CENTROID BODIES AND COMPARISON OF VOLUMES.

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Abstract. For $-1 < p < 1$ we introduce the concept of a polar p-centroid body $\Gamma_p^* K$ of a star body K. We consider the question of whether $\Gamma_p^* K \subset \Gamma_p^* L$ implies $\mathrm{vol}(L) \le \mathrm{vol}(K)$. Our results extend the studies by Lutwak in the case $p = 1$ and Grinberg, Zhang in the case $p > 1$.

1. Introduction

Let K be a star body in $\mathbb{R}^n$, then the centroid body of K is a convex body ΓK defined by its support function:

$$h_{\Gamma K}(\xi) = \frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)| dx, \quad \xi \in \mathbb{R}^n.$$

Let K and L be two origin-symmetric star bodies in $\mathbb{R}^n$ such that $\Gamma K \subset \Gamma L$, what can be said about the volumes of K and L? Lutwak [L] proved that, if L is a polar projection body then $\mathrm{vol}(K) \le \mathrm{vol}(L)$. On the other hand, if K is not a polar projection body, then there is a body L, so that $\Gamma K \subset \Gamma L$, but $\mathrm{vol}(K) > \mathrm{vol}(L)$. Since in $\mathbb{R}^2$ every convex body is a polar projection body [S], the results of Lutwak imply the following:

Suppose that K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$ such that $\Gamma K \subset \Gamma L$. If $n = 2$, then we necessarily have $\mathrm{vol}(K) \le \mathrm{vol}(L)$, while this is no longer true if $n \ge 3$.

Let K be a star body in $\mathbb{R}^n$ and $p \ge 1$, then the p-centroid body of K is the body $\Gamma_p K$ defined by:

$$h_{\Gamma_p K}(\xi) = \left(\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx \right)^{1/p}, \quad \xi \in \mathbb{R}^n. \tag{1}$$

Clearly, $h_{\Gamma_p K}$ is a homogeneous function of degree 1, and if $p \ge 1$, then this function is convex, and, therefore, $\Gamma_p K$ is well-defined. The polar of $\Gamma_p K$ is called the polar p-centroid body of K and denoted by $\Gamma_p^* K$. Since the support function of a body is the norm of its polar, $h = \|\cdot\|_*$, the polar p-centroid body of K is given by

$$\|\xi\|_{\Gamma_p^* K} = \left(\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx \right)^{1/p}, \quad \xi \in \mathbb{R}^n. \tag{2}$$

The p-centroid bodies and their polars have recently been studied by different authors, see e.g. [CG], [GZ], [L], [LYZ], [LZ]. In [GZ] Grinberg and Zhang generalized the results of Lutwak discussed in the beginning of this section. Namely, let K and L be two origin-symmetric star bodies in $\mathbb{R}^n$ such that for $p \geq 1$

$$\Gamma_p K \subset \Gamma_p L.$$

They prove that if the space $(\mathbb{R}^n, \|\cdot\|_L)$ embeds in L_p, then we necessarily have

$$\mathrm{vol}(K) \leq \mathrm{vol}(L).$$

On the other hand, if $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_p, then there is a body L so that $\Gamma_p K \subset \Gamma_p L$, but $\mathrm{vol}(K) \leq \mathrm{vol}(L)$.

Note, that if $p = 1$ the positive answer holds for all convex bodies in $\mathbb{R}^2$, while if $p > 1$ there is no dimension where this is always true. The preceding remark suggests considering $p < 1$ in order to make the answer affirmative in higher dimensions.

