A GUIDE TO MORGAN AND SHALEN

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Abstract. Given a closed 3-manifold and an action of $\pi_1(M)$ on a tree T, there is an equivariant weakly transverse map whose dual lamination has all leaves incompressible.

1. Maps, laminations determined by maps, and dual laminations

Definition 1.1. *$r : [0,1] \to T$ is rectifiable if the total lenghts of all partitions are uniformly bounded.*

Definition 1.2. *$f : \tilde{M} \to T$ is weakly transverse if at each point there is neighborhood U, homeomorphic to $D^{n-1} \times [0,1]$ on which $f = r \circ p$, $r : [0,1] \to T$ rectifiable, $p : U \to [0,1]$ projection.*

Definition 1.3. *Lamination: each flow box is homeomorphic to a product $U \times I$ of an open set $U \in \mathbb{R}^{n-1}$ and an open interval I. In each flow box lamination is of the form $U \times X$, where X is some closed subset of I. Different flow boxes should be compatible.*

Measured lamination: there exist integrals of transverse intervals, the weights of the intervals.

Definition 1.4. *The lamination $\widetilde{\mathcal{L}}$ determined by f is the lamination supported on the set C of points where f is not locally constant and the leaf of $\widetilde{\mathcal{L}}$ through a point $x \in C$ is the level set of f containing x.*

Let $f : \widetilde{M} \to T$ be a weakly transverse map which is equivariant with respect to an action of $\pi_1(M)$ on the tree T. Projecting $\widetilde{\mathcal{L}}$ to M, equivariance implies that if the projections of two leaves of $\widetilde{\mathcal{L}}$ intersect then their projections are identical. Since projection is a local homeomorphism on the interior of a fundamental domain, each leaf projects to a surface in M. Threfore $\widetilde{\mathcal{L}}$ induces a lamination $\mathcal{L}$ on M, the **dual lamination** to $\tilde{f}$.

Throughout we are using the following dictionary:

$$\begin{aligned} 3-\text{handles} &= \text{vertices}, \\ 2-\text{handles} &= \text{edges}, \\ 1-\text{handles} &= \text{faces}, \\ 0-\text{handles} &= \text{interior of tetraherda} \end{aligned}$$

or neighborhoods thereof.

We shall concentrate on weakly transverse maps with dual laminations that meet a given handle decomposition in specific ways. This will allow later for containing all such lamination in appropriate bundles.

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Definition 1.5. *Given a triangulation, a lamination is* **normal** *to it if:*

(1) *for every handle h each leaf of the lamination is transverse to the boundary and the co-cores of h*
(2) *for 0- and 1-handles the intersection of any leaf is, if not empty, an unfolded disc.*

In terms of a triangulation, this restricts the lamination to intersect each tetrahedron in one of the 7 ways, two of which are represented below.

