# NONSTANDARD DISCRETIZATION AND THE LOEB EXTENSION OF A FAMILY OF MEASURES

V. G. TROITSKIĬ

ABSTRACT. The present article consists of two parts. The first concerns a discretization of an integral operator and uses the discretization of the integral discovered by E. I. Gordon. The main result reads that we can approximate any integral operator to within an infinitesimal by a matrix of infinite size by replacing functions by vectors composed of their values at a finite (but unlimited) number of points. In the second part, we implement the Loeb construction for a random measure. We prove that the same object appears as a result if we consider the random measure as a vector one and construct the corresponding Loeb measure from the vector measure.

We use the language of the Kawai theory NST but all our reasoning remains correct within the classical Robinson nonstandard analysis. The objects under consideration are presumed to be internal until otherwise stated. For a standard set $A$, we denote by ${}^\circ A$ its standard core, i.e. the totality of all standard elements of $A$.

## 1. Discretization of an Integral Operator

In [1] the following theorem was proved:

**Theorem 1.** *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a standard $\sigma$-finite measure space. Then there exists a finite sequence $X = (x_1, \dots, x_N)$ of elements in $\mathcal{X}$ and a positive number $\Delta$ such that*

$$\int\limits_{\mathcal{X}} f\, d\mu = {}^\circ\Big(\Delta \sum_X f\Big)$$

*for every standard integrable function $f : \mathcal{X} \to \mathbb{R}$, where $\sum\limits_X f$ stands for $\sum\limits_{i=1}^N f(x_i)$.*

However, we need a stronger result:

**Theorem 2.** *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a standard $\sigma$-finite measure space. Then, for every positive infinitesimal $\varepsilon$, there exists a finite sequence $X = (x_1, \dots, x_N)$ of elements in $\mathcal{X}$ and a positive number $\Delta$ such that*

$$\Big|\int\limits_{\mathcal{X}} f\, d\mu - \Delta \sum_X f\Big| \leqslant \varepsilon$$

*for every standard integrable function $f : \mathcal{X} \to \mathbb{R}$, in which case we shall say that the couple $(X, \Delta)$ approximates the measure $\mu$ to within $\varepsilon$.*

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In the case of a finite measure Theorem 2 was obtained by E. I. Gordon in [1] in the process of proving Theorem 1. The proof of Theorem 2 in the case of a $\sigma$-finite measure is obtainable on a slight modification of that proof. We need some definitions and results of [2]. An arbitrary element $\tau$ is called *admissible* if it is a member of a standard set. An element $\xi$ is called *standard relative to an admissible element* $\tau$ *($\tau$-standard)* if there exists a standard function $f$ such that the set $f(b)$ is finite for each $b \in \operatorname{dom} f$, $\tau \in \operatorname{dom} f$, and $\xi \in f(\tau)$. It was shown in [2] that the transfer principle and the idealization principle remain valid if we replace each occurrence of the predicate "$X$ is standard" by the predicate "$X$ is $\tau$-standard" for any fixed admissible $\tau$. A real number $r$ is called *$\tau$-infinitesimal* ($r \overset{\tau}{\approx} 0$) if $|r| < t$ for every positive $\tau$-standard $t$; $r$ is called *$\tau$-infinite* if $r^{-1} \overset{\tau}{\approx} 0$.

It follows that if a sequence $(r_n)_{n\in\mathbb{N}}$ is standard relative to $\tau$, then $\lim\limits_{n\to\infty} r_n = r$ if and only if $r_N \overset{\tau}{\approx} r$ for every $\tau$-infinite $N$.

To prove Theorem 2, we also need the following result:

**Theorem 3** ([1]). *Let $\tau$ be an admissible set and let $(\mathcal{X}, \mathcal{A}, \mu)$ be a $\tau$-standard probability space. Then there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{X}$ such that*

$$\int\limits_{\mathcal{X}} f\, d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\xi_i)$$

*for every $\tau$-standard integrable function $f : \mathcal{X} \to \mathbb{R}$.*

*Proof of Theorem 2.* Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ be a standard sequence of measurable subsets of $\mathcal{X}$ such that $\mathcal{X}_n \subset \mathcal{X}_{n+1}$ , $\mu(\mathcal{X}_n) < \infty$ for all $n$, and $\mathcal{X} = \bigcup\limits_{n\in\mathbb{N}} \mathcal{X}_n$. The equality

$$\int\limits_{\mathcal{X}} f\, d\mu = \lim_{n\to\infty} \int\limits_{\mathcal{X}_n} f\, d\mu$$

and the fact that the sequence $\Big(\int\limits_{\mathcal{X}_n} f\, d\mu\Big)_{n\in\mathbb{N}}$ is standard (and, consequently, $\varepsilon$-standard) imply

$$\int\limits_{\mathcal{X}_N} f\, d\mu \overset{\varepsilon}{\approx} \int\limits_{\mathcal{X}} f\, d\mu$$

for every $\varepsilon$-infinite number $N$. Since the number $\varepsilon/2$ is $\varepsilon$-standard, we have

$$\Bigg| \int\limits_{\mathcal{X}_N} f\, d\mu - \int\limits_{\mathcal{X}} f\, d\mu \Bigg| \leqslant \frac{\varepsilon}{2}.$$

