# INFINITELY FINE PARTITIONS OF MEASURE SPACES

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## 1. Infinitely fine partitions

Throughout this paper $(\mathcal{X}, \mathcal{A})$ always stands for a measurable space, i.e., $\mathcal{X}$ is an arbitrary set and $\mathcal{A}$ is an algebra of its subsets.

By ${}^*\mathcal{X}$ we denote the nonstandard extension of $\mathcal{X}$. We assume that the nonstandard model is $\kappa$-saturated where $\kappa$ is the cardinality of $\mathcal{A}$.

1.1. **Definition.** A measurable partition $P$ of ${}^*\mathcal{X}$ is said to be an `infinitely fine partition` (abbreviated `ifp`) if ${}^*A = \bigcup\{p \in P \mid p \subseteq {}^*A\}$ for every measurable set $A$, or, equivalently, if $P$ is finer than every standard partition.

It follows from the saturation principle that every measurable space (and even a Boolean algebra) has a hyperfinite ifp. From now on the symbol $P$ stands for a fixed hyperfinite ifp of the measurable space $(\mathcal{X}, \mathcal{A})$. Infinitely fine partitions were originally introduced in [Loe72]. Some properties of ifp's were investigated in [Tro93].

Notice that if $\mathcal{A}$ contains all the singletons, then every standard singleton belongs to $P$.

1.2. **Definition.** We say that $p_1, p_2 \in P$ are `equivalent` (denoted $p_1 \sim p_2$) if $P \setminus \{p_1, p_2\} \cup \{p_1 \cup p_2\}$ is again an ifp.

1.3. **Theorem.** *An element of $P$ is not equivalent to any other element of $P$ if and only if it is standard.*

*Proof.* Obviously every standard elements of $P$ is not equivalent to another element of $P$. Suppose that $p \in P$ is non-standard. Consider



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the collection $q = \{A \in \mathcal{A} \mid p \subseteq {}^*A\}$. Let $\mathcal{S}$ be a finite subset of $q$ and put $S = \bigcap \mathcal{S}$, then $S \in \mathcal{A}$. Evidently $p \subseteq {}^*S$, but since $p$ is nonstandard, it follows that $p \neq {}^*S$. Therefore, there exists $p' \in P$ such that $p' \subset {}^*S$ and $p' \neq p$. Since $\mathcal{S}$ was chosen as an arbitrary finite sub-collection of $q$, by the saturation principle there exists an element $p_1 \in P$ distinct from $p$ contained in the extension of every set belonging to $q$, so that $p \sim p_1$. □

It was shown in [Loe72] that a standard bounded measurable function is approximately constant on every element of an ifp.

1.4. **Lemma.** *Let $p_1, p_2 \in P$, then $p_1 \sim p_2$ if and only if every standard bounded measurable function $f \colon X \to \mathbb{R}$ takes approximately equal values on $p_1$ and $p_2$.*

*Proof.* The implication from left to right follows from the fact that $P \setminus \{p_1, p_2\} \cup \{p_1 \cup p_2\}$ is an ifp. The converse can be easily obtained by considering characteristic functions of standard measurable sets. □

## 2. Nonatomicity

In this section we present some interpretations of nonatomicity in terms of infinitely fine partitions.

Suppose now that $(\mathcal{X}, \mathcal{A})$ is endowed with a standard finitely additive measure $\mu$. We introduce the following notations:

$$\begin{array}{rclcrcl} P_+ & = & \{p \in P \mid {}^*\mu(p) > 0\} & \qquad & P_0 & = & \{p \in P \mid {}^*\mu(p) = 0\} \\ \mathcal{A}_+ & = & \{A \in \mathcal{A} \mid \mu(A) > 0\} & \qquad & \mathcal{A}_0 & = & \{A \in \mathcal{A} \mid \mu(A) = 0\} \end{array}$$

The elements of $P_+$ will be referred to as the `essential`elements of $P$. Notice that every points of an essential element of $P$ is random (recall that a point $x \in {}^*\mathcal{X}$ is said to be `random` if $x \notin {}^*N$ for each $N \in \mathcal{A}_0$).