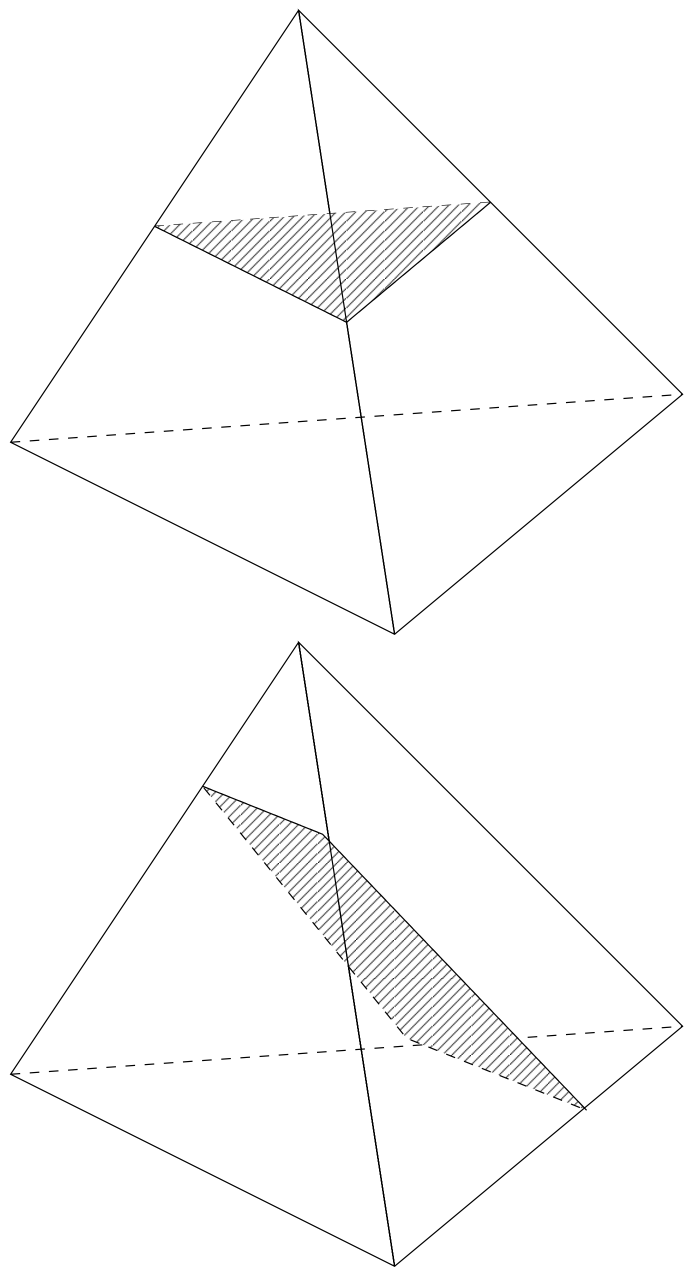

Figure 1

Theorem 1.6. *Given a triangulation of M and an action of $\pi_1(M)$ on T, there is a weakly transverse equivariant map $\tilde{f} : \widetilde{M} \to T$ whose dual lamination is measured, normal with respect to the triangulation, and nowhere dense.*

Proof. $\tilde{f}$ **on 3-handles:** Choose a fundamental domain whose boundary avoids all three handles. Define it to be an arbitrary constant on each 3-handle of a fundamental domain, then extend equivariantly to all 3-handles.

$\tilde{f}$ **on 2-handles:** Extend linearly on all 2-handles. ω_σ will denote the arc image (parametrized by arc lenght) of the 2-handle σ.

The paths ω_σ are clearly equivariant and they "match across 1-handles" (the properties (2.4) of [MSIII]) simply by virtue of the fact that they are paths joining three points on a tree. Redistribute the values on the edges in a Cantor fashion, making sure that close to the pre-image of the vertex, on all three edges, $\tilde{f}$ is constant .

$\tilde{f}$ **on 1-handles (faces):** For any two points on the edges with the same value, extend $\tilde{f}$ to have the same value on the segment joining them.

$\tilde{f}$ **on 0-handles:** Observe that by construction the map is monotone on 2-handles, therefore each leaf of the lamination intersects every edge at most once therefore each leaf intersects the boundaries of 0-handles along normal circles, see page 472 of [MSIII]. Therefore the lamination and the map extend inside 0-handles by attaching disks.

Define the measure of this lamination by

$$\int_\gamma \mu := \mathrm{length}\tilde{f}(\gamma).$$

$\square$

2. The functional I

Fix a triangulation on M. Let $(\mathcal{L}, \mu)$ be the normal measured lamination dual to a weakly transverse map $\tilde{f}$. If the **weight of** $(\mathcal{L}, \mu)$ **across an edge** σ is defined as $\int_\sigma \mu$, denote by $W(\mathcal{L}, \mu)$ the sum of all the weights of the lamination across the edges of the triangulation.

Now define the Euler characteristic of a parallel family of leaves to be:

$$\chi(\mathrm{leaf}) \cdot \int_\gamma \mu$$

for γ any tranverse path intersecting all leaves of the family. For non-parallel familes work on flow boxes and add up alternatingly.

Now for constants C, $K > 0$ define

$$I(\tilde{f}) = C(W(\mathcal{L}, \mu) - K\chi(\mathcal{L}, \mu))$$

as a functional on the set $\mathcal{F}$ of weakly transverse maps. (Observe that I can be defined on larger classes of laminations, such as transverse laminations.)

Also define a pre-oder "$\leq$" on the set $\mathcal{F}$:

$$\tilde{f}' \leq \tilde{f} \text{ if } d(\omega_\sigma(\tilde{f}), \omega_\sigma(\tilde{f}')) \leq I(\tilde{f}) - I(\tilde{f}').$$

for all σ. An element f_0 in $\mathcal{F}$ is called minimal if $f \leq f_0$ implies $f_0 \leq f$ for any f.