Consider the space $(\mathcal{X}_N, \mathcal{A}_N, \mu_N)$, where $\mathcal{A}_N = \{A \cap \mathcal{X}_N | A \in \mathcal{A}\}$ and $\mu_N = \frac{1}{\mu(\mathcal{X}_N)}\mu\big|_{\mathcal{A}_N}$ is a probability measure. The space is obviously $N$-standard; thus, according to Theorem 3, there exists an internal sequence $(\xi_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{X}$ such that, for all $N$-standard functions $g \in \mathcal{L}_1(\mathcal{X}_N, \mathcal{A}_N, \mu_N)$, we have

$$\int\limits_{\mathcal{X}_N} g\, d\mu_N = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(\xi_i),$$

i.e.

$$\lim_{n\to\infty} \frac{\mu(\mathcal{X}_N)}{n} \sum_{i=0}^{n-1} g(\xi_i) = \int\limits_{\mathcal{X}_N} g\, d\mu.$$

The sequence $\left(\frac{\mu(\mathcal{X}_N)}{n} \sum\limits_{i=0}^{n-1} g(\xi_i)\right)_{n\in\mathbb{N}}$ is $(\xi, N)$-standard. Therefore, if $K$ is a $(\xi, N)$-infinite number, then

$$\lim_{n\to\infty} \frac{\mu(\mathcal{X}_N)}{n} \sum_{i=0}^{n-1} g(\xi_i) \overset{(\xi,N)}{\approx} \frac{\mu(\mathcal{X}_N)}{K} \sum_{i=0}^{K-1} g(\xi_i).$$

Assign $X = (\xi_1, \dots, \xi_k)$ and $\Delta = \frac{\mu(\mathcal{X}_N)}{K}$. Then, $\int\limits_{\mathcal{X}_N} g\, d\mu \overset{(\xi,N)}{\approx} \Delta \sum\limits_X g$. Since the function $f$ is standard and, consequently, $N$-standard and the number $1/N$ is $(\xi, N)$-standard, we conclude that

$$\left| \int\limits_{\mathcal{X}_N} f\, d\mu - \Delta \sum_X f \right| \leqslant \frac{1}{N}.$$

The number $N$ is $\varepsilon$-infinite; in consequence, $1/N < \varepsilon/2$. Finally, we deduce

$$\left| \int\limits_{\mathcal{X}} f\, d\mu - \Delta \sum_X f \right| \leqslant \left| \int\limits_{\mathcal{X}} f\, d\mu - \int\limits_{\mathcal{X}_N} f\, d\mu \right| + \left| \int\limits_{\mathcal{X}_N} f\, d\mu - \Delta \sum_X f \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

During the proof of Theorem 2 we have only used the transfer and idealization principles; thus, we have

**Corollary 4.** *Let $\tau$ be an admissible set and let $(\mathcal{X}, \mathcal{A}, \mu)$ be a $\tau$-standard $\sigma$-finite measure space. Then, for every positive $\varepsilon$, there exists a finite sequence $X = (x_1, \dots, x_N)$ of elements in $\mathcal{X}$ and a positive number $\Delta$ such that*

$$\left| \int\limits_{\mathcal{X}} f\, d\mu - \Delta \sum_X f \right| \leqslant \varepsilon$$

*for every $\tau$-standard integrable function $f : \mathcal{X} \to \mathbb{R}$.*

Later we shall use the following lemma:

**Lemma 5.** *Let $\mathcal{Y}$ and $\mathcal{L}$ be standard sets, let $F : {}^\circ\mathcal{Y} \to \mathcal{P}^{Int}(\mathcal{L})$, where $\mathcal{P}^{Int}(\mathcal{L})$ stands for the set of all internal subsets of $(\mathcal{L})$, and let ${}^\circ\mathcal{L} \subset \bigcap\limits_{y \in {}^\circ\mathcal{Y}} F_y$. Then there exists an internal set $\mathcal{F}$ with the property ${}^\circ\mathcal{L} \subset \mathcal{F} \subset \bigcap\limits_{y \in {}^\circ\mathcal{Y}} F_y$.*

*Proof.* The function $F$ has an internal extension $F : \mathcal{Y} \to \mathcal{P}^{\mathrm{Int}}(\mathcal{L})$. (Within the Kawai theory the extension is provided by the idealization principle; within Robinson's analysis we need $\alpha^+$-saturation, where $\alpha$ is the cardinality of ${}^\circ\mathcal{Y}$ ). By assumption, we have $l \in F_y$ for every $y \in {}^\circ\mathcal{Y}$ and $l \in {}^\circ\mathcal{L}$, i.e. for all couples $(y, l) \in {}^\circ(\mathcal{Y} \times \mathcal{L})$. Let $n \in {}^\circ\mathbb{N}$, $(y_1, l_1), \dots, (y_n, l_n) \in {}^\circ(\mathcal{Y} \times \mathcal{L})$. If we set $\mathcal{F} = \bigcap\limits_{i=1}^{n} F_{y_i}$, then $l_i \in \mathcal{F} \subset F_{y_i}$ for every $i = 1, \dots, n$ and by idealization (or saturation), there exists an internal $\mathcal{F}$ such that $l \in \mathcal{F} \subset F_y$ for each $(y, l) \in {}^\circ(\mathcal{Y} \times \mathcal{L})$, i.e. ${}^\circ\mathcal{L} \subset \mathcal{F} \subset \bigcap\limits_{y \in {}^\circ\mathcal{Y}} F_y$. □

