UNIQUE DETERMINATION OF CONVEX POLYTOPES BY NON-CENTRAL SECTIONS

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Abstract. A question of Barker and Larman asks whether convex bodies that contain a sphere of radius t in their interiors are uniquely determined by the volumes of sections by hyperplanes tangent to the sphere. We affirmatively solve this problem for convex polytopes.

1. Introduction

One of the central questions in geometric tomography is the unique determination of convex bodies from the size of their sections, projections, or other lower dimensional data. A classical result in this area states that origin-symmetric convex bodies are uniquely determined by the volumes of their central sections, see Funk [5], Lifshitz and Pogorelov [11], as well as Gardner's book [7, Sec. 7.2] for additional details. It is also known that this theorem does not hold in the absence of symmetry. What kind of information about sections of a convex body, not necessarily symmetric, would allow to determine the body uniquely? It was proved independently by Falconer [4] and Gardner [6] that any convex body is uniquely determined by the volumes of hyperplane sections through any two points in the interior of the body. A generalization of this result for fractional derivatives of the section function was obtained by Koldobsky and Shane [10]. Böröczky and Schneider [2] showed that a convex body is determined by the volumes and centroids of its hyperplane sections through the origin. For other results in this area, we refer the reader to the book [7].

In [1] Barker and Larman asked the following question. Let P and Q be convex bodies in $\mathbb{R}^n$ containing a sphere of radius t in their interiors. Suppose that for every hyperplane H tangent to the sphere we have $\mathrm{vol}_{n-1}(P\cap H) = \mathrm{vol}_{n-1}(Q\cap H)$. Does this mean that $P = Q$? In [1] the authors obtained several partial results. They showed that in $\mathbb{R}^2$ the uniqueness holds if one of the bodies is a Euclidean disk. In $\mathbb{R}^n$ they proved that the answer to this conjecture is affirmative if hyperplanes are replaced by planes of a larger codimension. However, the answer to the original question is still unknown, even in dimension 2.

In this paper we show that the answer to the problem is affirmative if both P and Q are convex polytopes in $\mathbb{R}^n$. The case $n = 2$ of this result

The author was partially supported by a grant from NSERC.

was recently obtained by Xiong, Ma and Cheung [13], but for the sake of completeness we include a proof here.

Before we proceed to proving the result, let us say a few words about analytic aspects of the problem. If $t = 0$, which corresponds to the case of hyperplane sections through the origin, the standard solution to the problem is based on the injectivity properties of the spherical Radon transform. Alternatively, one can use Fourier transform techniques to prove this result, see [9, p.55]. When $t > 0$, many people tried to find an analytic approach to the problem, but so far all such attempts failed. That is why we take a different path and try methods of discrete geometry. Finally, let us emphasize once again that in our result the symmetry condition is not required, unlike in the case of sections through the origin.

For standard notions in geometric tomography and convex geometry the reader is referred to the books of Gardner [7], Gruber [8], Schneider [12].

2. Main result

Theorem 2.1. *Let P and Q be convex polytopes in $\mathbb{R}^n$ containing a sphere of radius t in their interiors. If*

$$\mathrm{vol}_{n-1}(P \cap H) = \mathrm{vol}_{n-1}(Q \cap H)$$

for every hyperplane H tangent to the sphere, then

$$P = Q.$$

Proof. i) We first consider the planar case. To reach a contradiction, suppose that the polygons P and Q are different. Since neither of them can be a subset of the other, there is a point where their boundaries intersect. Let u_1 be a common point of the boundaries of P and Q such that their edges meet transversally at u_1. Let these edges be given correspondingly by the lines

$$x = u_1 + l_1 s_1 \qquad \text{and} \qquad x = u_1 + m_1 \tau_1, \tag{1}$$

where $l_1, m_1 \in S^1$ are the directions of the edges, and s_1, τ_1 are the parameters along the lines. Note that by our assumption l_1 is not parallel to m_1.

