E_1$ and $P_n = E_n - E_{n-1}$ for $n>1$. Then $E_n=\sum_{k=1}^n P_k$ for each n. Observe that $P_nP_m=0$ whenever $n\neq m$. Indeed,

$$P_nP_m = (E_n-E_{n-1})(E_m-E_{m-1}) = E_nE_m - E_nE_{m-1} - E_{n-1}E_m + E_{n-1}E_{m-1}$$

so that $P_nP_m = E_n - E_n - E_{n-1} + E_{n-1} = 0$ when $n<m$ and $P_nP_m = E_m - E_{m-1} - E_m + E_{m-1} = 0$ when $m<n$. A sequence (d_n) in F is called a ***martingale difference sequence*** if $d_n\in\operatorname{Range}P_n$ for each n. Note that P_nd_k equals d_k if $n=k$ and zero otherwise. If we put $x_n=\sum_{k=1}^n d_k$ then one can easily verify that (x_n) is a martingale. Conversely, if (x_n) is a martingale, then the sequence (d_n) defined by $d_1=x_1$ and $d_n=x_n-x_{n-1}$ for $n>1$ is a martingale difference sequence. Observe also that if (E_n) is dense then for every $x\in F$ we have $\sum_{k=1}^n P_kx = E_nx\to x$, so that $x=\sum_{k=1}^\infty P_kx$, hence $F=\oplus_{k=1}^\infty\operatorname{Range}P_k$. Notice that if the ranges of P_k form a finite dimensional decomposition of F, then Proposition 11 guarantees that $M\big(F,(E_n)\big)$ is a Banach lattice.

Suppose that (d_k) is a martingale difference sequence. Fix a sequence of scalars (α_k), and put $x_n=\sum_{k=1}^n\alpha_kd_k$. Clearly, $X=(x_k)$ is a martingale, so that

$$\Big\|\sum_{k=1}^n \alpha_kd_k\Big\| = \|x_n\| \leqslant \|x_{n+1}\| = \Big\|\sum_{k=1}^{n+1}\alpha_kd_k\Big\|.$$

It follows that every martingale difference sequence is a monotone basic sequence in F.

14. Operators on martingales

Let F be a Banach lattice, and (E_n) a filtration on F. Let $T \in L(F)$, a bounded linear operator from F to F, such that the norm limit

$$\lim_{n\to\infty} E_m T x_n \tag{2}$$

exists for every bounded martingale $X = (x_n)$ and every $m \geqslant 1$. Denote this limit by y_m and put $Y = (y_m)$. Then Y is a martingale: if $k \leqslant m$ then

$$E_k y_m = \lim_{n\to\infty} E_k E_m T x_n = \lim_{n\to\infty} E_k T x_n = y_k.$$

Notice that $\|y_m\| \leqslant \|T\|\|X\|$ for every m, so that Y is bounded and $\|Y\| \leqslant \|T\|\|X\|$. It follows that T induces an operator in $L(M)$, denote it by $\widehat{T}$. Clearly, $\|\widehat{T}\| \leqslant T$.

If, in addition, (E_n) is dense, then we know that F can be identified with the subspace of M consisting of the convergent martingales of the form $(E_n x)$. In this case, $\widehat{T}_{|F} = T$, so that $\|\widehat{T}\| = T$

Before we discuss for which operators in $L(F)$ the limit in (2) exists and, therefore, $\widehat{T}$ is defined, observe that an extension of an opeator from $L(F)$ to $L(M)$ need not be unique. Indeed, when F is a KB-space and (E_n) is dense then $M = F \oplus F^d$ by Proposition 16. If F^d is non-trivial, one can define the extension arbitrarily on F^d.

Proposition 28. *If T commutes with E_n for all n then $\widehat{T}$ exists.*

Proof. Let $X = (x_n)$ be a bounded martingale. If $m \leqslant n$, then $E_m T x_n = T E_m x_n = T x_m$, so that $(T x_n)$ is a martingale. Put $y_n = T x_n$ and $Y = (y_n)$, then, in particular, $y_m = \lim_n E_m T x_n$ for every m, so that $Y = \widehat{T} X$. □

Proposition 29. *If F is a KB-space and either*

(i) *$T E_n \geqslant E_n T$ for all n, or*
(ii) *$T E_n \leqslant E_n T$ for all n,*

then $\widehat{T}$ exists.