From now on we assume $(\mathcal{X}, \mathcal{A}, \lambda_y)_{y \in \mathcal{Y}}$ to be a standard family of $\sigma$-finite measure spaces; i.e., $\mathcal{X}$ and $\mathcal{Y}$ are standard sets, $\mathcal{A}$ is a standard $\sigma$-algebra of subsets of $\mathcal{X}$, and $\lambda : \mathcal{A} \times \mathcal{Y} \to \mathbb{R}$ is a standard function such that, the function $\lambda_y = \lambda(\cdot, y) : \mathcal{A} \to \mathbb{R}$ is a $\sigma$-finite measure on $\mathcal{X}$ for every $y \in \mathcal{Y}$.

Introduce the notations $\mathcal{F}(\mathcal{Y}) = \mathbb{R}^{\mathcal{Y}}$ and $\mathcal{L}_1(\mathcal{X}) = \{f : \mathcal{X} \to \mathbb{R} : f$ is $\lambda_y$-integrable for all $y \in \mathcal{Y}\}$.

Given a finite sequence $X = (x_1, \dots, x_N)$ of elements in $\mathcal{X}$, denote by $\pi_X$ the "projection" from $\mathcal{L}_1(\mathcal{X})$ onto $\mathbb{R}^N$, which associates with a function $f \in \mathcal{L}_1(\mathcal{X})$ the vector $(f(x_1), \dots, f(x_N))$. Analogously, given a finite sequence $Y = (y_1, \dots, y_M)$ of elements in $\mathcal{Y}$, define $\pi_Y : \mathcal{F}(\mathcal{Y}) \to \mathbb{R}^M$ by $\pi_Y(F) = (F(y_1), \dots, F(y_M))$.

Denote by $T$ the pseudointegral operator that acts from $\mathcal{L}_1(\mathcal{X})$ into $\mathcal{F}(\mathcal{Y})$ upon the rule $(Tf)(y) = \int\limits_{\mathcal{X}} f \, d\lambda_y$.

**Theorem 6** (Discretization of a pseudointegral operator)**.** *There exist finite sequences $X = (x_1, \dots, x_N)$ and $Y = (y_1, \dots, y_M)$ of elements in $\mathcal{X}$ and $\mathcal{Y}$ respectively and an $N \times M$–matrix $\Lambda$ such that ${}^\circ\mathcal{Y} \subset Y$ and $\pi_Y(Tf) \approx \Lambda \pi_X(f)$ for every standard function $f \in \mathcal{L}_1(\mathcal{X})$; i.e., $\int f \, d\lambda_{y_j} \approx \sum\limits_{i=1}^{N} f(x_i)\Lambda_{ij}$ $(j = 1, \dots, M)$. In other words,*

$$\begin{array}{ccc} \mathcal{L}_1(\mathcal{X}) & \xrightarrow{\ T\ } & \mathcal{F}(\mathcal{Y}) \\ {\scriptstyle \pi_X}\big\downarrow & & {\scriptstyle \pi_Y}\big\downarrow \\ \mathbb{R}^N & \xrightarrow{\ \Lambda\ } & \mathbb{R}^M \end{array}$$

*is commutative to within an infinitesimal.*

*Proof.* Fix a positive infinitesimal $\varepsilon$. For every standard $y \in \mathcal{Y}$, the $\sigma$-finite measure $\lambda_y$ is standard; consequently, from Theorem 2 it follows that there exists a finite sequence $X_y$ of elements in $\mathcal{X}$ and a positive number $\Delta_y \in \mathbb{R}$ such that $\left| \int f\, d\lambda_y - \Delta_y \sum\limits_{X_y} f \right| \leqslant \varepsilon$ for every standard function $f \in \mathcal{L}_1(\mathcal{X})$.