2.1. **Definition.** Let $p$ be an essential element of $P$. We say that $p$ is `essentially joinable` if $p$ is equivalent to another essential element of $P$. We call $p$ `essentially divisible` if it can be written as a union of two disjoint sets of positive measure.

Recall that a set $F \in \mathcal{A}_+$ is said to be an atom of $\mu$ if for every measurable set $E \subseteq F$ we have either $\mu(E) = 0$ or $\mu(E) = \mu(F)$. A measure is said to be nonatomic if it has no atoms. A measure is said to be strongly continuous if for every $\varepsilon > 0$ there exists a finite measurable partition of $\mathcal{X}$ into sets of measure less than $\varepsilon$. A measure $\mu$ is said to be strongly nonatomic if for every set $F \in \mathcal{A}_+$ and every $0 < c < \mu(F)$ there exists a measurable set $E \subset F$ such that $\mu(E) = c$.

It was shown in [BRBR83] that strong nonatomicity implies strong continuity, strong continuity implies nonatomicity, and all the three properties are equivalent for sigma-additive measures.

2.2. **Theorem.** *The following statements are equivalent:*

(1) *every essential element of $P$ is essentially divisible;*
(2) *every essential element of $P$ is essentially joinable;*
(3) *$\mu$ is nonatomic.*

*Proof.* To prove the implication (1)⇒(2), consider an essential element $p \in P$. By the assumption, it is essentially divisible. Let $q = \{A \in \mathcal{A} \mid p \subset {}^*A\}$, and consider a finite sub-collection $\mathcal{S}$ of $q$. Let $S = \bigcap \mathcal{S}$. Evidently, $S$ is a standard set and $p \subset {}^*S$. Essential divisibility of $p$ implies that there exists $p_1 \in {}^*A_+$ such that $0 < {}^*\mu(p_1) < {}^*\mu(p) \leqslant \mu(S)$. By the transfer principle there exists a standard set $B \subset S$ such that the both $B$ and $S \setminus B$ are of positive measure. Assume without loss of generality that $p_1 \subset {}^*B$. Obviously there is an essential element $p_2 \subset {}^*S \setminus {}^*B \subset {}^*S$. Since $\mathcal{S}$ was chosen arbitrarily, it follows by saturation that there exists $p' \in P_+$ distinct from $p$, such that $p' \subseteq {}^*A$ for each $A \in q$, and it follows that $p' \sim p$.

To prove (2)⇒(3) take an arbitrary set $A \in \mathcal{A}_+$. There exists $p \in P_+$ such that $p \subseteq {}^*A$. Since $p$ is essentially joinable, then there is $p' \in P_+$ such that $p' \subset {}^*A \setminus p$ and it follows that ${}^*\mu(p) < \mu(A)$. By using the transfer principle we obtain the conclusion.

The implication (3)⇒(1) is straightforward. □

2.3. **Theorem.** *A measure is strongly continuous if and only if the measure of every element of* $P$ *is infinitesimal.*

*Proof.* The proof is elementary. $\square$

2.4. **Theorem.** *A measure* $\mu$ *is strongly nonatomic if and only if* $(\forall p \in P_+)(\forall \lambda \in {}^*[0,1])(\exists p' \in {}^*\mathcal{A}_+)(p' \subset p \ \& \ \frac{{}^*\mu(p')}{{}^*\mu(p)} = \lambda)$.

*Proof.* The implication from left to right is obvious. To prove the converse implication, fix $F \in \mathcal{A}_+$ and $c \in (0, \mu(F))$. Consider an internal measurable subset $E$ of ${}^*F$ such that

(1) $E = \bigcup\{p \in P \mid p \subset E\}$;
(2) ${}^*\mu(E) \leqslant c$;
(3) $p \nsubseteq E$ implies ${}^*\mu(E \cup p) > c$ for each $p \in P_+$.