Theorem 2.1. *For a given choice of C and K there is a minimal element.*

Proof. Zorn's lemma □

This gives a standard coercivity inequality for I: Either

$$d(\omega_\sigma(\tilde{f}), \omega_\sigma(\tilde{f_0})) \leq I(\tilde{f}) - I(\tilde{f_0})$$

or

$$d(\omega_\sigma(\tilde{f}), \omega_\sigma(\tilde{f_0})) = I(\tilde{f}) - I(\tilde{f_0}) = 0.$$

The main result in these notes is:

Theorem 2.2. *There are constants C and K such that the dual lamination of any minimal element of $I_{C,K}$ has all leaves incompressible.*

The proof of this theorem occupies the rest of these notes.
First comes a model version of the final Theorem:

Definition 2.3. *Given a triangulation, the multiplicity $\nu(D)$ of a disc D trasverse to the edges is the number of times it intersects the edges of the triangulation.*

Key Proposition: Assume that $C > K > 0$. The minimal element of any $I_{C,K}$, has dual lamination $\mathcal{L}_0$ whith all leaves satisfying the following: If the boundary circle of a disc is the disc's intersection with the lamination and $\nu(D) \leq \frac{K}{2} - 1$, the boundary can always be filled in on the leaf without increasing multiplicity.

Proof. If the boundary of D does not fill on the leaf l_0, extend D to D_0.
If arbitrarily close there are leaves l such that $D_0 \cap l$ bounds a disc in l, we could take the limit of such discs to get a disc on l_0. Therefore all leaves close to l_0 are intersected in circles that don't bound discs. Thicken up D_0 and operate to get a new lamination $\mathcal{L}_1$ with the property that

$$\chi(\mathcal{L}_1) = \chi(\mathcal{L}_0) + 2T$$

where T is the total weight of the surgery, since no leaves on which we operate bound discs.
Notation:

$$\begin{aligned} T_s &= \text{total weight of spheres,} \\ V_s' &= \text{weight of spheres across } 2-\text{handles away from surgery,} \\ W(\Lambda) &= \text{weight across all } 2-\text{handles of leaves in balls} \end{aligned}$$

Now throw away sphere leaves and leaves in balls to get a new lamination $\mathcal{L}'$ for which, after using that $\nu(D) \leq \dfrac{K}{2} - 1$,

$$I(\mathcal{L}_0) - I(\mathcal{L}') \geq 2\nu(D)(2T - 2T_s) + 2(V_s' + W(\Lambda))$$

Since the spheres constributing to the T_s can only come from tori Using again that the leaves we operate on bound no discs this is bounded below away form zero. On the other hand

$$2\nu(D)(2T - 2T_s) + 2(V_s' + W(\Lambda)) \geq d(\omega_\sigma(\mathcal{L}_0), \omega_\sigma(\mathcal{L}')).$$

by simple counting.

Therefore the final lamination has even smaller $I_{C,K}$ than the minimal element, a contradiction. □

3. The branched I-bundle N

Definition 3.1. *N is an I-branched bundle if it is a co-dimension 0 submanifold of M which is locally homeomorphic to the "standard model" in $\mathcal{R}^3$ with it vertical intervals. Different homeomorphisms to the standard model preserve verticality.*

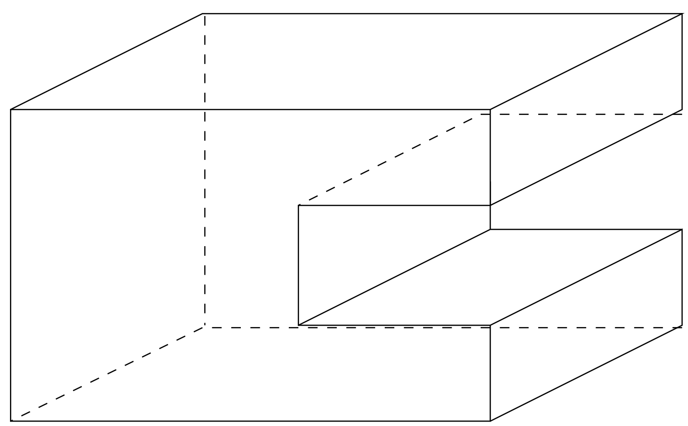

Figure 2

The boundary of N consists of horizontal and vertical boundary.