If $p < 1$, then the function $h_{\Gamma_p K}(\xi)$ in (1) is not necessarily convex, therefore it is not a support function, but the definition of the polar p-centroid body still makes sense, even though these bodies may be non-convex. So for all $p > -1$, $p \neq 0$ we define the polar p-centroid body of a star body K by the formula:

$$\|\xi\|_{\Gamma_p^* K} = \left(\frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx \right)^{1/p}, \quad \xi \in \mathbb{R}^n. \tag{3}$$

For $p = 0$, this definition looks as follows (if we send $p \to 0$):

$$\|\xi\|_{\Gamma_0^* K} = \exp\left(\frac{1}{\mathrm{vol}(K)} \int_K \ln|(x,\xi)| dx \right), \quad \xi \in \mathbb{R}^n. \tag{4}$$

Now we can ask the question discussed above for all $p > -1$. Namely, suppose that

$$\Gamma_p^* L \subset \Gamma_p^* K, \tag{5}$$

for origin-symmetric star bodies K and L. Does it follow that we have an inequality for the volumes of K and L? In this paper we show that if $(\mathbb{R}^n, \|\cdot\|_L)$ embeds in L_p, $p > -1$, then we have $\mathrm{vol}(K) \leq \mathrm{vol}(L)$. However if $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_p, we construct counterexamples to the latter result.

These results can also be reformulated as follows:
(i) If $0 < p < 1$, then in $\mathbb{R}^2$ the condition (5) implies that $\mathrm{vol}(K) \leq \mathrm{vol}(L)$, while this is no longer true in dimensions $n \geq 3$.
(ii) If $-1 < p \leq 0$, (5) implies that $\mathrm{vol}(K) \leq \mathrm{vol}(L)$ if and only if $n \leq 3$.

Clearly the integral in (3) diverges if $p \le -1$, but still we can make sense of this integral considering fractional derivatives. Indeed, if $-1 < p < 0$

$$\begin{aligned}\frac{1}{\mathrm{vol}(K)}\int_K |(x,\xi)|^p dx &= \frac{1}{\mathrm{vol}(K)}\int_{-\infty}^{\infty} |z|^p \int_{(x,\xi)=z} \chi(\|x\|_K) dx\ dz \\ &= \frac{1}{\mathrm{vol}(K)}\int_{-\infty}^{\infty} |z|^p A_{K,\xi}(z) dz \\ &= \frac{2\Gamma(p+1)}{\mathrm{vol}(K)} A_{K,\xi}^{(-p-1)}(0),\end{aligned}$$

where $A_{K,\xi}(z)$ is the parallel section function of K, and $A_{K,\xi}^{(-p-1)}(0)$ is its fractional derivative at zero. (For details on fractional derivatives, see e.g. [K5, Section 2.6]). So, in such terms our problem can be written as follows:

Suppose K and L are two origin-symmetric star bodies, so that for all $\xi \in S^{n-1}$:

$$\frac{A_{K,\xi}^{(-p-1)}(0)}{\mathrm{vol}(K)} \le \frac{A_{L,\xi}^{(-p-1)}(0)}{\mathrm{vol}(L)}.$$

Do we necessarily have an inequality for the volumes of K and L?

Note that Koldobsky already considered such inequalities (see e.g. [K4]) without dividing by volumes. So, for $-1 < p < 0$ the positive part of our results can also be obtained from the results of Koldobsky, but we give our own proof. The case $p = -1$ leads to the following modification of the Busemann-Petty problem. Let K and L be two convex origin-symmetric bodies in $\mathbb{R}^n$ such that

$$\frac{\mathrm{vol}_{n-1}(K \cap \xi^\perp)}{\mathrm{vol}(K)} \le \frac{\mathrm{vol}_{n-1}(L \cap \xi^\perp)}{\mathrm{vol}(L)}.$$

Does this imply an inequality for the volumes of K and L?

It is easy to show that in dimensions $n \le 4$ we have $\mathrm{vol}(L) \le \mathrm{vol}(K)$. The proof is almost identical to that of the original solution of the Busemann-Petty problem from [GKS]. The counterexamples in dimensions $n \ge 5$ from [GKS] also work in this situation.

In view of all these remarks one can consider our results as a certain bridge between the results of Lutwak-Grinberg-Zhang about p-centroid bodies and the results of Busemann-Petty type obtained by Koldobsky.

2. Centroid inequalities for $-1 < p < 1$, $p \neq 0$.

The *Minkowski functional* of a star-shaped origin-symmetric body $K \subset \mathbb{R}^n$ is defined as

$$\|x\|_K = \min\{a \ge 0 : x \in aK\}.$$

We denote by $(\mathbb{R}^n, \|\cdot\|_K)$ the Euclidean space equipped with the Minkowski functional of the body K. Clearly, $(\mathbb{R}^n, \|\cdot\|_K)$ is a normed space if and only if the body K is convex.