Such a set exists because ${}^*F$ is a union of a hyperfinite number of elements of $P$. We claim that ${}^*\mu(E) = c$. Indeed, if ${}^*\mu(E) < c$, then we could take any $p \subseteq {}^*F \setminus E$ and split it into the disjoint union of $p_1$ and $p_2$ so that ${}^*\mu(E \cup p_1) = c$, which would contradict to the assumptions on $E$. The transfer principle completes the proof. $\square$

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## 3. Representation of $L_\infty$

In the following section we extend a results obtained by P. Loeb in [Loe72]. For this subsection, let $\mu$ be a standard sigma-additive measure, $P$ a hyperfinite ifp for a standard measurable space $(\mathcal{X}, \mathcal{A})$, and $L_\infty$ stands for $L_\infty(\mathcal{X}, \mathcal{A}, \mu)$. In [Loe72] Loeb introduced a map $T_0 \colon L_\infty \to \mathbb{R}^P$ by the following rule. For every $p \in P_+$ pick a point $c_p \in p$. For $f \in L_\infty$ define

$$(T_0(f))_p = \begin{cases} {}^\circ f(c_p) & \text{if } p \in P_+; \\ 0 & \text{if } p \in P_0. \end{cases}$$

Loeb showed in [Loe72] that the variation of a standard bounded measurable function on each element of an ifp is infinitesimal, so that $T_0$ does not depend on the choise of $c_p$'s. Loeb proved that a vector

$\mathbf{v} \in \mathbb{R}^P$ equals $T_0(f)$ for some $f \in L_\infty$ if and only if the following three conditions hold:

(1) $\mathbf{v}_p = 0$ for all $p \in P_0$;
(2) $\max\limits_{p\in P}|\mathbf{v}_p|$ is nearstandard;
(3) for each $p$ in $P_+$ and for each $\varepsilon \in \mathbb{R}_+$ there exists a set $A \in \mathcal{A}$ such that $p \subseteq {}^*A$ and $|\mathbf{v}_p - \mathbf{v}_{p'}| < \varepsilon$ for every essential $p' \subseteq {}^*A$.

3.1. **Lemma.** *Condition* (3) *above is equivalent to the following statement:*

(3') $p_1 \sim p_2$ *implies* $\mathbf{v}_{p_1} \approx \mathbf{v}_{p_2}$ *for any* $p_1, p_2 \in P_+$.

*Proof.* Suppose that $p_1 \sim p_2$ for $p_1, p_2 \in P_+$. Then it follows from (3) that $|\mathbf{v}_{p_1} - \mathbf{v}_{p_2}| < \varepsilon$ for every $\varepsilon \in \mathbb{R}_+$, hence $\mathbf{v}_{p_1} \approx \mathbf{v}_{p_2}$, so that (3) implies (3'). To show the reverse implication, assume that there exists $p \in P_+$ and $\varepsilon \in \mathbb{R}_+$ such that for each $A \in \mathcal{A}$ satisfying $p \subseteq {}^*A$ one can find $p' \in P_+$ such that $p' \subseteq {}^*A$ and $|\mathbf{v}_p - \mathbf{v}_{p'}| > \varepsilon$. Let $q = \{A \in \mathcal{A} \mid p \subseteq {}^*A\}$ and consider a finite sub-collection $\mathcal{S}$ of $q$. Let $S = \bigcap \mathcal{S}$. By the assumption, there exists an essential $p' \subseteq {}^*S$ such that $|\mathbf{v}_p - \mathbf{v}_{p'}| \geqslant \varepsilon$. By the saturation principle there exists an element $p' \in P_+$, such that $p' \subseteq {}^*A$ and $|\mathbf{v}_p - \mathbf{v}_{p'}| \geqslant \varepsilon$, but this contradicts to (3'). □

## 4. Monads of ifp

Again, let $P$ be an ifp for a standard measurable space $(\mathcal{X}, \mathcal{A})$. For $p \in P$ we denote by $[p]$ the equivalence class of $p$, i.e., $[p] = \{p' \in P \mid p' \sim p\}$. The union of the elements of $[p]$ will be denoted by $m_p$ and referred to as a monad of $P$. Since "$\sim$" is an equivalence relation, $\mathfrak{M} = \{m_p\}_{p\in P}$ is an partition of ${}^*\mathcal{X}$. It is easy to see that $\mathfrak{M}$ is exactly the partition of $\mathcal{X}$ generated by all standard measurable sets. It implies, in particular, that the monads are independent of a particular ifp.

