SOME OPEN PROBLEMS AND CONJECTURES ASSOCIATED WITH THE INVARIANT SUBSPACE PROBLEM*

Y. A. ABRAMOVICH, C. D. ALIPRANTIS, G. SIROTKIN, AND V. G. TROITSKY

Abstract. There is a subtle difference as far as the invariant subspace problem is concerned for operators acting on real Banach spaces and operators acting on complex Banach spaces. For instance, the classical hyperinvariant subspace theorem of V. I. Lomonosov [10] while true for complex Banach spaces is false for real Banach spaces. When one starts with a bounded operator on a real Banach space and then considers some "complexification technique" to extend the operator to a complex Banach space, there seems to be no pattern that indicates any connection between the invariant subspaces of the "real" operator and those of its "complexifications."

The purpose of this note is to examine two complexification methods of an operator T acting on a real Banach space and present some questions regarding the invariant subspaces of T and those of its complexifications. *AMS Subject Classification Numbers:* 47A15, 47C05, 47L20, 46B99

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1. Introduction

For unexplained terminology in this paper, we refer the reader to [1]. If Y is an arbitrary (real or complex) Banach space, then $\mathcal{L}(Y)$ will denote the algebra of all bounded operators on Y. If $T \in \mathcal{L}(Y)$, then $\operatorname{Lat}(T)$ will denote the collection of all closed T-invariant subspaces of Y. Likewise, if $\mathcal{A}$ is an algebra of bounded operators on Y, i.e., a subalgebra of $\mathcal{L}(Y)$, then $\operatorname{Lat}\mathcal{A}$ is the collection of all closed subspaces of Y that are $\mathcal{A}$-invariant, i.e., invariant under every operator of $\mathcal{A}$. A subspace of Y is **non-trivial** if it is different from $\{0\}$ and Y. The famous invariant subspace problem can be stated as follows.

The Invariant Subspace Problem: *When does a bounded operator on a separable* (*real or a complex*) *Banach space have a non-trivial closed invariant subspace?*

For a complete discussion of the invariant subspace problem and its history, we refer the reader to [1, Chapter 10] and [13]. However, we remark that the problem is quite different for real and complex Banach spaces; see also [12] and [21].

Unless otherwise stated, throughout this work X will denote an infinite dimensional separable real Banach space (with norm dual X^*) and $T\colon X \to X$ a continuous operator on X without non-trivial closed invariant subspaces. As usual, $X_{\mathbf{c}}$ will denote the complexification of X. That is, $X_{\mathbf{c}}$ is the vector space

$$X_{\mathbf{c}} = X \oplus \imath X = \{x + \imath y\colon\ x, y \in X\}$$

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equipped with the norm $\|x + \imath y\| = \sup_{\theta\in[0,2\pi]} \|x\cos\theta + y\sin\theta\|$. With the standard algebraic operations, the normed space $(X_{\mathbf{c}}, \|\cdot\|)$ is a complex Banach space and the operator $T\colon X \to X$ has a natural continuous linear extension $T_{\mathbf{c}}\colon X_{\mathbf{c}} \to X_{\mathbf{c}}$, called the **complexification** of T, defined by $T_c(x + \imath y) = Tx + \imath Ty$. When we talk about spectral properties of the operator T we always refer to the complexification $T_{\mathbf{c}}$ of T. (See [18, § 3, p. 5] and [1, Section 1.1].)

For any scalar $\lambda \in \mathbb{R}$ the operators T and $T + \lambda I$ have the same invariant subspaces. Therefore, choosing $\lambda > 0$ large enough so that $-\lambda \notin \sigma(T_{\mathbf{c}})$ and replacing T by $T + \lambda I$ and scaling, we can assume without loss of generality that the bounded operator $T\colon X \to X$ is one-to-one, surjective and $\|T\| = 1$.

PROBLEM: *What can be said about the invariant subspaces of the operator $T_{\mathbf{c}}$?*

The questions and comments associated with this problem that will be listed below have been discussed extensively with our mentor, friend, and colleague Yuri Abramovich whose untimely death in February 5 of 2003 deprived us and our profession of a superb scientist.

2. Two Invariant Subspace Conjectures

We list in this section two general conjectures regarding the invariant subspace problem. We already know from the works of Enflo [7] and Read [14] that there are bounded operators acting on separable Banach spaces without non-trivial closed invariant subspaces.

Conjecture 1. *Every positive operator on a separable Banach lattice has a non-trivial closed invariant subspace.*

Note that Read's operator [14, 15, 16] is not positive. However, Read's construction can be modified in such a way that its negative part is just a rank-one operator. It was shown in [25] that the modulus of Read's operator [16] has invariant subspaces. We also mention that although Read's operator does not have non-trivial closed invariant subspaces, it has an invariant closed cone—which is a subcone of the standard cone of ℓ_1.