Throughout the proof of part (i) we will be using the following notation for a straight line orthogonal to $\xi \in S^1$ and distance t from the origin.

$$L(\xi) = \{x \in \mathbb{R}^2 : \langle x, \xi \rangle = t\}.$$

Let S_t denote the circle of radius t from the hypotheses of the theorem. Consider a line $L(\xi_0)$ through u_1 tangent to S_t. This line intersects the boundaries of P and Q at another common point u_2. If $\xi \in S^1$ is close to ξ_0 and the line $L(\xi)$ intersects the edges given by (1), then this line also intersects another pair of edges adjacent to u_2. Let these edges be given by the equations

$$x = u_2 + l_2 s_2 \qquad \text{and} \qquad x = u_2 + m_2 \tau_2, \tag{2}$$

where, as before, l_2 and m_2 are non-collinear unit vectors in the directions of the edges, and s_2 and τ_2 are parameters.

Let Λ denote an open arc of the unit circle S^1 comprising those vectors ξ that are sufficiently close to ξ_0 and such that the lines $L(\xi)$ intersect the edges of P and Q given by (1) and (2) and do not meet any vertices of P and Q.

Denoting by $p_1 = p_1(\xi)$, $p_2 = p_2(\xi)$, $q_1 = q_1(\xi)$, $q_2 = q_2(\xi)$ the points of intersection of $L(\xi)$, $\xi \in \Lambda$, and the corresponding edges of P and Q, we get

$$p_i = u_i + l_i \frac{t - \langle u_i, \xi\rangle}{\langle l_i, \xi\rangle}, \qquad q_i = u_i + m_i \frac{t - \langle u_i, \xi\rangle}{\langle m_i, \xi\rangle}, \qquad i = 1, 2.$$

Note that $P \cap L(\xi) = [p_1(\xi), p_2(\xi)]$ and $Q \cap L(\xi) = [q_1(\xi), q_2(\xi)]$. Now we can write the equality between the sections of P and Q as the equality between the vectors $p_1 - p_2$ and $q_1 - q_2$,

$$l_1 \frac{t - \langle u_1, \xi\rangle}{\langle l_1, \xi\rangle} - l_2 \frac{t - \langle u_2, \xi\rangle}{\langle l_2, \xi\rangle} = m_1 \frac{t - \langle u_1, \xi\rangle}{\langle m_1, \xi\rangle} - m_2 \frac{t - \langle u_2, \xi\rangle}{\langle m_2, \xi\rangle}. \tag{3}$$

Without loss of generality we may assume that the latter equality holds not only in Λ, but for all $\xi \in S^1 \setminus \{l_1^\perp, l_2^\perp, m_1^\perp, m_2^\perp\}$. Indeed, clearing the denominators and replacing ξ by $(\cos\phi, \sin\phi)$, we get two polynomials of $\cos\phi$ and $\sin\phi$ that are equal to each other in an open interval. Therefore, they must be equal everywhere.

Let us multiply equation (3) by $\langle l_1, \xi\rangle$ and consider ξ close to a unit vector $v \in l_1^\perp$. If none of the vectors l_2, m_1, m_2 is equal to l_1, then passing to the limit $\xi \to v$ we see that $l_1(t - \langle u_1, v\rangle) = 0$. The latter means that u_1 belongs to a line parallel to l_1 and tangent to S_t. This is impossible.

Therefore l_1 is parallel to one of the vectors l_2, m_1, m_2. By our assumption l_1 is not parallel to m_1. Suppose l_1 is parallel to m_2. By the same reasoning l_2 is parallel to m_1. From (3) we get

$$l_1 \frac{2t - \langle u_1 + u_2, \xi\rangle}{\langle l_1, \xi\rangle} = l_2 \frac{2t - \langle u_1 + u_2, \xi\rangle}{\langle l_2, \xi\rangle},$$

which means that l_1 and l_2 are parallel, and therefore l_1 and m_1 are parallel as well. This contradicts our assumption that the edges of P and Q meet transversally at u_1. Similarly, if we suppose that l_1 is parallel to l_2, then m_1 has to be parallel to m_2. As before, it is easy to see that all the vectors l_1, l_2, m_1, m_2 have to be parallel. We again reach a contradiction.