Definition 3.2. *The horizontal boundary of N, $\partial_h N$, is the boudary of N which is transversal to the vertical intervals of N.*

Definition 3.3. *The vertical boundary of N, $\partial_v N$ is the boundary which is parallel to the vertical intervals of N.*

Throughout, fix a triangulation of M. A branched I-bundle is **normal** if:

(1) N avoids all vertices.
(2) The vertical boundary $\partial_v N$ of N does not intersect the edges of the triangulation, and the horizontal boundary $\partial_h N$ is transverse to the edges.
(3) Both the vertical and horizontal boundaries are transverse to faces.
(4) The bundle meets each 2-handle as the following:

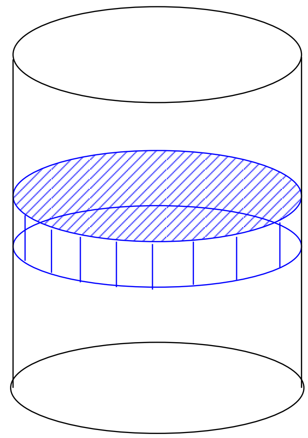

Figure 3

and it meets each 1-handle as:

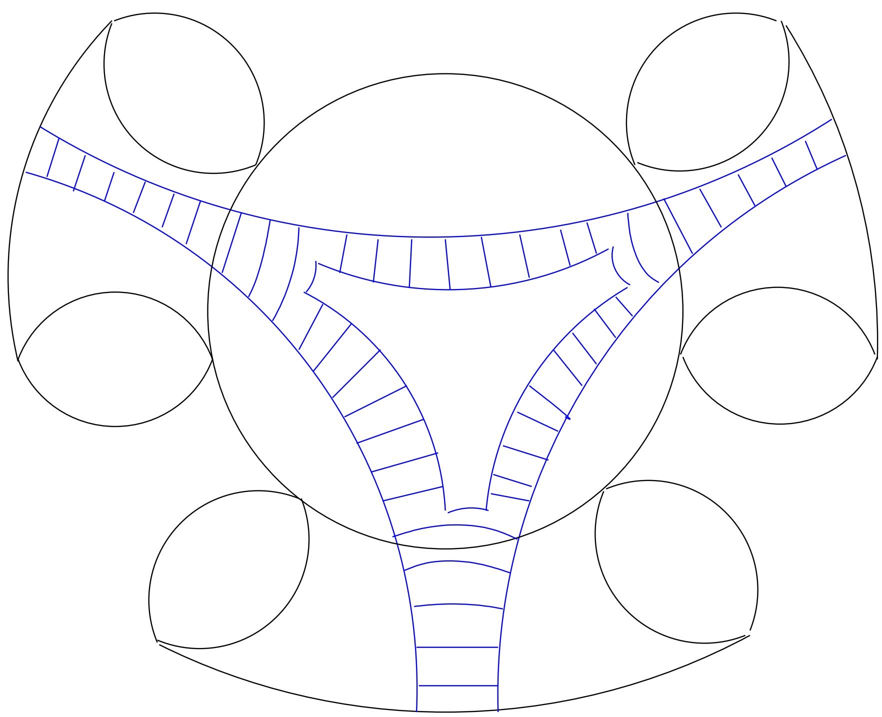

Figure 4

3.1. **The master bundles:**

(1) Draw the standard bundle on the faces.