The following conjecture was posed by Lomonosov in [11].

Conjecture 2. *Every adjoint operator has a non-trivial closed invariant subspace.*

This would imply in particular the existence of non-trivial closed invariant subspaces for all operators on reflexive Banach spaces, including L_p-spaces for $1 < p < \infty$ and Hilbert spaces.

3. The Complexification of Operators Without Invariant Subspaces

As stated in the introduction, $T\colon X \to X$ is a bounded operator on an infinite dimensional separable real Banach space without non-trivial closed invariant subspaces. As far as the invariant subspace problem is concerned, we can assume without loss of generality that T is also (besides being one-to-one) surjective and of norm one.

The conjecture we offer here regarding the operator T is the following.

Conjecture 3. *The operator $T_{\mathbf{c}}$ has no non-trivial closed invariant subspaces.*

If Z is a closed non-trivial invariant subspace of an operator R on a real Banach space X, then $Z + \imath Z$ is a closed non-trivial subspace of $X_{\mathbf{c}}$ invariant under $R_{\mathbf{c}}$. Note that the parameters in Read's example of an operator on ℓ_1 with no invariant subspace can be chosen in such a way that the example works both for the real and the complex case. The objective of this work is to discuss several questions associated with the following problem that is related to Conjecture 3.

Problem I: *Does the operator $T_{\mathbf{c}}$ have a minimal non-zero closed invariant subspace?*

Recall that an invariant closed subspace V of $X_{\mathbf{c}}$ is said to be **minimal** if it follows from $U \subseteq V$ and U a $T_{\mathbf{c}}$-invariant closed subspace of $X_{\mathbf{c}}$ that either $U = \mathbf{0}$ or $U = V$. Of course, if Conjecture 3 is true, then $X_{\mathbf{c}}$ is automatically a non-zero minimal invariant closed vector subspace.

We shall state below several properties of the invariant subspaces of $T_{\mathbf{c}}$. To this end, let W be a non-trivial closed $T_{\mathbf{c}}$-invariant subspace of $X_{\mathbf{c}}$. We shall present the properties of W in the form of displayed statements.

1. *The vector subspace W is infinite dimensional.*

To see this, assume by way of contradiction that W is finite dimensional. Pick a basis $\{z_1, z_2, \ldots, z_n\}$ for W and let $z_k = x_k + \imath y_k$ for each k. If Y is the finite dimensional subspace in X generated by $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, then Y is a non-zero (and hence non-trivial since X is infinite dimensional) closed T-invariant subspace of X, which is a contradiction. Hence, W is infinite dimensional.

2. *If $z = x + \imath y \in W$ and either $x = 0$ or $y = 0$, then $z = 0$.*

To see this, let $V = \{y \in X\colon\ 0 + \imath y \in W\}$. Clearly, V is a closed subspace of X which is also T-invariant. (Indeed, notice that for each $y \in V$ we have $0 + \imath Ty = T_{\mathbf{c}}(0 + \imath y) \in W$, and so $Ty \in V$.)

Next, we claim that $V = \{0\}$. If $V \neq \{0\}$, then (since T does not have any non-trivial closed invariant subspaces) $V = X$. This implies that for each $x \in X$ we have $x + \imath 0 = -\imath(0 + \imath x) \in W$. In particular, for each $z = x + \imath y \in X_{\mathbf{c}}$ we have $z = (x + \imath 0) + (0 + \imath y) \in W$, and so $W = X_{\mathbf{c}}$ which is a contradiction. Therefore, $V = \{0\}$, and from this the validity of (**2**) follows.

3. *If $x \in X$, then there exists at most one $y \in X$ such that $x + \imath y \in W$. If $z = x + \imath y \in W$, then this unique y will be denoted by Sx, i.e., $y = Sx$ and $x + \imath Sx \in W$.*

If $x + \imath y,\ x + \imath y_1 \in W$, then $0 + \imath(y - y_1) = (x + \imath y) - (x + \imath y_1) \in W$, and so by part (**2**) we must have $y = y_1$.

Next, we define the following vector subspace of X:

$$\Delta = \big\{x \in X\colon\ \exists\ y \in X \text{ such that } x + \imath y \in W\big\}.$$

By (**3**) we know that there exists a mapping $S\colon \Delta \to X$ defined for each $x \in \Delta$ by letting Sx be the unique vector such that $x + \imath Sx \in W$. In particular, we have

$$W = \big\{x + \imath Sx\colon\ x \in \Delta\big\}.$$

4. *The mapping $S\colon \Delta \to X$ is a linear operator with range Δ. Moreover, $S^2 = -I_\Delta$ on Δ (and so the operator $S\colon \Delta \to \Delta$ is invertible).*

The linearity of the mapping $S\colon \Delta \to X$ follows immediately from the definitions of addition and scalar multiplication:

$$\begin{aligned}(x_1 + \imath Sx_1) + (x_2 + \imath Sx_2) &= (x_1 + x_2) + \imath(Sx_1 + Sx_2)\\ \alpha(x_1 + \imath Sx_1) &= \alpha x_1 + \imath(\alpha Sx_1)\,.\end{aligned}$$

Now for each $x \in \Delta$ we have $Sx + \imath(-x) = -\imath(x + \imath Sx) \in W$. This implies $Sx \in \Delta$ and that $S^2x = -x$ for each $x \in \Delta$.