The functions $X : {}^\circ\mathcal{Y} \to \bigcup\limits_{n\in\mathbb{N}} \mathcal{X}^n$ and $\Delta : {}^\circ\mathcal{Y} \to \mathbb{R}$ have internal extensions $X : \mathcal{Y} \to \bigcup\limits_{n\in\mathbb{N}} \mathcal{X}^n$ and $\Delta : \mathcal{Y} \to \mathbb{R}$ respectively. Denote by $\Phi(y, f)$ the internal formula $\left| \int f\, d\lambda_y - \Delta_y \sum\limits_{X_y} f \right| \leqslant \varepsilon$ and let $F_y$ denote the internal set $\{f \in \mathcal{L}_1(\mathcal{X}) \,|\, \Phi(y, f)\}$. Then ${}^\circ\mathcal{L}_1(\mathcal{X}) \subset F_y$ for every standard $y \in \mathcal{Y}$. By Lemma 5, there exists an internal $\mathcal{F}$ such that ${}^\circ\mathcal{L}_1(\mathcal{X}) \subset \mathcal{F} \subset \bigcap\limits_{y\in{}^\circ\mathcal{Y}} F_y$, in particular, $\forall^{\mathrm{st}} y \in \mathcal{Y}\ \forall f \in \mathcal{F}\ \Phi(y, f)$.

Assign $Y_\circ = \{y \in \mathcal{Y} \mid \forall f \in \mathcal{F}\ \Phi(y, f)\}$. The set is internal and, moreover, ${}^\circ\mathcal{Y} \subset Y_\circ$. As is known, there exists an internal finite set $\widetilde{\mathcal{Y}}$ such that ${}^\circ\mathcal{Y} \subset \widetilde{\mathcal{Y}} \subset \mathcal{Y}$. Let $Y_1 = Y_\circ \cap \widetilde{\mathcal{Y}}$. Then $Y_1$ is a finite internal set and $\forall y \in Y_1\ \forall f \in \mathcal{F}\ \Phi(y, f)$. Since ${}^\circ\mathcal{L}_1(\mathcal{X}) \subset \mathcal{F}$, it follows that $\forall y \in Y_1\ \forall^{\mathrm{st}} f \in \mathcal{L}_1(\mathcal{X})\ \Phi(y, f)$.

Take as $Y$ any sequence $(y_1, \dots, y_M)$ composed of all elements of $Y_1$ and take as $X$ the concatenation $X_{y_1} \oplus X_{y_2} \oplus \dots \oplus X_{y_M}$, of the sequences $X_{y_1}, X_{y_2}, \dots, X_{y_M}$, i.e. the sequence composed of consecutive elements of the sequences $X_{y_1}, X_{y_2}, \dots, X_{y_M}$. Assume $X = (x_1, \dots, x_N)$ and set

$$\Lambda_{nm} = \begin{cases} \Delta_{y_m} & \text{if } \sum\limits_{j=1}^{m-1} N_j < n \leqslant \sum\limits_{j=1}^{m} N_j, \\ 0 & \text{otherwise}, \end{cases}$$

where $N_j$ stands for the length of the sequence $X_{y_j}$. Then, for every standard function $f \in \mathcal{L}_1(\mathcal{X})$, we have

$$\left| \int f\, d\lambda_{y_j} - \sum_{i=1}^{N} f(x_i)\Lambda_{ij} \right| = \left| \int f\, d\lambda_{y_j} - \sum_{X_{y_j}} f \cdot \Delta_{y_j} \right| \leqslant \varepsilon \quad (j = 1, \dots, M),$$

i.e. $\int\limits_{\mathcal{X}} f\, d\lambda_{y_j} \approx \sum\limits_{i=1}^{N} f(x_i)\Lambda_{ij}$. □

We now consider a particular instance of a pseudointegral operator given by an integral operator. Suppose that under the hypotheses of Theorem 6 we have a $\sigma$-finite measure $\mu$ on the $\sigma$-algebra $\mathcal{A}$; such that, for every $y \in \mathcal{Y}$, the measure $\lambda_y$ is absolutely continuous with respect to $\mu$ with the Radon-Nikodym derivative $K : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ and the function $K_y = K(\,\cdot\,, y)$ belongs to $\mathcal{L}_\infty(\mathcal{X}, \mathcal{A}, \mu)$ for every $y \in \mathcal{Y}$. In this situation the pseudointegral operator turns out to be the one integral:

$$T : \mathcal{L}_1(\mathcal{X}) \to \mathcal{F}(\mathcal{Y}), \qquad (Tf)(y) = \int_{\mathcal{X}} f\, d\lambda_y = \int_{\mathcal{X}} f \cdot K_y\, d\mu.$$

Fix a positive infinitesimal $\varepsilon$. Theorem 2 ensures existence for a finite sequence $X = (x_1, \dots, x_N)$ of elements in $\mathcal{X}$ and a positive $\Delta \in \mathbb{R}$ such that

$$\Bigl| \int_{\mathcal{X}} f\, d\mu - \Delta \sum_X f \Bigr| \leqslant \varepsilon$$

for every standard function $f \in \mathcal{L}_1(\mathcal{X}, \mathcal{A}, \mu)$.