Thus, we conclude that P and Q are equal to each other.

ii) We now consider the case of $\mathbb{R}^n$, $n \geq 3$. To prove that P and Q are identical, we will show that P and Q have exactly the same vertices.

Suppose the contrary. Let u be a vertex of P that is not a vertex of Q. Throughout the proof the hyperplanes tangent to the sphere of radius t centered at the origin will be denoted by

$$H(\xi)$$

Consider a hyperplane $\langle x, \xi_0\rangle = t$ that contains u and is chosen in such a way that there is a facet F of P adjacent to u that has no common points with $\langle x, \xi_0\rangle = t$, other than u. The existence of such a hyperplane can be seen from the following argument. Consider a facet F containing u and consider the hyperplane E that contains F. Since F is a convex polytope in E and u is its vertex, there is a supporting $(n-2)$-dimensional plane to F in E that intersects F only at u. Define $\langle x, \xi_0\rangle = t$ to be a hyperplane in $\mathbb{R}^n$ that contains this supporting plane and is tangent to S_t, the sphere of radius t. After a small perturbation of ξ_0, if necessary, we can also assume that the hyperplane $\langle x, \xi_0\rangle = t$ does not contain any vertex of either P or Q, other than u, and that the normal vector to the facet F is not perpendicular to ξ_0.

For simplicity we will assume that $\xi_0 = e_n$. Let Λ be a spherical cap of small radius centered at e_n such that, for all $\xi \in \Lambda$, the hyperplanes $\langle x, \xi\rangle = t$ do not contain any vertex of P or Q except possibly u.

Note that $\mathrm{vol}_{n-1}(P \cap \{\langle x, \xi\rangle = t\}) = \mathrm{vol}_{n-1}(Q \cap \{\langle x, \xi\rangle = t\})$, $\xi \in \Lambda$, if and only if the projections of the sets $P \cap \{\langle x, \xi\rangle = t\}$ and $Q \cap \{\langle x, \xi\rangle = t\}$ onto the hyperplane $x_n = 0$ have the same $(n-1)$-dimensional volume. We will denote these projections by P_ξ and Q_ξ and their volumes by $V_P(\xi)$ and $V_Q(\xi)$ correspondingly. Let us compute the latter functions.

Denote the edges of P that intersect the hyperplane $x_n = t$ and do not contain the vertex u by

$$x = u_i + l_i s_i, \qquad i = 1, ..., K,$$

where l_i is a unit vector in the direction of the corresponding edge, s_i is a parameter, and u_i is a vertex of P.

The edges of P adjacent to the vertex u will be denoted by

$$x = u + l_i s_i, \qquad i = K+1, ..., M.$$

Since the hyperplane $x_n = t$ contains u and no other vertex of P, none of the latter edges lies in the hyperplane $x_n = t$ and so some of these edges lie above this hyperplane and some below the hyperplane. Let the former edges be indexed by $i = K+1, ..., L$ and the latter ones by $i = L+1, ..., M$.

The edges of Q that intersect the hyperplane $x_n = t$ we denote by

$$x = v_i + m_i \tau_i, \qquad i = 1, ..., N.$$

Denote by Λ_+ (correspondingly, Λ_-) the subset of those vectors $\xi \in \Lambda$ for which the hyperplane $\langle x, \xi\rangle = t$ does not contain u and intersects all the edges of P numbered $i = K+1, ..., L$ (correspondingly, numbered $i = L+1, ..., M$). In other words, for any $\xi \in \Lambda_+$, the hyperplane $H(\xi)$ intersects all the edges of P adjacent to u that are above the hyperplane $x_n = t$ and none of the ones that are below. Similarly, for any $\xi \in \Lambda_-$, the hyperplane $H(\xi)$ intersects all the edges of P adjacent to u that are below the hyperplane $x_n = t$ and none of the ones that are above. Note that Λ can be written as a union of mutually disjoint subsets

$$\Lambda = \Lambda_+ \cup \Lambda_- \cup \{\xi \in \Lambda : u \in H(\xi)\}.$$

Let $\bar{u}$, $\bar{u}_i$, $\bar{l}_i$, $\bar{v}_i$, $\bar{m}_i$, considered as $(n-1)$-dimensional vectors, be the orthogonal projections of the vectors u, u_i, l_i, v_i, m_i onto the hyperplane $e_n^\perp$.