Proof. Let $X = (x_n)$ be a bounded martingale in F. It suffices to show that the limit in (2) exists for all m. By Theorem 7, X is regular, so that we can assume without loss of generality that $X \geqslant 0$. Notice also that $\|E_m T x_n\| \leqslant \|T\|\|X\|$, so that the sequence $(E_m T x_n)_{n=1}^{\infty}$ is norm bounded. If $m \leqslant n$ then in case (i) we have

$$E_m T x_n = E_m T E_n x_{n+1} \geqslant E_m E_n T x_{n+1} = E_m T x_{n+1},$$

while in case (ii) we have

$$E_mTx_n = E_mTE_nx_{n+1} \leqslant E_mE_nTx_{n+1} = E_mTx_{n+1},$$

In either case the sequence $(E_mTx_n)_{n=1}^{\infty}$ is monotone. Since F is a KB-space, the sequence converges. □

15. Notes, remarks, and questions

The results of the present paper can be applied to vector-valued martingales based on vector measures. A beautiful exposition of vector-valued martingales can be found in [DU77]. In particular, there are deep results characterizing certain properties of a Banach space X via properties of X-valued martingales in $L_p(\Omega; X)$. Notice, that if F is a Banach lattice, then $L_p(\Omega; F)$ is again a Banach lattice, so that an F-valued martingale in the sense of [DU77] can be viewed as a martingale in $L_p(\Omega; F)$ in the sense of the present paper. Note that $L_p(\Omega; F)$ inherits many Banach lattice properties of F. In particular, one can easily verify that $L_p(\Omega; F)$ has order continuous norm iff F has order continuous norm, and $L_p(\Omega; F)$ is a KB-space iff F is a KB-space. It should be noted, however, that a vector-valued martingale which is uniformly integrable with respect to a vector measure in the sense of [DU77] need not be relatively weakly compact, so that Theorem 17 might not apply. Many interesting results about spaces $L_p(\Omega; F)$ of Banach lattice valued martingales can be found in [SW76, Szul78, Szul79].

R. DeMarr in [DeM66] introduced a martingale in a vector lattice F as double sequences (x_n, E_n) where x_n is an element of F, E_n is a positive linear projection, $E_nE_m = E_{n\wedge m}$ and $E_nx_m = x_n$ whenever $n \leqslant m$. DeMarr then generalized the *almost everywhere* part of Doob's Convergence Theorem. Namely, he proved that under certain special conditions, a martingale in a vector lattice is order convergent.

Dor and Odell in [DO75] studied sequences of projections on $L_p(\mu)$ satisfying $E_nE_m = E_{n\wedge m}$ and showed that such a sequence can often be reduced to a classical filtration.

The theory of non-commutative martingales is currently an active and promising subject, see [Um54, PX97, Jun02, Ran02].

We proved in this present paper that in several important special cases $M\big(F, (E_n)\big)$ and $M_r\big(F, (E_n)\big)$ are Banach lattices. However, we don't have an example of a bounded non-regular martingale. Neither do we know an example where $M\big(F, (E_n)\big)$ is not a Banach lattice.

Let F be a Banach lattice with order continuous norm and a weak order unit. It is known (see, e.g., [LT79, Theorem 1.b.14]) that F is order isometric to a norm dense ideal of $L_1(P)$ for some probability space $(\Omega, \mathcal{F}, P)$. It would be interesting to know if this representation could be chosen in such a way that a sequence in F is a martingale if and only if its image in $L_1(P)$ is a martingale in the classical sense?

In the classical case, given a martingale $X = (x_n)$ in $L_1(P)$, one can recover a filtration with respect to which X is a martingale. Namely, one can take $\mathcal{F}_n$ to be the smallest σ-algebra that makes $x_1, \dots, x_n$ measurable. In this approach a martingale is a primary object, while the filtration is recovered from the martingale. It is interesting if there is a similar procedure in the Banach lattice case. That is, given a sequence (x_n) of elements of a Banach lattice is a martingale, when can one find a filtration (E_n) with respect to which (x_n) is a martingale?

There are several important martingale inequalities in the classical martingale theory involving maximal function, martingale transform, subordinate martingales, and escape numbers, see [BDG72, Bur81, Bur84, Bur01]. It would be interesting to find analogues of these inequalities in Banach lattice setting.

In the present paper we considered spaces of martingales relative to a fixed filtration. It would seem natural to consider the set of all martingales corresponding to all filtrations on a given Banach lattice and to try to establish some order structure on this set. However, there are certain difficulties on the way, even in the classical setting. For example, let $X = (x_n)$ be the double-or-nothing martingale on the unit interval, and let $Y = (y_n)$ where $y_n = x_{n+1}$, so that Y is just a "shift" of X. Then X and Y are not only distinct, but even non-comparable in M_1. We don't see any reasonable way to define the $X \vee Y$.

In this paper we only dealt with discrete martingales, ordered by positive integers. Most of the results can be easily generalized to continuous martingales.

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