**Theorem 7** (Discretization of an integral operator). *There exists a finite sequence $Y = (y_1, \dots, y_M)$ of elements in $\mathcal{Y}$ such that ${}^\circ\mathcal{Y} \subset Y$ and $\pi_Y(Tf) \approx \Lambda \pi_X(f)$ for every standard function $f \in \mathcal{L}_1(\mathcal{X})$, where $\Lambda_{ij} = \Delta \cdot K(x_i, y_j)$.*

*Proof.* The reasoning is mainly the same as in the proof of Theorem 6. Let $y \in {}^\circ\mathcal{Y}$. Denote by $\Psi(y, f)$ the internal formula $\bigl|\int f\, d\lambda_y - \Delta \sum_X f \cdot K_y\bigr| \leqslant \varepsilon$. Since $K_y$ is standard, it follows that

$$\Bigl| \int_{\mathcal{X}} f\, d\lambda_y - \Delta \sum_X f \cdot K_y \Bigr| = \Bigl| \int_{\mathcal{X}} f \cdot K_y\, d\mu - \Delta \sum_X f \cdot K_y \Bigr| \leqslant \varepsilon$$

for every standard function $f \in \mathcal{L}_1(\mathcal{X})$.

Assign $F_y = \{f \in \mathcal{L}_1(\mathcal{X}) \mid \Psi(y, f)\}$. By Lemma 5, there is an internal set $\mathcal{F}$ such that $\forall^{\mathrm{st}} y \in \mathcal{Y}\ \forall f \in \mathcal{F}\ \Psi(y, f)$ and ${}^\circ\mathcal{L}_1(\mathcal{X}) \subset \mathcal{F}$. In this case there exists a finite internal $Y_1 \subset \mathcal{Y}$ such that ${}^\circ\mathcal{Y} \subset Y_1$ and $\forall y \in Y_1\ \forall f \in \mathcal{F}\ \Psi(y, f)$. Since ${}^\circ\mathcal{L}_1(\mathcal{X}) \subset \mathcal{F}$, it follows $\forall y \in Y_1\ \forall^{\mathrm{st}} f \in \mathcal{L}_1(\mathcal{X})\ \Psi(y, f)$. If we set $Y = (y_1, \dots, y_M)$, where $(y_1, \dots, y_M)$ are all elements of $Y_1$, then

$$\int_{\mathcal{X}} f\, d\lambda_{y_j} \approx \Delta \sum_X f \cdot K_{y_j} = \sum_{i=1}^{N} f(x_i) \Lambda_{ij} \quad (j = 1, \dots, M).$$

□

The following remarks clarify Theorems 6 and 7.

**Remark 8.** The proofs of Theorems 6 and 7 provide a stronger result: for every positive infinitesimal $\varepsilon$, there exist $X$, $Y$, and $\Lambda$, described in the statements of the theorems, such that

$$\Bigl| \int_{\mathcal{X}} f\, d\lambda_{y_j} - \sum_{i=1}^{N} f(x_i) \Lambda_{ij} \Bigr| \leqslant \varepsilon \qquad (j = 1, \dots, M)$$

for every standard $f \in \mathcal{L}_1(\mathcal{X})$.

**Remark 9.** The proofs also ensure that the internal finite sequence $Y$ constructed there contains the standard core of $\mathcal{Y}$ and, therefore, inherits a number of its properties. For example, we have $\sup\limits_{y\in\mathcal{Y}} F(y) = {}^\circ\!\sup\limits_{j=1,\dots,M} F(y_j) = {}^\circ\max \pi_Y(F)$ for every standard bounded function $F : \mathcal{Y} \to \mathbb{R}$.

**Remark 10.** The projection $\pi_X$ in Theorem 7 preserves the $L_1$-norm of a standard integrable function $f : \mathcal{X} \to \mathbb{R}$:

$$\int\limits_{\mathcal{X}} f\, d\mu = {}^\circ\left(\Delta \sum_X f\right) = {}^\circ\left(\Delta \cdot \sum_{i=1}^{N} (\pi_X(f))_i\right).$$

**Remark 11.** By requiring absolute continuity with respect to $\mu$ for each $\lambda_y$ we obtain a more "explicit" construction for $X$ and $\Lambda$ than that of Theorem 6, namely: $X$ approximates the measure $\mu$ and $\Lambda$ is the matrix of values of the function $\Delta \cdot K$ on the finite grid $X \times Y$.

If we define the norm in $\mathbb{R}^M$ by $\| v \| = \max\limits_{j=1,\dots,M} |v_j|$, then the result obtained in Theorem 7 can be rewritten as follows: for every couple $(X, \Delta)$ approximate to the measure $\mu$ to within $\varepsilon$, there exists a finite sequence $Y = (y_1, \dots, y_M)$ such that ${}^\circ\mathcal{Y} \subset Y$ and $\| \pi_Y(Tf) - \Lambda\pi_X(f) \| \leqslant \varepsilon$.

**Theorem 12.** *For every finite sequence $Y = (y_1, \dots, y_M)$ of elements in $\mathcal{Y}$, there exists a couple $(X, \Delta)$ approximate to $\mu$ such that $\| \pi_Y(Tf) - \Lambda\pi_X(f) \| \leqslant \varepsilon$ for every $f \in {}^\circ\mathcal{L}_1(\mathcal{X})$. (The matrix $\Lambda$ is the same as in Theorem 7.)*

*Proof.* For every $y \in Y$, the function $K_y$ belongs to the set $\{K_{y_1}, \dots, K_{y_M}\}$ and, consequently, is $Y$-standard. The space $(\mathcal{X}, \mathcal{A}, \mu)$ is standard and thus $Y$-standard. Corollary 4 provides existence for a finite sequence $X$ and a positive $\Delta$ such that

$$\left| \int\limits_{\mathcal{X}} g\, d\mu - \Delta \sum_X g \right| \leqslant \varepsilon$$

for every $Y$-standard integrable function $g$.