Let $Q$ be the Stone space of the algebra $\mathcal{A}$. Recall that the points of $Q$ are the ultra-filters on $\mathcal{A}$. It is well known that there exists a canonical boolean algebra isomorphism $\iota$ between $\mathcal{A}$ and the algebra $\operatorname{Clop} Q$ of all clopen subsets of $Q$ given by $\iota(A) = \{q \in Q \mid A \in q\}$.

Notice that there is a one-to-one correspondences between the monads of $P$, the ultra-filters on $\mathcal{A}$, the zero-one measures on $\mathcal{A}$, and the points of $Q$. Consider a monad $m$ of $P$. The corresponding ultra-filter $q_m$ is $\{A \in \mathcal{A} \mid m \subseteq {}^*A\}$, i.e., the elements of $q_m$ are exactly the standard measurable sets containing the monad $m$. On the other hand, since $m = \bigcup[p]$ where $[p]$ is exactly the equivalence class of those elements of $P$ which are contained in each $A$ in $q_m$, we have $m = \bigcap q_m$.

Any monad $m$ corresponds to some zero-one measure on $\mathcal{A}$ defined by

$$\delta_m(A) = \begin{cases} 1 & \text{if } m \subseteq {}^*A, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $q_m = \{A \in \mathcal{A} \mid \delta_m(A) = 1\}$. Finally, for any monad $m$ we can consider the ultra-filter $q_m$ as a point of the Stone space $Q$ of $\mathcal{A}$. Notice that the set $\bigcup\{\iota(p) \mid p \in P, p \subseteq m\}$ is the usual topological monad of $q_m$ in $Q$. Notice also, that $\{q_m \mid m \subset {}^*A\} = \iota(A)$ for any $A$ in $\mathcal{A}$.

It follows from [Loe72, Proposition 4.2] that there is a one-to-one correspondence between the monads of $P$ and the multiplicative linear functionals on $L_\infty(X)$. Namely, if $m$ is a monad of $P$, then the map $f \mapsto {}^\circ({}^*f)_{|m}$ is the multiplicative linear functional corresponding to $m$.

Consider a monad $m$ of $P$. Since $q_m$ is a standard ultra-filter on $\mathcal{A}$, it follows that ${}^*q_m$ contains exactly one element of $P$. This element will be referred as the `central`element of $m$, we will denote it by $p_m$. Obviously $p_m \subseteq m$. We mentioned before that $\{\iota(p) \mid p \in P, p \subseteq m\}$ form a partition of the topological monad of the (standard) point $q_m$ in $Q$. Then $p_m$ can be characterized as the element of $P$ for which $q_m \in {}^*\iota(p_m)$. Finally, central elements can also be characterized in terms of zero-one measures. Namely, $p = p_m$ if and only if ${}^*\delta_m(p) = 1$. Denote by $P_c$ the collection of all central elements of $P$.

It will be shown that properties of measures are essentially determined by their values at the central elements. For the rest of this section we suppose that $\mu$ is a standard finitely additive measure on $\mathcal{A}$. We assume for simplicity that $\mu$ is finite.