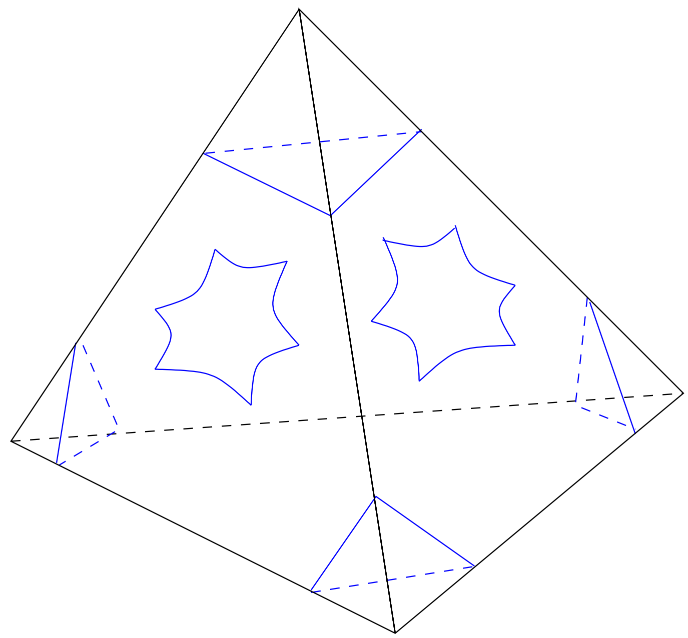

Figure 5

(2) Extend according to how the interior stays parallel to the faces.

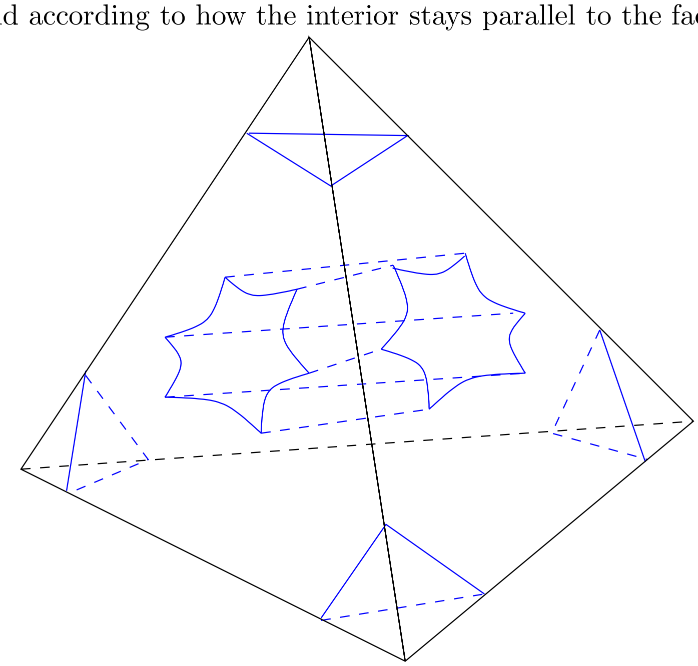

Figure 6

(3) By construction all normal bundles either homotopic to or subbundles of one of these models.

3.2. **Incompressible I-bundles:**

Definition 3.4. (1) *No discs of contact. (Discs with boundary on ∂_v and interior in the interior of N.)*

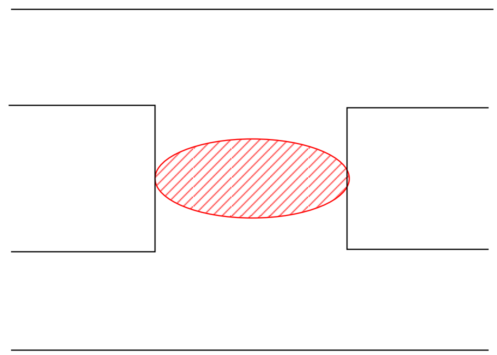

Figure 7

(2) *No compressing discs. (Discs with boundary on ∂_h, but not homotopic to any dic on ∂_h while keeping the boundary fixed.) Given that we can ussually make ∂_h consist of leaves, this agrees with the defintion of compressing discs for the leaves when the compressing discs stay around the horizontal boundary - let's call these N-compressing.*

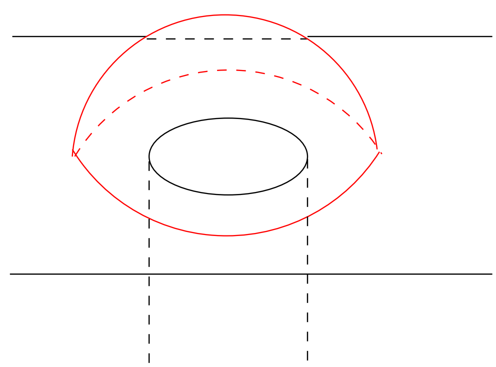

Figure 8

(3) *No return discs. (Discs with boundary consisting of a vertical component and a horizontal component, and the interior in the complement of the bundle.*