The *support function* of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max_{\xi \in K}(x,\xi), \quad x \in \mathbb{R}^n.$$

If K is origin-symmetric, then h_K is the Minkowski norm of the polar body K^*.

A well-known result going back to P.Lévy, (see [BL, p. 189] or [K5, Section 6.1]), is that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p > 0$ if and only if there exists a finite Borel measure μ on the unit sphere so that, for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu(\xi). \tag{6}$$

On the other hand, this can be considered as the definition of embedding in L_p, $-1 < p < 0$ (cf. [K2]).

It was proved in [K1] that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p, $p > 0$, $p \notin 2\mathbb{N}$ if and only if the Fourier transform of the function $\Gamma(-p/2)\|x\|^p$ (in the sense of distributions) is a positive distribution outside of the origin. If $-n < p < 0$ a similar fact was proved in [K2]: a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p if and only if the Fourier transform of $\|\cdot\|^p$ is a positive distribution in the whole $\mathbb{R}^n$.

Now we are ready to prove our first result.

Theorem 2.1. *Let $-1 < p < 1$, $p \neq 0$. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, so that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p and*

$$\Gamma_p^* K \subset \Gamma_p^* L. \tag{7}$$

Then $\mathrm{vol}(L) \leq \mathrm{vol}(K)$.

Proof. First let us prove the case $0 < p < 1$. Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p, there exists a measure μ_K on the unit sphere S^{n-1} such that

$$\|x\|_K^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu_K(\xi).$$

Note that (7) can be written as

$$\frac{1}{\mathrm{vol}(L)} \int_L |(x,\xi)|^p dx \leq \frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx. \tag{8}$$

Integrating both sides of the last inequality over S^{n-1} with the measure μ_K, we get

$$\frac{1}{\mathrm{vol}(L)} \int_{S^{n-1}} \int_L |(x,\xi)|^p dx \; d\mu_K(\xi) \leq \frac{1}{\mathrm{vol}(K)} \int_{S^{n-1}} \int_K |(x,\xi)|^p dx \; d\mu_K(\xi).$$

Applying Fubini's Theorem,

$$\frac{1}{\mathrm{vol}(L)} \int_L \|x\|_K^p dx \leq \frac{1}{\mathrm{vol}(K)} \int_K \|x\|_K^p dx. \tag{9}$$

Note that

$$\begin{aligned}\int_K \|x\|_K^p dx &= \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} \|r\theta\|_K^p \; r^{n-1} dr \right) d\theta \\ &= \frac{1}{n+p} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta = \frac{n}{n+p} \mathrm{vol}(K).\end{aligned}$$

Therefore, (9) can be rewritten as

$$\frac{1}{\mathrm{vol}(L)} \int_L \|x\|_K^p dx \le \frac{n}{n+p}.$$

Using the inequality

$$\frac{1}{\mathrm{vol}(L)} \int_L \|x\|_K^p dx \ge \frac{n}{n+p} \left(\frac{\mathrm{vol}(L)}{\mathrm{vol}(K)} \right)^{p/n} \tag{10}$$

from [MP, Section 2.2], we get

$$\frac{n}{n+p} \quad \ge \quad \frac{1}{\mathrm{vol}(L)} \int_L \|x\|_K^p dx \ge \frac{n}{n+p} \left(\frac{\mathrm{vol}(L)}{\mathrm{vol}(K)} \right)^{p/n},$$

therefore $\mathrm{vol}(L) \le \mathrm{vol}(K)$, which proves the theorem for $0 < p < 1$.