If $f \in {}^\circ\mathcal{L}_1(\mathcal{X})$, then $f \cdot K_y$ is a $Y$-standard integrable function; consequently,

$$\left| \int\limits_{\mathcal{X}} f\, d\lambda_y - \Delta \sum_X f \cdot K_y \right| = \left| \int\limits_{\mathcal{X}} f \cdot K_y\, d\mu - \Delta \sum_X f \cdot K_y \right| \leqslant \varepsilon,$$

which proves the theorem. □

## 2. A Loeb Random Measure

**Definition 13.** Let $(\mathcal{X}, \mathcal{A})$ be a space $\mathcal{X}$ with algebra $\mathcal{A}$ of its subsets and let $(\mathcal{Y}, \mathcal{B}, \nu)$ be a measure space $\mathcal{Y}$ with algebra $\mathcal{B}$ and finitely-additive measure $\nu$. A function $\lambda : \mathcal{A} \times \mathcal{Y} \to \mathbb{R}$ is called a (*finitely-additive*) *random measure* if

1) the function $\lambda_A = \lambda(A, \cdot) : \mathcal{Y} \to \mathbb{R}$ is $\mathcal{B}$-measurable for each $A \in \mathcal{A}$;

2) there exists a subset $\overline{\mathcal{Y}} \subset \mathcal{Y}$ such that $\nu_L(\mathcal{Y} \setminus \overline{\mathcal{Y}})$ and, for every $y \in \overline{\mathcal{Y}}$, the function $\lambda_y = \lambda(\cdot, y)$ is a (finitely-additive) measure on $\mathcal{A}$.

To emphasize that $\mathcal{Y}$ is equipped with the algebra $\mathcal{B}$, we shall write $\lambda : \mathcal{A} \times \mathcal{Y}_{\mathcal{B}} \to \mathbb{R}$.

Throughout the section, the spaces $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ and a finitely-additive random measure $\lambda$ are assumed to be internal. For the measure $\nu$, we construct the Loeb measure $\nu_L : L(\mathcal{B}, \nu) \to {}^{\circ}\overline{\mathbb{R}}$. We shall write $L(\mathcal{B})$ rather than $L(\mathcal{B}, \nu)$ henceforth.

For each $y \in \overline{\mathcal{Y}}$, from the measure $\lambda_y$ we also construct the Loeb measure $(\lambda_y)_L : L(\mathcal{A}, \lambda_y) \to {}^{\circ}\overline{\mathbb{R}}$. Denote by $\sigma(\mathcal{A})$ the smallest external $\sigma$-algebra that contains $\mathcal{A}$. Obviously, $\sigma(\mathcal{A}) \subset L(\mathcal{A}, \lambda_y)$ for every $y \in \overline{\mathcal{Y}}$.

Define a function $\lambda^L : \sigma(\mathcal{A}) \times \mathcal{Y} \to {}^{\circ}\overline{\mathbb{R}}$ as follows: for every $y \in \overline{\mathcal{Y}}$ and $A \in \sigma(\mathcal{A})$, put $\lambda^L(A, y) = (\lambda_y)_L(A)$, and define $\lambda^L$ on $\mathcal{Y} \setminus \overline{\mathcal{Y}}$ arbitrarily.

**Theorem 14** (A Loeb random measure)**.** *The function $\lambda^L$ constructed above is an external random measure $\lambda^L : \sigma(\mathcal{A}) \times \mathcal{Y}_{L(\mathcal{B})} \to {}^{\circ}\overline{\mathbb{R}}$.*

*Proof.* First, observe that $\lambda^L_y = (\lambda_y)_L$ and $\nu_L(\mathcal{Y} \setminus \overline{\mathcal{Y}})$. Therefore, $\lambda^L_y$ is a measure for $\nu_L$-almost all $y \in \mathcal{Y}$. Denote by $\mathfrak{M}$ the set of those $A \in \sigma(\mathcal{A})$ for which the function $\lambda^L_A$ is $L(\mathcal{B})$-integrable.

Take $A \in \mathcal{A}$; then $\lambda^L_A(y) = \lambda^L_y(A) = {}^{\circ}\lambda_y(A) = {}^{\circ}\lambda_A(y)$ for all $y \in \overline{\mathcal{Y}}$. Therefore, $\lambda_A$ is a lifting of $\lambda^L_A$. Since $\lambda_A$ is $\mathcal{B}$-measurable, it follows that $\lambda^L_A$ is $L(\mathcal{B})$-measurable by the Lifting Theorem of [3]; i.e., $\mathcal{A} \subset \mathfrak{M}$. (The Lifting Theorem was formulated in [3] only for a finite measure; however, the proof given there works in the desired direction as well.)