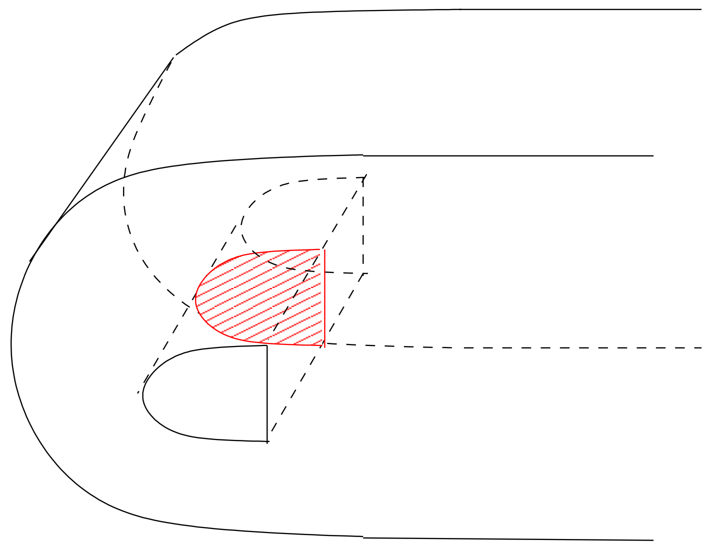

Figure 9

(4) *No trivial components.*

4. Laminations in branched I-bundles

4.1. **Cutting N along a lamination:** A lamination $\mathcal{L}$ is carried by N with positive weights if:

(1) $\mathcal{L}$ is contained in N.
(2) All leaves of $\mathcal{L}$ are transverse to vertical intervals.
(3) $\mathcal{L}$ intersects each vertical interval with positive weight. (I.e. vertical intervals intersect sufficiently many leaves of $\mathcal{L}$.)

To cut N along $\mathcal{L}$, consider the disjoint union of two sets, $N \setminus \mathcal{L}$ and the set of points on leaves with orientations (germs of components of $N \setminus \mathcal{L}$). Call the result $N_{\mathcal{L}}$. With a suitable topology, it is a (not branched) I-bundle:

First bring the ∂_h onto leaves. This can be done since the weights are positive: Every vertical interval starting from horizontal boundary intersects a first leaf. Use the vertical intervals to homotope this leaf to the boundary.

Then use the standard model for the branched bundle and the fact that the horizontal boundary is on leaves to cut N into five parallelepiped blocks.