5. *The subspace Δ is T-invariant and S and T commute on Δ. In particular, Δ is dense in X.*

If $x \in \Delta$, then $x + \imath Sx \in W$, and from the $T_{\mathbf{c}}$-invariance of W we get $Tx + \imath T(Sx) \in W$. This implies $Tx \in \Delta$ and that $TSx = STx$. Therefore, Δ is T-invariant and S and T commute on Δ. Since $\Delta \neq \{0\}$ and T has no non-trivial closed invariant subspaces, it follows that Δ is dense in X.

6. *The invertible operator $S\colon \Delta \to \Delta$ is a closed operator.*[1]

If $\big(x_n, Sx_n\big) \to (x, y)$ in $X \times X$, then $x_n + \imath Sx_n \to x + \imath y$ in $X_{\mathbf{c}}$, and so (from the closedness of W) we infer that $x + \imath y \in W$. This implies $x \in \Delta$ and $y = Sx$. Therefore, the operator $S\colon \Delta \to \Delta$ is closed.

7. *The vector space Δ under the norm $|||x||| = \|x\| + \|Sx\|$ is a Banach space.*

This follows immediately from the fact that S is a closed operator.

8. *The operator $S\colon \Delta \to \Delta$ is continuous if and only if $\Delta = X$.*

If $\Delta = X$, then the continuity of S follows from the closed graph theorem. For the converse, assume that $S\colon \Delta \to \Delta$ is continuous. Then, S as an operator from Δ to X is uniformly continuous, and so (since Δ is dense in X) it has a continuous linear extension $S_1\colon X \to X$. Now let $x \in X$. Pick a sequence $\{x_n\} \subseteq \Delta$ such that $x_n \to x$, and note that the sequence $\{x_n + \imath Sx_n\} \subseteq W$ satisfies $x_n + \imath Sx_n \to x + \imath S_1x$ in $X_{\mathbf{c}}$. This implies $x + \imath S_1x \in W$, and so $x \in \Delta$. Therefore, $\Delta = X$.

We are now in the position to state two consequences of the preceding discussion.

Lemma 4. *If the operator $S\colon \Delta \to \Delta$ is continuous, then $T_{\mathbf{c}}\colon W \to W$ has no non-trivial closed invariant subspaces, i.e., W is a minimal closed invariant subspace of $T_{\mathbf{c}}$.*

Proof. Since, S is continuous, it follows from (**8**) that

$$W = \big\{x + \imath Sx\colon\ x \in X\big\}\,.$$

Now assume that a non-zero closed subspace W_1 of W is $T_{\mathbf{c}}$-invariant. As before, there is a dense vector subspace Δ_1 of X and a linear operator $S_1\colon \Delta_1 \to \Delta_1$ such that $W_1 = \big\{x + \imath S_1x\colon\ x \in \Delta_1\big\}$. It follows that $S_1x = Sx$ for each $x \in \Delta_1$. This implies that $S_1\colon \Delta_1 \to \Delta_1$ is continuous, and as in (**8**) we must have $\Delta_1 = X$. Therefore, $W_1 = \big\{x + \imath Sx\colon\ x \in X\big\} = W$, and so W is minimal. ■

Lemma 5. *Conjecture* 3 *is false if and only if there exists a closed operator $S\colon \Delta \to \Delta$ that commutes with T and satisfies $S^2 = -I$.*[2]

[1] Recall that an operator $R\colon V \to X$, where V is a vector subspace of X, is said to be **closed** if its graph $\{(v, Rv)\colon\ v \in V\}$ is a closed subset of $X \times X$.

[2] As usual, an operator $R\colon V \to V$, where V is a vector subspace of X, is said to **commute** with T if V is T-invariant and $ST = TS$ holds true on V.

Proof. The "only if" part follows from the above discussion. Now assume the existence of the operator $S\colon \Delta \to \Delta$ with the above properties. Let

$$W = \{x + \imath Sx\colon\ x \in \Delta\}.$$

Since S is closed, it follows that W is a non-zero closed vector subspace of $X_{\mathbf{c}}$ that is different than $X_{\mathbf{c}}$. Now note that W is $T_{\mathbf{C}}$-invariant. ∎

4. Complex structures and Conjecture 3

We shall discuss in this section the concept of a complex structure for a real Banach space and present its connection with our basic problem. We start with its definition.