4.1. **Theorem.** *If $p$ is a central element with ${}^*\mu(p)$ standard, then*

(1) ${}^*\mu(p') = 0$ *for any noncentral $p'$ equivalent to $p$;*
(2) *There exists a set $A \in \mathcal{A}$ such that $p \subseteq {}^*A$ and $\mu(A) = {}^*\mu(p)$.*

*Proof.* It follows from $p \in P_c$ that $p = p_m$ for some monad $m$ of $P$. Since $p \in {}^*q_m$ and ${}^*\mu(p)$ is standard, then by the transfer principle we can find a standard set $A \in q_m$ such that $\mu(A) = {}^*\mu(p)$. This proves (2). Finally, (1) follows immediately from (2). □

4.2. **Corollary.** *Suppose that $p$ is a central element of measure zero. If $p' \sim p$ then $p'$ is of measure zero.*

4.3. **Theorem.** *If $p$ is a central element of $P$, then for every standard $\varepsilon > 0$ there exists a standard set $A \in \mathcal{A}$ such that $m_p \subseteq {}^*A$ and $\mu(A) \leqslant {}^*\mu(p_m) + \varepsilon$.*

*Proof.* Let $\lambda = {}^*\mu(p) + \frac{\varepsilon}{2}$, then ${}^*\mu(p) < {}^\circ\lambda$. Since $p \in {}^*q_m$ then, by the transfer principle, there exists a standard set $A \in q_m$ such that $\mu(A) < {}^\circ\lambda < {}^*\mu(p_m) + \varepsilon$. □

4.4. **Corollary.** *The measure of every noncentral element is infinitesimal. Moreover, the union of any collection of non-central elements of $m$ has infinitesimal measure. Furthermore, if $m$ is a monad of $P$ and $D \subset m$ for some $D \in {}^*\mathcal{A}$ such that $q_m \notin {}^*\iota(D)$, then ${}^*\mu(D) \approx 0$.*

4.5. **Remark.** There is another important consequence of Theorem 4.3. We have noticed before that the monads do not depend on a particular ifp, but only on the algebra. Now it follows immediately from Theorem 4.3 that if $m$ is a monad, then ${}^\circ\big({}^*\mu(p_m)\big)$ does not depend on a particular ifp either.

4.6. **Lemma.** *If $p$ is a central element of $P$ then ${}^*\mu(p) \geqslant {}^\circ\big({}^*\mu(p)\big)$.*

*Proof.* For every standard $A \in q_{m_p}$ we have $p \subset A$, and, therefore, $^\circ\big({}^*\mu(p)\big) \leqslant \mu(A)$. By the transfer principle it follows that $^\circ\big({}^*\mu(p)\big) \leqslant {}^*\mu(p)$. $\square$

## 5. Sobczik-Hammer Decomposition Theorem

The technique of ifp gives us an opportunity to present a simple proof of the Sobczik-Hammer Decomposition Theorem.

5.1. **Theorem** (Sobczik-Hammer Decomposition Theorem)**.** *Let $\mu$ be a finite finitely additive measure on a measurable space $(\mathcal{X},\mathcal{A})$. Then there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ of distinct zero-one measures on $\mathcal{A}$, a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers, and a strongly continuous measure $\bar{\mu}$ on $\mathcal{A}$, such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\mu = \bar{\mu} + \sum_{n=1}^{\infty} a_n\delta_n$. Further, this decomposition is unique up to the order of terms.*

*Proof.* Let $P$ be a hyperfinite ifp for $(\mathcal{X},\mathcal{A})$. Let $p_1$ be an element of $P$ of maximum measure, put $a_1 = {}^\circ\big(({}^*\mu(p_1))\big)$. For $\delta_1$ we take the zero-one measure corresponding to $m_{p_1}$. If $\mu$ is strongly continuous, then $a_1 = 0$ by Theorem 2.3. Otherwise, it follows from Corollary 4.4 that $p_1$ is central. Then Lemma 4.6 implies that $a_1 \leqslant {}^*\mu(p_1)$. It follows that $\mu_1 = \mu - a_1\delta_1$ is a standard measure which is nonnegative on each element of $P$, hence $\mu_1$ is nonnegative. We proceed in the similar fashion defining $\mu_{n+1} = \mu_n - a_n\delta_n$, etc. This process may stop after a finite number of steps if $a_n = 0$ for some $n$. In this case we get $\mu = \sum_{i=1}^{n} a_i\delta_i + \mu_n$ where $\mu_n$ is a strongly continuous measure, so that the conclusion of the theorem is satisfied. Otherwise, we and obtain a decreasing sequence $(a_n)_{n\in\mathbb{N}}$ of standard non-negative reals, a sequence $(\delta_n)_{n\in\mathbb{N}}$ of standard distinct zero-one measures, and a sequence $(\mu_n)_{n\in\mathbb{N}}$ of standard non-negative measures, such that $\mu_n = \mu - \sum_{i=1}^{n} a_i\delta_i$ for every $n \in \mathbb{N}$. In particular, $\mu(\mathcal{X}) \geqslant \sum_{i=1}^{n} a_i\delta_i(\mathcal{X}) = \sum_{i=1}^{n} a_i$ for every natural $n \in \mathbb{N}$; it follows that $\sum_{i=1}^{\infty} a_i < \infty$.