Let $(A_n)_{n \in {}^{\circ}\mathbb{N}}$ be a monotone sequence of sets in $\mathfrak{M}$ and let $A = \lim\limits_{n \to \infty} A_n$. Then $A \in \sigma(\mathcal{A})$. Since we have $\lambda^L_y(A) = \lim\limits_{n \to \infty} \lambda^L_y(A_n)$ for every $y \in \overline{\mathcal{Y}}$, the function $\lambda^L_A$ is $L(\mathcal{B})$-measurable as a limit of the sequence of $L(\mathcal{B})$-measurable functions $(\lambda^L_{A_n})_{n \in {}^{\circ}\mathbb{N}}$. Thus, $\mathfrak{M}$ is a monotone class containing $\mathcal{A}$. By [4, p. 27], any monotone class containing an algebra contains the $\sigma$-algebra generated by the algebra; therefore, $\sigma(\mathcal{A}) \subset \mathfrak{M}$. But $\mathfrak{M} \subset \sigma(\mathcal{A})$ by definition, which completes the proof. □

We have deliberately examined $\lambda^L$ only on $\sigma(\mathcal{A})$. Suppose that the algebras $L(\mathcal{A}, \lambda_y)$ coincide for all $y \in \mathcal{Y}$ and denote the common algebra by $L(\mathcal{A})$. Even in this situation the function $\lambda^L : L(\mathcal{A}) \times \mathcal{Y}_{L(\mathcal{B})} \to {}^\circ\overline{\mathbb{R}}$ can not be a random measure:

*Example* 15. Fix an infinite $\eta \in \mathbb{N}$, put $\Delta t = \eta^{-1}$ and $\mathcal{Y} = \{0, \Delta t, 2\Delta t, \dots, \eta \cdot \Delta t = 1\}$, and let $\mathcal{B}$ be the algebra $\mathcal{P}^{\mathrm{Int}}(\mathcal{Y})$ of all internal subsets of $\mathcal{Y}$. Let $\nu$ be the counting measure on $\mathcal{Y}$: $\nu(A) = |A|/|\mathcal{Y}|$ for every internal $A \subset \mathcal{Y}$, where $|A|$ stands for the cardinality of $A$. It was shown in [3] that the Loeb algebra $L(\mathcal{B}, \nu)$ differs from $\mathcal{P}(\mathcal{Y})$. Fix a $\nu_L$-nonmeasurable set $N$. Assign $\mathcal{X} = \mathcal{Y}$ and $\mathcal{A} = \mathcal{B} = \mathcal{P}^{\mathrm{Int}}(\mathcal{Y})$ and, for $y \in \mathcal{Y}$ and $A \in \mathcal{A}$, set

$$\lambda(A, y) = \chi_A(y) = \begin{cases} 1 & \text{if } y \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\lambda$ is a random measure, $\lambda : \mathcal{A} \times \mathcal{Y}_{\mathcal{B}} \to \mathbb{R}$.

It is readily verified that $L(\mathcal{A}, \lambda_y) = \mathcal{P}(\mathcal{Y})$ for every $y \in \mathcal{Y}$; therefore, $N \in L(\mathcal{A}) = \mathcal{P}(\mathcal{Y})$. But the function $\lambda^L_N = \chi_N$ is not $L(\mathcal{B})$-measurable; i.e., $\lambda^L : L(\mathcal{A}) \times \mathcal{Y} \to {}^\circ\mathbb{R}$ is not a random measure.

Let $M(\mathcal{Y}, \mathcal{B}, \nu)$ be the space of cosets of almost everywhere equal measurable functions from $\mathcal{Y}$ into $\mathbb{R}$; and let $M(\mathcal{Y}, L(\mathcal{B}), \nu_L)$ denote the space of cosets of $\nu_L$-almost everywhere equal $\nu_L$-almost everywhere bounded $L(\mathcal{B})$-measurable functions from $\mathcal{Y}$ into ${}^\circ\overline{\mathbb{R}}$. In the sequel we consider the measure $\nu$ finite. As usually, we shall sometimes identify cosets and functions.

Let $\mathcal{N}_\approx$ be the order ideal in $M(\mathcal{Y}, \mathcal{B}, \nu)$ consisting of all functions nearly equal to zero everywhere except a $\nu_L$-null set:

$$\mathcal{N}_\approx = \{\widetilde{F} \in M(\mathcal{Y}, \mathcal{B}, \nu) \mid (\forall F \in \widetilde{F})(\exists N)((\nu_L(N) = 0)\&((\forall y \notin N)F(y) \approx 0))\}.$$

Denote by $\widetilde{F}/\mathcal{N}_\approx$ the element, of the quotient space $M(\mathcal{Y}, \mathcal{B}, \nu)/\mathcal{N}_\approx$, which corresponds to $\widetilde{F} \in M(\mathcal{Y}, \mathcal{B}, \nu)$. By the Lifting Theorem of [3], we have