Consider $\xi \in \Lambda_+$. Finding the points of intersection of $\langle x, \xi\rangle = t$ and the edges of P, and projecting them onto the hyperplane $e_n^\perp$, we get the vertices of the polytope P_ξ,

$$p_i = \bar{u}_i + \bar{l}_i \frac{t - \langle u_i, \xi\rangle}{\langle l_i, \xi\rangle}, \qquad i = 1, ..., K,$$
$$p_i = \bar{u} + \bar{l}_i \frac{t - \langle u, \xi\rangle}{\langle l_i, \xi\rangle}, \qquad i = K+1, ..., L.$$

Similarly, for $\xi \in \Lambda_-$, the vertices of P_ξ are

$$p_i = \bar{u}_i + \bar{l}_i \frac{t - \langle u_i, \xi\rangle}{\langle l_i, \xi\rangle}, \qquad i = 1, ..., K,$$
$$p_i = \bar{u} + \bar{l}_i \frac{t - \langle u, \xi\rangle}{\langle l_i, \xi\rangle}, \qquad i = L+1, ..., M.$$

For $\xi \in \Lambda$, the vertices of Q_ξ are given by

$$q_i = \bar{v}_i + \bar{m}_i \frac{t - \langle v_i, \xi\rangle}{\langle m_i, \xi\rangle}, \qquad i = 1, ..., N.$$

In order to compute $V_Q(\xi)$ we first fix a triangulation of Q_{ξ_0}. We will split the $(n-1)$-dimensional polytope Q_{ξ_0} into simplices in such a way that each simplex has one vertex at the origin and the other vertices belonging to the vertex set of the polytope Q_{ξ_0}. This can be done by first triangulating the $(n-2)$-skeleton of Q_{ξ_0} (see [8, p.257] for details) and then connecting the origin with the vertices of this triangulation.

Two polytopes are said to be combinatorially equivalent if there is a bijection between their vertex sets that maps (the sets of vertices of) faces to (the sets of vertices of) faces. If ξ is close to ξ_0, then $Q \cap \{\langle x, \xi\rangle = t\}$ is combinatorially equivalent to $Q \cap \{\langle x, \xi_0\rangle = t\}$. The bijection between their vertices is given by the incidence to the same edge of Q. Consequently, Q_ξ and Q_{ξ_0} are also combinatorially equivalent, and therefore Q_ξ inherits the triangulation from Q_{ξ_0}.

If a simplex Δ_{n-1} in the triangulation of Q_ξ has, for example, points q_1, q_2,..., q_{n-1} and the origin as its vertices, then its volume is given by the determinant

$$\frac{1}{n-1}|[q_1, q_2, \ldots, q_{n-1}]|.$$

The points q_i are assumed to have $n-1$ coordinates and be ordered in such a way that the determinant is positive.

Therefore,

$$V_Q(\xi) = \frac{1}{n-1} \sum_I |[q_{i_1}, q_{i_2}, \dots, q_{i_{n-1}}]|,$$

where the summation runs over all $(n-1)$-tuples of indices $(i_1, ..., i_{n-1})$ that correspond to the simplices of the chosen triangulation. Note that this formula holds for all $\xi \in \Lambda$, since the combinations of indices that correspond to the simplices of the triangulation of Q_ξ are the same for all $\xi \in \Lambda$.

In order to compute $V_P(\xi)$ we will need a triangulation of P_ξ. We use the idea from [8, p.257], which, as indicated there, originally belongs to Edmonds [3]. Recall that we ordered the vertices of P_ξ as follows: first we put the vertices that are associated with the edges of P that do not contain u, and then those that do contain. The triangulation is done by induction. If the $(k-1)$-skeleton is triangulated, then the triangulation of the k-skeleton is performed as follows. In each k-dimensional face G take the vertex with the lowest index, let us call this vertex p, and consider convex hulls of p and the simplices of the triangulation of the $(k-1)$-dimensional faces of G, disjoint from p. The triangulation thus constructed agrees with the triangulation of the $(k-1)$-dimensional faces of G that contain p. Using this procedure, we triangulate the $(n-2)$-skeleton of P_ξ. Finally, we connect the origin with the simplices of the triangulation of the $(n-2)$-skeleton of P_ξ and obtain a triangulation of P_ξ.