Definition 6. *A real Banach space X is said to admit a **complex structure**, if the real scalar multiplication $(\lambda, x) \mapsto \lambda x$ of X can be extended to a complex multiplication on X so that:*

(a) *X with the extended scalar multiplication and the original addition operation is a complex vector space, and*
(b) *the complex vector space X is a Banach space under a new norm that when restricted to the real Banach space X is equivalent to the original norm of X.*

That is, a complex structure on a real Banach space X is achieved if one can define a complex multiplication on X (i.e., a map $(\lambda, x) \mapsto \lambda x$ from $\mathbb{C} \times X$ to X) making X a Banach space over $\mathbb{C}$ in such a way that the new multiplication agrees with the original on X and its norm induces an equivalent norm on X. It should be obvious that every complex Banach space X considered as a real space admits a complex structure—namely, its original structure.

The real Banach spaces that admit a complex structure are characterized as follows.

Lemma 7. *A real Banach space X admits a complex structure if and only if there exists a bounded operator $S\colon X \to X$ satisfying $S^2 = -I$. Moreover, we have the following.*

(1) *If X admits a complex structure, then the operator $S\colon X \to X$, defined by $Sx = \imath x$, is a bounded operator on X and satisfies $S^2 = -I$.*
(2) *If an operator $S\colon X \to X$ satisfies $S^2 = -I$, then by defining*
   (a) *the complex scalar product by $(\alpha + \imath\beta)x = \alpha x + \beta Sx$, i.e., by letting $\imath x = Sx$, and*
   (b) *the norm on X by $\|x\|_{\mathbb{C}} = \sup\{\|e^{\imath\theta}x\|\colon\ 0 \le \theta \le 2\pi\}$,*

   we obtain a complex structure on X.

Proof. The proof is straightforward. Notice however that from the definition of the norm $\|\cdot\|_{\mathbb{C}}$ for each $x \in X$ we have

$$\|x\| \le \|x\|_{\mathbb{C}} \le (1 + \|S\|)\|x\|\,, \tag{✠}$$

and so $\|\cdot\|_{\mathbb{C}}$ indeed induces an equivalent norm to $\|\cdot\|$ on the real vector space X. ∎

Thus, there is a one-to-one correspondence between the complex structures of a real Banach space X and the bounded operators $S \in \mathcal{L}(X)$ satisfying $S^2 = -I$. Moreover, if a bounded operator $S\colon X \to X$ satisfies $S^2 = -I$, then the complex structure it generates on X is given by

$\imath x = Sx$. In other words, the operator S applied to any vector x plays the role of multiplication by $\imath$ which is interpreted geometrically as the "rotation" of the vector x by 90^{o}. For this reason, we shall call any bounded operator $S\colon X \to X$ on a real vector space X satisfying $S^2 = -I$ a **90^{o}-rotation**.

Definition 8. *If $S\colon X \to X$ is a 90^{o}-rotation on a real Banach space X, then the complex structure generated on X by S will be denoted by X_S.*

Two 90^{o}-rotations $S, T\colon X \to X$ on a real Banach space are ***comparable*** *(resp.* ***incomparable****) if the complex Banach spaces X_S and X_T are isomorphic (resp. non-isomorphic).*

In the class of finite dimensional vector spaces, only the ones with even dimension admit complex structures—in which case all 90^{o}-rotations are comparable. Indeed, if $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is a 90^{o}-rotation, then (viewing S as an $n \times n$ real matrix) we get $0 < (\det S)^2 = \det(S^2) = \det(-I) = (-1)^n$, from which it follows that n is even. The matrices below are 90^{o}-rotations in $\mathbb{R}^2$ and $\mathbb{R}^4$.

$$S_1 = \begin{bmatrix} \sqrt{3} & 2 \\ -2 & -\sqrt{3} \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} \sqrt{8} & 3 \\ -3 & -\sqrt{8} \end{bmatrix},$$

$$R_1 = \begin{bmatrix} \sqrt{3} & 2 & 0 & 0 \\ -2 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 2 \\ 0 & 0 & -2 & -\sqrt{3} \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} \sqrt{3} & 2 & 0 & 0 \\ -2 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{8} & 3 \\ 0 & 0 & -3 & -\sqrt{8} \end{bmatrix}.$$

For a Banach space with infinitely many non-comparable 90^{o}-rotation see [4].

Now let $S\colon X \to X$ be a 90^{o}-rotation on a real Banach space. Clearly, a subspace W of X is S-invariant if and only if it is invariant under complex multiplication on X_S or, equivalently, if W is a vector subspace of X_S. Thus, we have the following result.