It can be easily verified that $\bar{\mu}$ defined by $\bar{\mu} = \mu - \sum_{n=1}^{\infty} a_n\delta_n$ is a standard nonnegative measure, and $\bar{\mu} \leqslant \mu_n$ for all $n \in \mathbb{N}$. Assume that $\bar{\mu}$ is not strongly continuous, then there exists $p \in P$ such that $\operatorname{st}({}^*\bar{\mu}(p)) > 0$. Since $\lim_n a_n = 0$, we can find $n \in \mathbb{N}$ such that $\operatorname{st}({}^*\bar{\mu}(p)) > a_n \geqslant \operatorname{st}({}^*\mu_{n-1}(p))$; but this contradicts to the fact that $\bar{\mu} \leqslant \mu_n$ for each $n \in \mathbb{N}$.

We see that the numbers $a_n$ and measures $\delta_n$ are completely determined by the values of $\mu$ on the central elements of $P$ up to the order. Thus, the constructed decomposition is unique. □

The concept of an infinitely fine partition also gives us an opportunity to prove the Sobczik-Hammer Decomposition Theorem for vector measures. Let $Y$ be a standard Banach space, and let $F\colon \mathcal{A} \to Y$ be a standard $Y$-valued measure on $(\mathcal{X}, \mathcal{A})$. Recall that the variation of $F$ is a real-valued measure given by $|F|(A) = \sup_\pi \sum_{B\in\pi} \|F(B)\|$, where sup is taken over all finite measurable partitions $\pi$ of $A$. A vector measure is said to be nonatomic (strongly continuous, strongly nonatomic) if the same is true for its variation.

To prove a vector analogue of Sobczik-Hammer Theorem we need the following lemma. As usually, we assume that $P$ is a hyperfinite ifp for $(\mathcal{X}, \mathcal{A})$.

5.2. **Lemma.** *If $F$ is a standard vector measure of bounded variation on $(\mathcal{X}, \mathcal{A})$; then $|{}^*F|(p) \approx \|{}^*F(p)\|$ for each $p \in P$.*

*Proof.* Clearly $|F|(A) \geqslant \|F(A)\|$ for any measurable $A$. If $p \in P \setminus P_c$ then $|{}^*F|(p) \approx 0$ by Corollary 4.4, so that $|{}^*F|(p) \approx \|{}^*F(p)\|$ holds trivially.

Suppose that $p \in P_c$. Fix an infinitesimal $\varepsilon > 0$ and consider a hyperfinite measurable partition $\pi$ of $p$ such that $|{}^*F|(p) \leqslant \sum_{p'\in\pi} \|{}^*F(p')\| + \varepsilon$. Then $P \setminus \{p\} \cup \pi$ is again a hyperfinite ifp. Let $p_\pi$ be the central element of this new ifp, corresponding to the monad of $p$. By Corollary 4.4 $|{}^*F|(p \setminus p_\pi) \approx 0$, so that

$$\begin{aligned}\big\|{}^*F(p_\pi)\big\| \leqslant |{}^*F|(p) \leqslant \sum_{p'\in\pi} \big\|{}^*F(p')\big\| + \varepsilon \leqslant \sum_{p'\in\pi, p'\neq p_\pi} |{}^*F|(p') + \big\|{}^*F(p_\pi)\big\| + \varepsilon \\ = |{}^*F|(p \setminus p_\pi) + \big\|{}^*F(p_\pi)\big\| + \varepsilon \approx \big\|{}^*F(p_\pi)\big\|.\end{aligned}$$