One key feature of this triangulation is that each simplex has one vertex at the origin and all other vertices belong to the vertex set of P_ξ. Another feature can be described as follows. Consider two polytopes P_{ξ_1}, $\xi_1 \in \Lambda_+$, and P_{ξ_2}, $\xi_2 \in \Lambda_-$. There is a correspondence between those faces of P_{ξ_1} and P_{ξ_2} that contain only vertices with indices not exceeding K, i.e. not adjacent to u. This correspondence is given by a one-to-one map between their vertices, which sends vertices with the same index to each other, and the triangulations of the faces agree under this map.

We now turn to $V_P(\xi)$. If $\xi \in \Lambda_+$ (respectively, Λ_-), then we will write $V_P(\xi) = V_+(\xi) + V(\xi)$, (respectively, $V_P(\xi) = V_-(\xi) + V(\xi)$) where V_+ (respectively, V_-) is the total volume of the simplices in the triangulation of P_ξ that have at least one vertex p_i with index i larger that K, and V is the total volume of all other simplices. Note that, because of the choice of the triangulation, $V(\xi)$ has the same formula for both Λ_+ and Λ_-.

Since $V_P(\xi) = V_Q(\xi)$ for all $\xi \in \Lambda$, we have $V_+(\xi) + V(\xi) = V_Q(\xi)$ for $\xi \in \Lambda_+$, and $V_-(\xi)+V(\xi) = V_Q(\xi)$ for $\xi \in \Lambda_-$. Moreover, we can assume that these equalities hold for all $\xi \in S^{n-1}$ without a finite number of great subspheres. Indeed, consider, for example, the equality $V_+(\xi) + V(\xi) = V_Q(\xi)$. Both sides are rational functions of $\xi_1, \dots, \xi_n$. Clearing the denominators and moving everything to one side, we get a polynomial in these variables that is equal to zero in an open subset of the sphere. To reach a contradiction, suppose that this polynomial is not equal to zero at some point on the sphere. Consider the intersection of the sphere and a two-dimensional plane that passes through this point and the interior of Λ_+. The problem reduces to a polynomial of two variables restricted to a circle. Since this polynomial is equal to zero on an open arc, it is zero on the whole circle, as we saw above in part (i). Contradiction. Thus, we conclude that the equality $V_+(\xi) + V(\xi) = V_Q(\xi)$ holds for all ξ on the sphere except for those where the denominators vanish.

In view of these comments we get that $V_+(\xi) + V(\xi) = V_-(\xi) + V(\xi)$, or simply

$$V_+(\xi) = V_-(\xi)$$

for all $\xi \in S^{n-1}$ except finitely many great subspheres.

Recall that $V_+(\xi)$ and $V_-(\xi)$ are given by the sums of the determinants of the form

$$|[\bar{u}_{i_1} + \bar{l}_{i_1} \frac{t - \langle u_{i_1}, \xi \rangle}{\langle l_{i_1}, \xi \rangle}, \ldots, \bar{u}_{i_{n-1}} + \bar{l}_{i_{n-1}} \frac{t - \langle u_{i_{n-1}}, \xi \rangle}{\langle l_{i_{n-1}}, \xi \rangle}]|$$

Let η be the normal vector to the facet F, fixed in the beginning. Consider the following curve on the sphere:

$$\xi(\epsilon) = \frac{\eta + \lambda\epsilon}{|\eta + \lambda\epsilon|},$$

for small enough ϵ and a vector λ chosen in such a way that it is orthogonal to η, transversal to the hyperplanes $\langle l_i, \xi \rangle = 0$ for all l_i that are orthogonal to η, and if the tail of λ is placed at u, then the tip of λ lies inside the facet F.

Now put $\xi(\epsilon)$ into the equality

$$V_+(\xi) = V_-(\xi),$$

multiply both sides by ϵ^{n-1}, and send $\epsilon \to 0$.