Lemma 9. *If $S\colon X \to X$ is a 90^{o}-rotation on a real Banach space, then:*

(a) *The vector subspaces of X_S are exactly the S-invariant subspaces of X.*
(b) *If X has dimension greater then two, then S has non-trivial closed invariant subspaces. In fact, for every non-zero $x \in X$ the linear span of x and Sx in X is invariant under S.*

The commutant of a 90^{o}-rotation coincides with $\mathcal{L}(X_S)$.

Lemma 10. *Let S be a 90^{o}-rotation on a real a Banach space X. Then a bounded operator T on X defines a bounded operator on the complex Banach space X_S if and only if T commutes with S. In particular, we have $\mathcal{L}(X_S) = \{S\}'$, the commutant of S in $\mathcal{L}(X)$.*

Proof. Note that an operator $T \in \mathcal{L}(X)$ belongs to $\mathcal{L}(X_S)$ if and only if it is complex-linear (boundedness follows from (✠) in the proof of Lemma 7), i.e., if and only if $T(\imath x) = \imath Tx$. However, the latter is equivalently to $TSx = STx$ for every $x \in X$, i.e., $TS = ST$. ■

Corollary 11. *If a bounded operator on an infinite dimensional real Banach space has no non-trivial closed invariant subspaces, then it does not commute with any 90^{o}-rotation of the space.*

Note also that if T satisfies an irreducible quadratic equation with real coefficients, then for some $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$ the operator $S = \alpha T + \beta I$ satisfies $S^2 = -I$. This implies that if

$T \in \mathcal{L}(X)$ satisfies an irreducible quadratic equation and X has dimension greater than two, then Lemma 9(b) guarantees that T has plenty of closed invariant subspaces, including 2-dimensional ones.

Now let us connect the preceding discussion with Lemma 4. If the operator S is continuous (and hence defined on all of X), then since $S^2 = -I$, it determines a complex structure on X. Furthermore, since T commutes with S we have $T \in \mathcal{L}(X_S)$. Conversely, if there is a complex structure on X given by some $S \in \mathcal{L}(X)$ such that $T \in \mathcal{L}(X_S)$, then T commutes with S, and the non-trivial closed subspace $W = \{x + \imath Sx\colon\ x \in X\}$ of $X_{\mathbf{c}}$ is $T_{\mathbf{c}}$-invariant.

The preceding discussion can be summarized in the following result.

Theorem 12. *Suppose that T is a continuous operator on a real Banach space with no invariant subspaces. Then the following are true.*

(1) *The non-trivial closed invariant subspaces of $T_{\mathbf{c}}$ are precisely the graphs of the closed operators S satisfying*

$$\operatorname{Dom} S = \operatorname{Range} S, \quad ST = TS, \quad \textit{and} \quad S^2 = -I\,.$$

In particular, $T_{\mathbf{c}}$ has no non-trivial closed invariant subspaces if and only if there are no such operators S.

(2) *If there is a continuous operator $S \in \{T\}'$ satisfying $S^2 = -I$, then its graph is a minimal non-trivial closed invariant subspace of $T_{\mathbf{c}}$.*

Regarding the connection between complex structures and operators satisfying $S^2 = -I$ see, in particular, [4, 9, 22, 23, 24]. It is proved in [8, 22] that Lomonosov's Theorem [10] remains valid for an operator on a real Banach spaces if and only if the operator satisfies no irreducible quadratic equation. For some extensions of Lomonosov's invariant subspace theorem to the setting of Banach lattices see [2, 3] and [1].

Here is an example where Lomonosov's Theorem is not applicable neither in the real nor in the complex case.

Example 13. Let $S\colon X \to X$ be any operator satisfying $S^2 = -I$. Pick any non-zero vector $x \in X$ and let $f \in X^*$ be a functional satisfying $f(x) = 1$ and $f(Sx) = 0$. Consider the finite rank operator K defined by $K = x \otimes S^* f + Sx \otimes f$.

Notice that S and K commute, and K is compact. Suppose that T is an operator commuting with S. At a first glance, T seems to satisfy the hypotheses of Lomonosov's theorem. However, in the real case, one cannot employ Lomonosov's theorem to find invariant subspaces for T, because S satisfies the irreducible quadratic equation $S^2 = -I$. If one goes to the complex structure X_S on X generated by S, then K is a compact linear operator on X_S, T is a continuous linear operator on X_S, and we still have $TS = ST$ and $SK = KS$. However, Lomonosov's Theorem is still not applicable, because S is now the scalar operator $\imath I$, i.e., $Sx = \imath x$ for all $x \in X$! ■

Finally, suppose there is a closed operator S as in Theorem 12(1). Let $\Delta = \operatorname{Dom} S = \operatorname{Range} S$. Define a new norm on Δ by

$$\|x\|_S = \sup\bigl\{\|ax + bSx\|\colon\ a, b \in \mathbb{R} \text{ and } a^2 + b^2 = 1\bigr\}\,.$$

Clearly, this is a norm on Δ and $\|S\|_S = 1$. Moreover, S defines a complex structure on $(\Delta, \|\cdot\|_S)$. Finally, since $\|x\| \le \|x\|_S$, the inclusion map from $(\Delta, \|\cdot\|_S)$ to X is continuous.