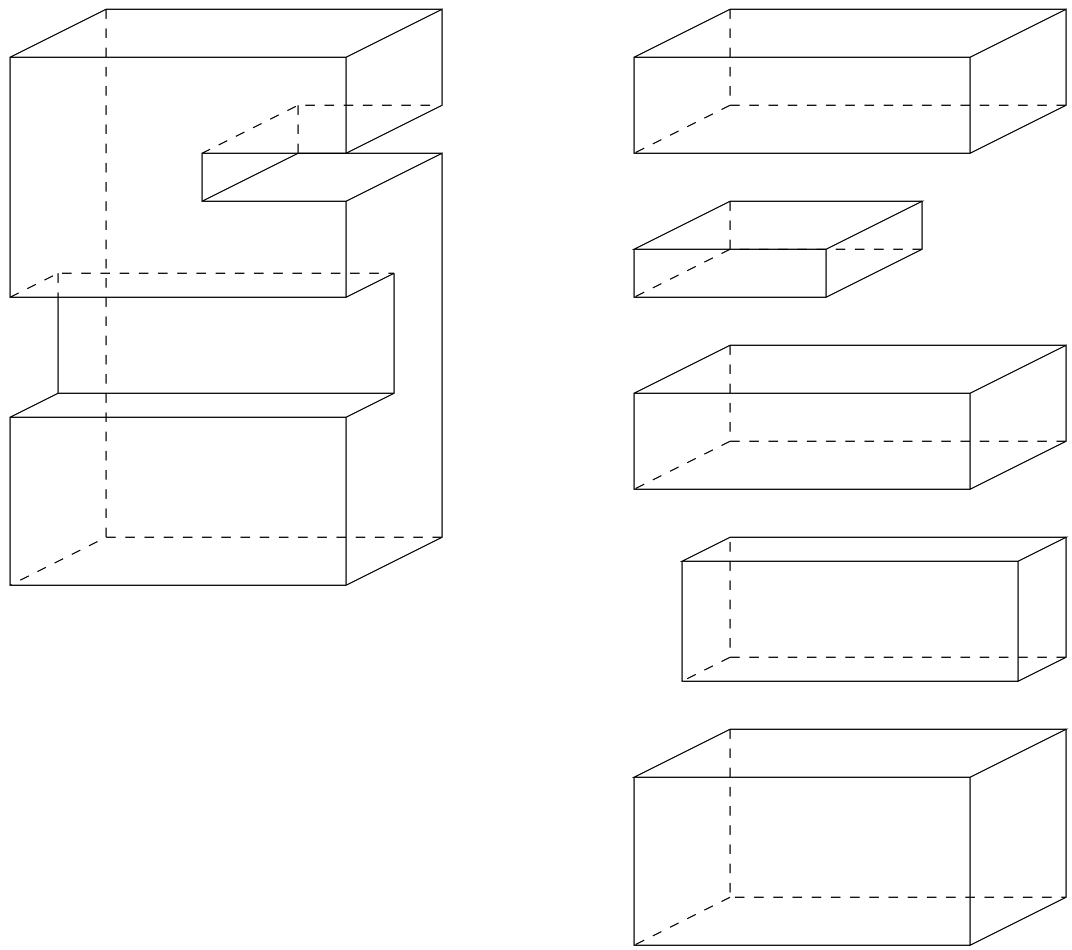

Figure 10

In each block, pick a point x in the complement of the leaves. Thought this point there is a vertical interval and, since the lamination is closed, there is a whole subinterval containing x that does not intersect the lamination. The neighborhood of a maximal such subinterval gives the local I-bundle structure.

(The I-bundle structure is obvious from the flow-boxes of the lamination. The fussing in [MSII] happens to be able to tell which parts of the leaves the different components of the I-bundle are over.)

4.2. **Incompressible laminations from incompressible branched bundles.** N-compressing discs are taken care of since the bundle is incompressible. This takes care of the l-compressing that are not N-compressing.

4.2.1. *Step 1:* $\pi_1(l)$ *injects into* $\pi_1(N)$. It is enough to show that in the universal cover of N and show that leaf is simply connected.

(Indeed, we prove that the sequence $\pi_1(\tilde{l}) \to \pi_1(l) \to \pi_1(N)$ is exact, i.e. $\delta_1 : \pi_1(\tilde{l}) \to \pi_1(l)$, $\delta_2 : \pi_1(l) \to \pi_1(N)$ and we want to prove that $\mathrm{Im}\delta_1 = \mathrm{Ker}\delta_2$.

Take $\tilde{\gamma} \in \pi_1(\tilde{l})$. $\tilde{\gamma} \subset \tilde{N}$ is closed, therefore its projection is trivial in $\pi_1(N)$ because nontrivial loops in $\pi_1(N)$ are not closed in $\tilde{N}$. So, for any $\tilde{\gamma} \in \pi_1(\tilde{l})$ $\delta_2(\delta_1(\tilde{\gamma})) = e$ or $\delta_1(\tilde{\gamma}) \in \mathrm{Ker}\delta_2$, therefore $\mathrm{Im}\delta_1 \subset \mathrm{Ker}\delta_2$.

Take $\gamma \in \mathrm{Ker}\delta_2 \subset \pi_1(l)$. γ is the image of $\tilde{\gamma}$ under the map from the manifold to the universal cover. $\tilde{\gamma}$ is closed (because $\gamma \in \mathrm{Ker}\delta_2$, so $\delta_2(\gamma)$ is trivial in N and trivial loops in N have closed images in the universal cover). So, $\tilde{\gamma} \in \pi_1(\tilde{l})$ and we have that for any $\gamma \in \mathrm{Ker}\delta_2$ we can find $\gamma \in \pi_1(l)$ i.e. $\mathrm{Ker}\delta_2 \subset \mathrm{Im}\delta_1$.

Therefore, $\mathrm{Im}\delta_1 = \mathrm{Ker}\delta_2$ and the sequence is exact.

If we prove that $\pi_1(\tilde{l}) = \{e\}$, then $\mathrm{Im}\delta_1 = \{e\} = \mathrm{Ker}\delta_2$ and this means that δ_2 is injection.)