Clearly, we are interested only in those terms that have a product of $n-1$ factors in the denominator:

$$|[\bar{l}_{i_1}, \ldots, \bar{l}_{i_{n-1}}]| \cdot \frac{t - \langle u_{i_1}, \xi \rangle}{\langle l_{i_1}, \xi \rangle} \cdots \frac{t - \langle u_{i_{n-1}}, \xi \rangle}{\langle l_{i_{n-1}}, \xi \rangle}, \tag{4}$$

all others will just vanish.

Among such terms those that survive must have all l_{i_1},..., $l_{i_{n-1}}$ perpendicular to η. The simplex with the vertices

$$u_{i_1} + l_{i_1} \frac{t - \langle u_{i_1}, \xi \rangle}{\langle l_{i_1}, \xi \rangle}, \ldots, u_{i_{n-1}} + l_{i_{n-1}} \frac{t - \langle u_{i_{n-1}}, \xi \rangle}{\langle l_{i_{n-1}}, \xi \rangle}$$

lies in some facet of P. This facet must also contain the edges with equations

$$u_{i_1} + l_{i_1} s_{i_1}, \ldots, u_{i_{n-1}} + l_{i_{n-1}} s_{i_{n-1}}. \tag{5}$$

By the definition of V_+ and V_-, the vertex u belongs to at least one of these edges. The only facet of P orthogonal to η and containing u is F. Therefore, the vertex u belongs to all of the edges (5), and thus, $u_{i_1} = \cdots = u_{i_{n-1}} = u$.

In the limit all terms of the form (4) give

$$|[\bar{l}_{i_1}, \dots, \bar{l}_{i_{n-1}}]| \cdot \frac{(t - \langle u, \eta\rangle)^{n-1}}{\langle l_{i_1}, \lambda\rangle \cdots \langle l_{i_{n-1}}, \lambda\rangle},$$

Note that $t - \langle u, \eta\rangle \neq 0$, otherwise this would mean that the hyperplane with normal η through u, which contains the facet F, is tangent to the sphere of radius t. This is impossible since P is convex and the sphere is contained in the interior of P.

Second of all, $\langle l_{i_k}, \lambda\rangle > 0$ for all k, due to the choice of λ.

Third, all the determinants $|[\bar{l}_{i_1}, \dots, \bar{l}_{i_{n-1}}]|$ are non-zero and have the same sign. To show this, recall that the vectors $\bar{u} + \bar{l}_{i_k} \frac{t - \langle u_{i_k}, \xi\rangle}{\langle l_{i_k}, \xi\rangle}$ in the determinants

$$|[\bar{u} + \bar{l}_{i_1} \frac{t - \langle u_{i_1}, \xi\rangle}{\langle l_{i_1}, \xi\rangle}, \dots, \bar{u} + \bar{l}_{i_{n-1}} \frac{t - \langle u_{i_{n-1}}, \xi\rangle}{\langle l_{i_{n-1}}, \xi\rangle}]| \tag{6}$$

were ordered in such a way that the determinants were positive. These vectors connect the origin with the vertices of the triangulation of the projection of $F \cap \{\langle x, \xi\rangle = t\}$ onto $e_n^\perp$. On the other hand, the vectors $\bar{l}_{i_k}$ connect the point $\bar{u}$ with the vertices of the same triangulation. Therefore, depending on the parity of the dimension and on whether the $(n-2)$-dimensional plane containing $\mathrm{proj}_{e_n^\perp}(F \cap \{\langle x, \xi\rangle = t\})$ separates $\bar{u}$ and the origin, all the determinants $|[\bar{l}_{i_1}, \dots, \bar{l}_{i_{n-1}}]|$ will either have the same sign as (6) or all will have the opposite sign.

Since the face F contributes either to V_+ or V_-, but not both, one side in the equality $\lim_{\epsilon\to 0} V_+(\xi(\epsilon)) = \lim_{\epsilon\to 0} V_-(\xi(\epsilon))$ is zero and the other is not. Contradiction. Therefore, our assumption that P and Q have different vertices is wrong, which means that these polytopes are identical.

□

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