5. An Algebraic Approach via Transitive algebras

In this section we mention several related problems in the algebraic version of the Invariant Subspace Problem. We remind the reader that X denotes an infinite dimensional separable Banach space. An algebra $\mathcal{A}$ of operators on a Banach space is said to be **transitive** if for every two non-zero vectors x and y and every $\varepsilon > 0$ there exists an operator $A \in \mathcal{A}$ such that $\|Ax - y\| < \varepsilon$. It is easy to see that $\mathcal{A}$ is transitive if and only if it has no common non-trivial closed invariant subspaces, i.e., $\operatorname{Lat}\mathcal{A} = \{0, X\}$. The orbit of a vector x under an algebra $\mathcal{A}$ is the set $\mathcal{A}x = \{Ax\colon\ A \in \mathcal{A}\}$. Clearly, $\mathcal{A}$ is transitive if and only if the orbit of every non-zero vector is dense. It is easy to see that a bounded operator T has no invariant subspaces if and only if the algebra $\operatorname{Alg} T$ generated by T (consisting of all the polynomials of T) is transitive. Furthermore, T has no hyperinvariant subspaces if and only if $\{T\}'$ is transitive.

We can view $X_{\mathbf{c}}$ as $X \oplus X$, and complex multiplication as the "complex multiplication" operator $(x, y) \mapsto \mathcal{J}(x, y) = (-y, x)$. Then W is a (complex) vector subspace of $X_{\mathbf{c}}$ if and only if W is a vector subspace of $X \oplus X$ that is invariant under $\mathcal{J}$. Furthermore, $W \in \operatorname{Lat} T_{\mathbf{c}}$ implies that W is invariant under $T^{(2)} = T \oplus T$. Thus, $\operatorname{Lat} T_{\mathbf{c}} = \operatorname{Lat}\mathcal{A}$, where $\mathcal{A}$ is the subalgebra of $\mathcal{L}(X \oplus X)$ generated by $\mathcal{J}$ and $T^{(2)}$. In connection with Theorem 12(1), it should be mentioned that it is well known (see [5, 13]) that given a continuous operator T and a closed operator S, then S commutes with T if and only if the graph of S is invariant under $T^{(2)}$.

More generally, given an operator T in $\mathcal{L}(X)$ and $n \in \mathbb{N}$, let $T^{(n)} = T \oplus \cdots \oplus T$, the direct sum of n copies of T, so that $T^{(n)} \in \mathcal{L}(X^n)$. If $\mathcal{A}$ is a subalgebra of $\mathcal{L}(X)$, then $\mathcal{A}^{(n)} = \{A^{(n)} \mid A \in \mathcal{A}\}$ is a subalgebra of $\mathcal{L}(X^n)$. We say that $\mathcal{A}$ is n**-transitive** if the orbit under $\mathcal{A}^{(n)}$ of every linearly independent n-tuple in X^n is dense. Equivalently, for every set $\{x_1, \dots, x_n\}$ of n linearly independent vectors, every collection $\{y_1, \dots, y_n\}$ of n vectors, and every $\varepsilon > 0$ there exists $A \in \mathcal{A}$ such that $\|Ax_i - y_i\| < \varepsilon$ for each $i = 1, \dots, n$. Clearly, $\mathcal{A}$ is transitive if and only if it is 1-transitive.

Similarly, we say that an algebra $\mathcal{A}$ is **strictly transitive** if for every two non-zero vectors x and y there exists an operator $A \in \mathcal{A}$ such that $Ax = y$. Again, it is rather easy to check that $\mathcal{A}$ is strictly transitive if and only if it has no common invariant vector (not necessarily closed) subspaces (and also if and only if $\mathcal{A}x = X$ for every non-zero vector x). Clearly, strict transitivity implies transitivity. Given $n \in \mathbb{N}$ and a subalgebra $\mathcal{A}$ of $\mathcal{L}(X)$, we say that $\mathcal{A}$ is n**-strictly transitive** if the orbit under $\mathcal{A}^{(n)}$ of every linearly independent n-tuple is all of X^n. Equivalently, for every set $\{x_1, \dots, x_n\}$ of n linearly independent vectors and every collection $\{y_1, \dots, y_n\}$ of n vectors there exists $A \in \mathcal{A}$ such that $Ax_i = y_i$ for each $i = 1, \dots, n$. Clearly, $\mathcal{A}$ is strictly transitive if and only if it is 1-strictly transitive.