Now consider $-1 < p < 0$. In this case (7) is equivalent to

$$\frac{1}{\mathrm{vol}(L)} \int_L |(x,\xi)|^p dx \ge \frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx. \tag{11}$$

Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds into L_p, $p > -1$, there exists a measure μ_K on the unit sphere such that

$$\|x\|_K^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu_K(\xi).$$

Integrating both sides of (11) over S^{n-1} with the measure μ_K and using the same argument as in the first part of the proof, we get

$$\frac{1}{\mathrm{vol}(L)} \int_L \|x\|_K^p dx \ge \frac{n}{n+p}. \tag{12}$$

Passing to spherical coordinates and applying Hölder's inequality

$$\begin{aligned}\int_L \|x\|_K^p dx &= \int_{S^{n-1}} \left(\int_0^{\|\theta\|_L^{-1}} r^{n+p-1} \|\theta\|_K^p dr \right) d\theta \\ &= \frac{1}{n+p} \int_{S^{n-1}} \|\theta\|_L^{-n-p} \|\theta\|_K^p d\theta \\ &\le \frac{1}{n+p} \left(\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta \right)^{(n+p)/n} \left(\int_{S^{n-1}} \|\theta\|_K^{-n} d\theta \right)^{-p/n} \\ &= \frac{n}{n+p} \left(\mathrm{vol}(L)\right)^{(n+p)/n} \left(\mathrm{vol}(K)\right)^{-p/n}.\end{aligned}$$

So (12) can be written as

$$\begin{aligned} 1 &\le \frac{1}{\mathrm{vol}(L)} \left(\mathrm{vol}(L)\right)^{(n+p)/n} \left(\mathrm{vol}(K)\right)^{-p/n} \\ &= \left(\mathrm{vol}(L)\right)^{p/n} \left(\mathrm{vol}(K)\right)^{-p/n}. \end{aligned}$$

Therefore, using the fact that $p < 0$, we get $\mathrm{vol}(L) \le \mathrm{vol}(K)$.

□

Since all 2-dimensional spaces embed in L_1, and therefore in L_p with $-2 < p < 1$ (see e.g. [K5, Chapter 6]), and all 3-dimensional spaces embed in L_0, and therefore in L_p with $-3 < p < 0$ (see [KKYY]), we have the following

Corollary 2.2. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, so that $\Gamma_p^* K \subset \Gamma_p^* L$. Then*

i) if $0 < p < 1$, we necessarily have $\mathrm{vol}(L) \le \mathrm{vol}(K)$ in dimension $n = 2$,

ii) if $-1 < p < 0$, we necessarily have $\mathrm{vol}(L) \le \mathrm{vol}(K)$ in dimensions $n = 2$ and 3.

In order to show a negative counterpart of Theorem 2.1, we need some lemmas. The following Lemma is [K5, Corollary 3.15] with $k = 0$ and $p = -q - 1$.

Lemma 2.3. *Let $-1 < p < 1$, $p \neq 0$. For an origin-symmetric convex body K in $\mathbb{R}^n$ we have*

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = -\frac{\pi}{2\Gamma(p+1)\sin(\pi p/2)} \int_{S^{n-1}} |(\theta,\xi)|^p \ \|\theta\|_K^{-n-p} d\theta.$$

We will use this formula in the following form:

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = -\frac{\pi(n+p)}{2\Gamma(p+1)\sin(\pi p/2)} \int_K |(x,\xi)|^p dx.$$

Also we can write this formula in terms of fractional derivatives of the parallel section function of K. Recall that the parallel section function of a an origin-symmetric star body K is defined by

$$A_{K,\xi}(z) = \int_{(x,\xi)=z} \chi(\|x\|_K) dx.$$

For $-1 < q < 0$ the fractional derivative of this function at zero is defined by

$$A_{K,\xi}^{(q)}(0) = \frac{1}{2\Gamma(-q)} \int_{-\infty}^{\infty} |z|^{-1-q} A_{K,\xi}(z) dz = \frac{1}{2\Gamma(-q)} \int_K |(x,\xi)|^{-1-q} dx.$$

In fact one can see that this is analytically extendable to $q < -1$. Therefore Lemma 2.3 can be reformulated as follows. Let $-1 < p < 1$, $p \neq 0$, then

$$\left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = -\frac{\pi(n+p)}{\sin(\pi p/2)} A_{K,\xi}^{(-p-1)}(0).$$

Note, that for $-1 < p < 0$ this formula was proved in [GKS].