A disc D in $\tilde{N}$ with boundary on a leaf $\tilde{l}$ intersects all leaves in circles by transversality.

If there are circles that do not bound discs on their leaves, take an innermost one of this kind. If there are no more circles inside it, deform to the leaf using the I-bundle structure (see below). If there are more circles, then by construction they bound discs. Change the outermost ones to be on their leaves. Use again the I-bundles to deform.

How to deform using I-bundles: The only obstruction to deforming could be coming from touching the horizontal boundary. When this happens, and since N is incompressible, we are at the boundary of a hole. Fill in the holes and use the I-bundles of $\tilde{X}$ to deform.

One more case to consider: When the boundary ∂D of the disc D is the limit of circles that bound discs on their leaves, then ∂D itself bounds a disc on the leaf (perhaps other than D).

The following shows that incompressible I-bundles as defined in 3.4 implies that there are no compressing discs in N in the usual sense.

4.2.2. *Step 2: There is a surface S in N carried with positive weights such that any loop in l is homotopic in N to a loop is S.*

4.2.3. *Step 3: If N is incompressible, then S is incompressible.* [FO].

Now start with γ on l, trivial in M. By Step 2 γ is homotopic to γ' on S, still trivial in M, therfore trivial on S since S is incompressible, therefore trivial in N since S is a subset of N. Then Step 1 gives that γ is trivial on l.

5. An incompressible branched I-bundle

Minimize the functional $I_{C,K}$ for the correct values of C and K to get a weakly transverse map. Its dual lamination lives in at least one of the $N_i's$. Chose the one of minimum complexity, call it N_I.

Theorem 5.1. *N is incompressible.*

Proof. **No discs of contact:** This is the opposite of the key proposition. Using the vertical structure, we have a deformation of any such disc away from leaves of the lamination. Then we have discs whose bounday is on vertical boundary and have nothing to do with the leaves. This contradicts the fact that the bundle has minimum complexity.

No N-compressing discs: If it does, work on the one with minimum multiplicity. Then the key proposition applies because of the choice of K. Therefore the disc can be deformed to a disc D' on the the leaf. If $D' \cap \partial_v$ is empty the original disc is not an N-compressing disc, since it completely lies on ∂_h. Therefore $D' \cap \partial_v$ is nonempty, hence a circle. Then there is a disc of contact whose boundary is homotopic to this circle. (Uses the choice of C and K.)

No return discs: The idea is to show that a return disc forces the leaves to intersect 2-handles. If there are return discs, the bundle had a boundary component that consists of a horizontal annulus and a vertical annulus glued together, otherwise there would be compressing discs. This means that the leaves of $\mathcal{L}$ are flat rather than of positive curvature (they cannot fold inside of the tetrahedron). This forces them to intersect 2-handles. Therefore, pushing the lamination away minimizes the W-part of the functional while keeping the χ-part the same. M-S also showe that this component of the boundary is a torus, therefore there can be no leaves contained in its interior, hence they can isotope away from the horizontal boundary of the return. □

6. Choice of C and K

$$K = 2\left(\max_i\{\delta_i, \Delta_i, d_i\} + 1\right)$$

where

$$\begin{aligned} d_i &= \text{minimum } \nu \text{ of discs of contact,} \\ \Delta_{\mathrm{i}} &= \text{maximum } \nu \text{ of horizontal discs,} \\ \delta_{\mathrm{i}} &= \text{minimum } \nu \text{ of compressing discs} \end{aligned}$$

where i keeps track of: the subbundles, up to isomorphism, of the master bundle that support normal laminations, the subbundles resulting from removing the disc of contact of minimal multiplicity, the subbundles of that, the new bundles after removing discs of contact of minimal multiplicity, and so on.

References

[FO] W.Floyd and U.Oertel, *Incompressible surfaces via branched surfeces*, Topology **23** (1984), 117-125

[MSII] J.W.Morgan and P.B.Shalen *Degenerations of hyperbolic structures, II. Measured laminations in* 3*-manifolds* Annals of Mathematics, **127** (1988), 403-456

[MSIII] J.W.Morgan and P.B.Shalen, *Degeneration of hyperbolic structures, III: Action of 3-manifold groups on trees and Thurston's compactness theorem*, Annals of Mathematics, **127** (1988), 457-519