A classical theorem of Rickart [17] and Yood [26] asserts that a strictly transitive algebra of operators on a complex Banach space is WOT-dense[3] in $\mathcal{L}(X)$. Here is an outline of the proof given in [13]. It is shown that if $\mathcal{A}$ is n-strictly transitive for some $n \in \mathbb{N}$, then it is $(n+1)$-transitive. It follows by induction that a strictly transitive algebra is n-strictly transitive and, therefore, n-transitive for every $n \in \mathbb{N}$. To complete the proof, one applies the fact that if $\mathcal{A}$ is n-transitive for all n, then $\mathcal{A}$ is WOT-dense in $\mathcal{L}(X)$.

A similar reasoning won't work for transitive algebras because n-transitivity does not necessarily imply $(n+1)$-transitivity. Indeed, the algebra generated by an operator with no invariant subspaces is 1-transitive. It is not 2-transitive, however, because no commutative algebra is 2-transitive ([5],

[3]By WOT we denote the weak operator topology on $\mathcal{L}(X)$.

see also [19, Theorem 4.9]). It is not known whether this is the case for $n > 1$. The following conjecture is known as $(n+1)$-transitivity problem.

Conjecture 14. *If $\mathcal{A}$ is n-transitive for some $n > 1$, then it is $(n+1)$-transitive.*

As far as we know, it is not known whether the theorem of Rickart and Yood remains true for real Banach spaces.

Conjecture 15. *If $\mathcal{A}$ is a strictly transitive algebra of operators on a real Banach space X then $\mathcal{A}$ is WOT-dense in $\mathcal{L}(X)$.*

Finally, notice that for every operator T on a real or complex Banach space, the commutant $\{T\}'$ of T is a WOT-closed algebra, containing the WOT-closed algebra generated by T, i.e., $\overline{\operatorname{Alg} T}^{WOT} \subseteq \{T\}'$.

Conjecture 16. *If T has no non-trivial closed invariant subspaces, then $\overline{\operatorname{Alg} T}^{WOT} = \{T\}'$.*

As far as we know, this conjecture is open for Read's operators too. However, the converse is false: $\overline{\operatorname{Alg} T}^{WOT} = \{T\}'$ is satisfied for the right shift on ℓ_p.

6. The Closure Property

Let X be a real or complex Banach space. Given a subalgebra $\mathcal{A}$ of $\mathcal{L}(X)$ and a linear operator $S\colon Y \to X$ defined on a linear (not necessarily closed) subspace Y of X, we say that $\mathcal{A}$ commutes with S if Y is invariant under $\mathcal{A}$ and $ASx = SAx$ for every $x \in Y$ and $A \in \mathcal{A}$. An algebra $\mathcal{A}$ is said to have the **Closure Property** if every linear operator commuting with $\mathcal{A}$ is closable. Similarly, an operator $T \in \mathcal{L}(X)$ has the **Closure Property** if every linear operator commuting with T is closable or, equivalently, if the algebra generated by T has the Closure Property. The Closure Property was introduced in [20] and the following conjecture was posed there.

Conjecture 17. *Every transitive algebra with the Closure Property is WOT-dense.*

The motivation for this conjecture is the following. It is known (see, e.g., [13]) that every transitive algebra of operators on a Hilbert space that contains a maximal Abelian self-adjoint algebra (*m.a.s.a.*) is WOT-dense in $\mathcal{L}(H)$. It is also known that every m.a.s.a. has the Closure Property. It is natural to ask, therefore, if the Closure Property alone is already sufficient to guarantee that a transitive algebra is WOT-dense. Furthermore, this conjecture makes sense for Banach spaces too.

Notice that an affirmative answer to Conjecture 17 would imply the affirmative answer to the following conjecture.

Conjecture 18. *No commutative algebra with the Closure Property is transitive.*

We claim that this conjecture is true for algebras generated by a single operator.

Lemma 19. *Suppose that T is an operator on a real or complex Banach space with no non-trivial closed invariant subspaces. Suppose that $f(T)x = 0$ for an entire function[4] f and some non-zero vector x. Then f is identically zero.*

[4] In the real case we assume that f has only real coefficients in its Taylor series expansion at zero, as $f(T)$ makes no sense otherwise.

Proof. Suppose $f(z) = \sum_{i=0}^{\infty} a_i z^i$ and note that $f(T) = \sum_{i=0}^{\infty} a_i T^i$ commutes with T. Therefore, the kernel of $f(T)$ is T-invariant. Since it is non-empty, it must be the whole space, that is, $f(T) = 0$ holds.