It follows that $\|{}^*F(p_\pi)\| \approx |{}^*F|(p)$. On the other hand, it follows from ${}^*F(p) = {}^*F(p_\pi) + {}^*F(p \setminus p_\pi)$ and $\|{}^*F(p \setminus p_\pi)\| \leqslant |{}^*F|(p \setminus p_\pi) \approx 0$ that $\|{}^*F(p)\| \approx \|{}^*F(p_\pi)\|$, so that $\|{}^*F(p)\| \approx |{}^*F|(p)$. ☐

Now we can proceed with the vector-valued version of the Sobczik-Hammer Theorem. A similar result can be found in [KM89], but the proof presented there is based on completely different concepts.

5.3**. Theorem** (Sobczik-Hammer Decomposition Theorem for vector measure)**.** *Let $F$ be a standard $Y$-valued measure of bounded variation on $(\mathcal{X}, \mathcal{A})$ such that the range of $F$ is relatively compact. Then there exists a strongly additive vector measure $\bar{F}$, a sequence $(x_n)_{n\in\mathbb{N}} \subset E$, and a sequence $(\delta_n)_{n\in\mathbb{N}}$ of distinct zero-one measures on $(\mathcal{X}, \mathcal{A})$, such that $F = \bar{F} + \sum_{n=1}^{\infty} x_n\delta_n$. Further, this decomposition is unique up to the order of terms.*

*Proof.* The proof is analogous to the proof of Theorem 5.1. Consider a hyperfinite ifp $P$ for $(\mathcal{X}, \mathcal{A})$. Let $p_1$ be an element of $P$ of maximum value of $|{}^*F|$. Since $F$ has compact range we are guaranteed that ${}^*F(p_1)$ is nearstandard. Let $x_1 = \mathrm{st}({}^*F(p_1))$, let $\delta_1$ be the zero-one measure corresponding to $[p]$, and let $F_1 = F - x_1\delta_1$. It follows that $|{}^*F_1|(p) \approx \|{}^*F_1(p_1)\| = \|{}^*F(p_1) - \mathrm{st}(F(p_1))\| \approx 0$. Further, follows from the transfer principle, that $|F_1| \leqslant |F|$.

Iterating this process we obtain standard sequences $(F_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, and $(\delta_n)_{n\in\mathbb{N}}$ such that $F_n = F - \sum_{i=1}^{n} x_i\delta_i$, $|F_{n+1}| \leqslant |F_n| \leqslant |F|$, and $\|x_{n+1}\| \leqslant \|x_n\|$ for each $n \in \mathbb{N}$. Let $\bar{F} = F - \sum_{i=1}^{\infty} x_i\delta_1$, then an argument similar to the one in the proof of Theorem 5.1 shows that $\bar{F}$ is strongly continuous. ☐

## 6. Horn-Tarsky Theorem

6.1**. Theorem.** *Let $\mathcal{A}$ be an algebra of subsets of a set $\mathcal{X}$, $\mathcal{C}$ a subalgebra of $\mathcal{A}$, and $\mu$ a finitely additive measure on $\mathcal{C}$. Then $\mu$ can be extended to a finitely additive measure on $\mathcal{A}$.*

*Proof.* Let $P_{\mathcal{A}}$ and $P_{\mathcal{C}}$ be hyperfinite ifp's for $\mathcal{A}$ and $\mathcal{C}$ respectively. Without loss of generality we can assume $P_{\mathcal{A}}$ is a refinement of $P_{\mathcal{C}}$. Take $p \in P_{\mathcal{C}}$ and let $p_1, p_2, \ldots, p_N$ be a hyperfinite partition of $p$ into elements of $P_{\mathcal{A}}$. Next, we assign a weight $w(p_i) \in {}^*\mathbb{R}_+$ to each $p_1$ as $1 \leqslant i \leqslant N$ so that $\sum_{i=1}^{N} w(p_i) = {}^*\mu(p)$.