**Lemma 16.** *Let $\varphi(\widetilde{F}/\mathcal{N}_\approx) = {}^\circ\widetilde{F}$ for $\widetilde{F}/\mathcal{N}_\approx \in M(\mathcal{Y}, \mathcal{B}, \nu)/\mathcal{N}_\approx$. Then $\varphi$ is a linear isomorphism between $M(\mathcal{Y}, \mathcal{B}, \nu)/\mathcal{N}_\approx$ and $M(\mathcal{Y}, L(\mathcal{B}), \nu_L)$.*

The quotient space of $M(\mathcal{Y}, \mathcal{B}, \nu)$ by the subspace $\mathcal{N}_\approx$, where $M(\mathcal{Y}, \mathcal{B}, \nu)$ is regarded as a vector space over ${}^\circ\mathbb{R}$, is an analog of the nonstandard hull of a normed space (cf., for instance [5]), where the collection of all elements with infinitesimal norm is taken as a subspace. If we had taken, instead of $M(\mathcal{Y}, \mathcal{B}, \nu)/\mathcal{N}_\approx$, the nonstandard hull of $\operatorname{fin} M(\mathcal{Y}, \mathcal{B}, \nu)$ with respect to the seminorm $\|F\| = \operatorname{ess}_\nu \sup|F|$, then the mapping $\varphi$

would have been only an ephimorphism. For example, if we take the unit segment [0,1] endowed with the Lebesgue measure, then the functions $F_1 \equiv 0$ and $F_2 = \chi_{[0,\varepsilon]}$, where $\varepsilon$ is a positive infinitesimal, have the respective cosets in the hull distinct; nevertheless, $\varphi$ carries both the functions into zero.

Let $V$ be an internal vector space and let $\mathcal{N} \subset V^{\approx}$ be its external subspaces. We shall call the quotient space $V^{\approx}/\mathcal{N}$ the *nonstandard hull* of $V$ and denote it by the symbol $\widehat{V}$. For $v \in V^{\approx}$, we denote the corresponding coset in $\widehat{V}$ by $\hat{v}$.

Let $\mathcal{A}$ be an internal algebra of subsets of $\mathcal{X}$ and let $F : \mathcal{A} \to V$ be an internal finitely-additive vector measure such that $\operatorname{im} F \subset V^{\approx}$. Define ${}^{\circ}F : \mathcal{A} \to \widehat{V}$ by ${}^{\circ}F(A) = \widehat{F(A)}$. It is natural to define the Loeb vector measure $L(F) : L(\mathcal{A}, F) \to \widehat{V}$ as the completion of an extension of the measure ${}^{\circ}F$ onto $\sigma(\mathcal{A})$, in case such an extension exists. Unfortunately, the question of existence of the extension is complicated.

Observe that the space $M(\mathcal{Y}, \mathcal{B}, \nu)/\mathcal{N}_{\approx}$ considered above is the nonstandard hull of $M(\mathcal{Y}, \mathcal{B}, \nu)$ with respect to the ideal $\mathcal{N}_{\approx}$; moreover, the space $M(\mathcal{Y}, \mathcal{B}, \nu)$ itself is taken as $(M(\mathcal{Y}, \mathcal{B}, \nu))^{\approx}$.

In addition to the two random measures $\lambda : \mathcal{A} \times \mathcal{Y}_{\mathcal{B}} \to \mathbb{R}$ and $\lambda^L : \sigma(\mathcal{A}) \times \mathcal{Y}_{L(\mathcal{B})} \to {}^{\circ}\overline{\mathbb{R}}$ considered above, we introduce two vector measures $\widetilde{\lambda} : \mathcal{A} \to M(\mathcal{Y}, \mathcal{B}, \nu)$ and $\widetilde{\lambda}^L : \sigma(\mathcal{A}) \to M(\mathcal{Y}, L(\mathcal{B}), \nu_L)$ which are defined by $\widetilde{\lambda}(A) = \lambda_A$ and $\widetilde{\lambda}^L(A) = \lambda^L_A$.

**Theorem 17.** *For the vector measure $\widetilde{\lambda}$, there exists a Loeb measure $L(\widetilde{\lambda})$. Furthermore, $L(\widetilde{\lambda})$ agrees with $\widetilde{\lambda}^L$ on $\sigma(\mathcal{A})$ up to the isomorphism $\varphi$ described in Lemma 16.*

*Proof.* Let $A \in \mathcal{A}$, then $\varphi({}^{\circ}\widetilde{\lambda}(A)) = \varphi(\widehat{\widetilde{\lambda}(A)}) = {}^{\circ}(\widetilde{\lambda}(A)) = {}^{\circ}\lambda_A = \lambda^L_A = \widetilde{\lambda}^L(A)$; i.e., the measures $(\varphi \circ {}^{\circ}\widetilde{\lambda})$ and $\widetilde{\lambda}^L$ agree on $\mathcal{A}$. Hence, $\widetilde{\lambda}^L$ gives an extension of ${}^{\circ}\widetilde{\lambda}$ onto $\sigma(\mathcal{A})$ up to the isomorphism $\varphi$. □

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