According to [6, Theorem VII.3.16], $f(T) = 0$ implies that $f(z)$ is zero on an open set containing $\sigma(T)$ except for a finite set of poles F, and f vanishes at the poles.[5] If $\sigma(T) \setminus F$ is non-empty, then f vanishes on an open set, hence f is identically zero, and we are done. Therefore, $\sigma(T) = F$.

Let $\lambda \in F$. Then $f(\lambda) = 0$, so that $f(z) = (z-\lambda)^n h(z)$ for some $n > 0$ and some entire function $h(z)$ such that $h(\lambda) \neq 0$. In the complex case, this yields $0 = f(T) = (T - \lambda I)^n h(T)$. It follows that $h(T) = 0$, as otherwise $T - \lambda I$ has nontrivial kernel, which would be invariant under T.

Similarly, in the real case we would have $(T_{\mathbf{c}} - \lambda I)^n h(T_{\mathbf{c}}) = 0$, and this would again imply $h(T_{\mathbf{c}}) = 0$. Indeed, otherwise there exists $y \in X_{\mathbf{c}}$ such that $T_{\mathbf{c}} y = \lambda y$. Then $\operatorname{span}\{\Re y, \Im y\}$ is a non-trivial closed subspace of X invariant under T. In either the real or the complex case, this implies that h vanishes on F, contrary to $h(\lambda) \neq 0$. ∎

Theorem 20. *Suppose that T is an operator on a real or a complex Banach space with no non-trivial closed invariant subspaces. If $\mathcal{A}$ is the algebra of all polynomials of T, then for every non-zero vector x there exists a non-zero vector y such that $\mathcal{A}x \cap \mathcal{A}y = \{0\}$.*

Proof. We can assume without loss of generality that $\|T\| \leq 1$. Given a non-zero x, put $x_i = T^i x$ for $i = 0, 1, \dots$, then $\mathcal{A}x = \operatorname{span}\{x_i \colon i \geq 0\}$. Put $y = e^T x$ and note that by Lemma 19 we have $y \neq 0$. Suppose by way of contradiction that $\mathcal{A}x \cap \mathcal{A}y \neq \{0\}$. Then there exist polynomials p and q such that $p(T)x = q(T)y = q(T)e^T x$. Let $f(z) = p(z) - q(z)e^z$. It follows from Lemma 19 that $f(z)$ is identically zero, contradiction. ∎

We will also use the following observation from [20]. We include the proof for the convenience of the reader.

Lemma 21 ([20]). *Let $\mathcal{A}$ be a commutative transitive subalgebra of $\mathcal{L}(X)$ satisfying the Closure Property, and let x and y be two non-zero vectors. Then $\mathcal{A}x \cap \mathcal{A}y \neq \{0\}$.*

Proof. Suppose that $\mathcal{A}$ has the Closure Property, and that there exist non-zero x and y such that $\mathcal{A}x \cap \mathcal{A}y = \{0\}$. Let $Y = \mathcal{A}x \oplus \mathcal{A}y$, then Y is a vector subspace of X. Define $S \colon Y \to Y$ via $S \colon Ax + By \mapsto Bx + Ay$ for any $A, B \in \mathcal{A}$. It is easy to see that S is a well defined linear operator, and S commutes with $\mathcal{A}$. It follows that S is closable, hence $\overline{S}$ is a closed linear operator. Fix $A \in \mathcal{A}$ and note that $S(Ax + Ay) = Ax + Ay$. This implies that $\overline{S}$ has a non-trivial eigenspace. This eigenspace is closed because $\overline{S}$ is closed. However, one can easily verify that $\overline{S}$ also commutes with $\mathcal{A}$, so that its eigenspaces are invariant under $\mathcal{A}$. This contradicts the transitivity of $\mathcal{A}$. ∎

Theorem 20 and Lemma 21 yield the following.

Corollary 22. *If a bounded operator on a real or complex Banach space satisfies the Closure Property, then it has a non-trivial closed invariant subspace.*

[5] Recall that if T is an operator on a real Banach space, then $\sigma(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T_{\mathbf{c}}$ is not invertible on $X_{\mathbf{c}}$.

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Y.A. Abramovich (1945–2003)

C. D. Aliprantis, Department of Economics, Krannert School of Management, Purdue University, 403 West State Street, W. Lafayette, IN 47907-2056, USA
E-mail address: aliprantis@mgmt.purdue.edu

G. Sirotkin, Department of Mathematics, Northern Illinois University, DeKalb, IL 60115, USA
E-mail address: sirotkin@math.niu.edu

V. G. Troitsky, Department of Mathematics, University of Alberta, Edmonton, AB T6G 2G1, CANADA
E-mail address: vtroitsky@math.ualberta.ca