Now recall a version of Parseval's formula on the sphere proved by Koldobsky [K3].

Lemma 2.4. *If K and L are origin-symmetric infinitely smooth bodies in $\mathbb{R}^n$ and $0 < p < n$, then $(\|x\|_K^{-p})^\wedge$ and $(\|x\|_L^{-n+p})^\wedge$ are continuous functions on S^{n-1} and*

$$\int_{S^{n-1}} \left(\|x\|_K^{-p}\right)^\wedge (\xi) \left(\|x\|_L^{-n+p}\right)^\wedge (\xi) d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p} \|x\|_L^{-n+p} dx.$$

Remark 2.5. A proof of this formula via spherical harmonics was given in [K4]. Repeating this proof word by word and using the above definition of the fractional derivative of order $q < -1$, one can easily extend this result to $-1 < p < 0$.

Now we prove a negative counterpart of Theorem 2.1.

Theorem 2.6. *Let L be an infinitely smooth origin-symmetric strictly convex body in $\mathbb{R}^n$, for which $(\mathbb{R}^n, \|\cdot\|_L)$ does not embed in L_p, $-1 < p < 1$, $p \neq 0$. Then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that*

$$\Gamma_p^* K \subset \Gamma_p^* L.$$

but

$$\mathrm{vol}(L) > \mathrm{vol}(K).$$

Proof. First consider $0 < p < 1$. Since $(\mathbb{R}^n, \|\cdot\|_L)$ does not embed in L_p, there exists a $\xi \in S^{n-1}$ such that $\left(\|x\|_L^p\right)^\wedge (\xi)$ is positive; for more details see [K1]. Because $\left(\|x\|_L^p\right)^\wedge (\theta)$ is a continuous function on S^{n-1}, there exists a neighborhood of ξ where it is positive. Define

$$\Omega = \{\theta \in S^{n-1} : \left(\|x\|_L^p\right)^\wedge (\theta) > 0\}.$$

Choose a non-positive infinitely-smooth even function v supported in Ω. Extend v to a homogeneous function $|x|_2^{-n-p} v(x/|x|_2)$ of degree $-n-p$ on $\mathbb{R}^n$. By [K5, Lemma 3.16], the Fourier transform of $|x|_2^{-n-p} v(x/|x|_2)$ is equal to $|x|_2^p\, g(x/|x|_2)$ for some infinitely smooth function g on S^{n-1}.

Define a body K by

$$\|x\|_K^{-n-p} = \|x\|_L^{-n-p} + \epsilon |x|_2^{-n-p} g(x/|x|_2)$$

for some small ϵ so that the body K is convex (see e.g. the perturbation argument from [K5, p.96]). Applying the Fourier transform to both sides we get

$$\left(\|x\|_K^{-n-p}\right)^\wedge (\xi) = \left(\|x\|_L^{-n-p}\right)^\wedge (\xi) + \epsilon (2\pi)^n |\xi|_2^p v(\xi/|\xi|_2).$$

So using the formula from Lemma 2.3

$$\left(\|x\|_K^{-n-p}\right)^\wedge (\xi) = \Gamma(-p) \sin\left(\frac{\pi(p+1)}{2}\right) \int_K |(x,\xi)|^p dx$$

we have

$$\int_L |(x,\xi)|^p dx < \int_K |(x,\xi)|^p dx. \tag{13}$$

Consider the integral

$$\begin{aligned}
&\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi) \left(\|x\|_K^{-n-p}\right)^\wedge(\xi)d\xi \\
&= \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi) \left(\|x\|_L^{-n-p}\right)^\wedge(\xi)d\xi + \epsilon(2\pi)^n \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)v(\xi)d\xi \\
&< \int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi) \left(\|x\|_L^{-n-p}\right)^\wedge(\xi)d\xi \\
&= (2\pi)^n \int_{S^{n-1}} \|x\|_L^p \|x\|_L^{-n-p} dx = (2\pi)^n n\mathrm{vol}(L). \qquad (14)
\end{aligned}$$

Here we used a version of Parseval's formula (Lemma 2.4 and Remark 2.5) and the fact that v is negative on Ω.