We apply this procedure to every $p \in P_{\mathcal{C}}$. Now each element of $P_{\mathcal{A}}$ is assigned a weight. For a set $A \in \mathcal{A}$ we put

$$\lambda(A) = \mathrm{st}\Bigg(\sum_{p \in P_{\mathcal{A}},\ p \subseteq {}^*A} w(p)\Bigg).$$

It can be easily verified that $\lambda$ is a standard finitely additive measure and $\lambda_{|\mathcal{C}} = \mu$. ☐

Obviously, the same reasoning can be used to prove Horn-Tarsky Theorem for a Banach-valued measure, but again we would need the range of the measure be relatively compact so that we could take standard parts when defining $\lambda$.

## 7. Ergodic transformations

Let $(\mathcal{X}, \mathcal{A})$ again be a measurable space, and let $\tau \colon \mathcal{X} \to \mathcal{X}$ be a measurable transformation, i.e., $\tau^{-1}(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$. We say that a set $A$ is $\tau$–invariant if $\tau^{-1}(A) = A$. If $P$ is an ifp for $(\mathcal{X}, \mathcal{A})$, we say that $P$ is $\tau$–invariant if $\tau^{-1}(p) \in P$ for each $p \in P$. In this case $\tau^{-1}$ induces a permutation of elements of $P$. Let $\mu$ be a probability measure on $(\mathcal{X}, \mathcal{A})$. Recall that a measurable transformation $\tau$ is said to be measure-preserving if $\mu\big(\tau^{-1}(A)\big) = \mu(A)$ for each $A \in \mathcal{A}$, and ergodic if each $\tau$-invariant set has measure zero or one. If $P$ is an ifp, we write $P_+ = \{p \in P \mid {}^*\mu(p) > 0\}$.

7.1. **Proposition.** *If $\tau$ is ergodic and measure preserving, and $P$ is a $\tau$-invariant ifp, then $P_+$ is hyperfinite and $\tau$-invariant, all the elements of $P_+$ have the same measure, and $\tau^{-1}$ induces a cyclic permutation of elements of $P_+$.*

*Proof.* Fix $p \in P_+$ and consider the sequence $\{\tau^{-k}(p) \mid k \in {}^*\mathbb{N}\}$, i.e., the orbit of $p$ in $P$ under $\tau^{-1}$. Since $\tau$ is measure-preserving, all the sets in the orbit have the same measure. If all of them were distinct, hence disjoint, this would contradict to $\mu$ being a probability measure. Thus, we have $\tau^{-n}(p) = \tau^{-m}(p)$ for some distinct $m$ and $n$. Without loss of generality we can assume that $m = 0$, i.e., $\tau^{-n}(p) = p$ for some $n \in {}^*\mathbb{N}$. Then $A = \bigcup_{k=0}^{n-1} \tau^{-k}(p)$ is a $\tau$-invariant set of positive measure. Since $\tau$ is ergodic, we have $\mu(A) = 1$. This implies that every element of $P_+$ is contained in $A$, so that $P_+ = \{\tau^{-k}(p) \mid k = 0, \dots, n-1\}$. It follows immediately that $P_+$ is hyperfinite and that $\tau$ acts as a cyclic permutation of the elements of $P_+$. $\square$

Unfortunately, one cannot always find an invariant ifp. For example, it is easy to see that no ifp is invarial under a strongly mixing transformation. Recall, that a measure-preserving transformation $\tau$ is called a `strong mixing` if $\mu\big(A \cap \tau^{-n}(B)\big) \to \mu(A)\mu(B)$ for any two $A, B \in \mathcal{A}$. Indeed, if $P$ were a $\tau$-invariant ifp, then by Proposition 7.1 $P$ has to be hyperfinite, and $\tau^{-1}$ induces a cyclic permutation of the elements of $P_+$. But then $\mu\big(p_1 \cap \tau^{-n}(p_2)\big)$ does not converge for any $p_1, p_2 \in P$, a contradiction.

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