On the other hand, again using Parseval's formula and (10)

$$\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi) \left(\|x\|_K^{-n-p}\right)^\wedge(\xi)d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_L^p \|x\|_K^{-n-p} dx$$

$$= (2\pi)^n (n+p) \int_K \|x\|_L^p dx \geq (2\pi)^n n\mathrm{vol}(K) \left(\frac{\mathrm{vol}(L)}{\mathrm{vol}(L)}\right)^{p/n}. \tag{15}$$

Combining (14) and (15) we get

$$\mathrm{vol}(K) < \mathrm{vol}(L). \tag{16}$$

Now from (16) and (13) it follows that

$$\frac{1}{\mathrm{vol}(L)} \int_L |(x,\xi)|^p dx \leq \frac{1}{\mathrm{vol}(K)} \int_K |(x,\xi)|^p dx,$$

which is equivalent to

$$\Gamma_p^* K \subset \Gamma_p^* L.$$

Now consider the case $-1 < p < 0$. Since $(\mathbb{R}^n, \|\cdot\|_L)$ does not embed in L_p, there exists a $\xi \in S^{n-1}$ such that $\left(\|x\|_L^p\right)^\wedge(\xi)$ is negative, see [K2, Theorem 1]. Define

$$\Omega = \{\theta \in S^{n-1} : \left(\|x\|_L^p\right)^\wedge(\theta) < 0\}$$

and choose $v(\theta)$ the same way as in the first part.

Define a body K by

$$\frac{\|x\|_K^{-n-p}}{\mathrm{vol}(K)} = \frac{\|x\|_L^{-n-p}}{\mathrm{vol}(L)} + \epsilon |x|_2^{-n-p} g(x/|x|_2)$$

for some small ϵ so that the body K is convex. Applying Fourier transform to both sides we get

$$\frac{1}{\mathrm{vol}(K)} \left(\|x\|_K^{-n-p}\right)^\wedge(\xi) = \frac{1}{\mathrm{vol}(L)} \left(\|x\|_L^{-n-p}\right)^\wedge(\xi) + \epsilon(2\pi)^n |\xi|_2^p v(\xi/|\xi|_2).$$

Again using the formula from Lemma 2.3 and the fact that $v(\theta)$ is non-positive, we have

$$\frac{1}{\mathrm{vol}(K)}\int_K |(x,\xi)|^p dx < \frac{1}{\mathrm{vol}(L)}\int_L |(x,\xi)|^p dx,$$

which is the same as

$$\Gamma_p^* K \subset \Gamma_p^* L,$$

since $-1 < p < 0$.

Consider the integral

$$\frac{1}{\mathrm{vol}(K)}\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)\left(\|x\|_K^{-n-p}\right)^\wedge(\xi)d\xi$$

$$= \frac{1}{\mathrm{vol}(L)}\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)\left(\|x\|_L^{-n-p}\right)^\wedge(\xi)d\xi + \epsilon(2\pi)^n\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)v(\xi)d\xi$$

$$> \frac{1}{\mathrm{vol}(L)}\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)\left(\|x\|_L^{-n-p}\right)^\wedge(\xi)d\xi = (2\pi)^n n. \qquad (17)$$

Here we used Parseval's formula and the fact that v is negative on Ω.

On the other hand, again using Parseval's formula and Hölder's inequality

$$\begin{aligned}\int_{S^{n-1}} \left(\|x\|_L^p\right)^\wedge(\xi)\left(\|x\|_K^{-n-p}\right)^\wedge(\xi)d\xi &= (2\pi)^n\int_{S^{n-1}} \|x\|_L^p\|x\|_K^{-n-p}dx \\ &\le (2\pi)^n\left(\int_{S^{n-1}}\|x\|_L^{-n}dx\right)^{-p/n}\left(\int_{S^{n-1}}\|x\|_K^{-n}dx\right)^{(n+p)/n} \\ &= (2\pi)^n n\left(\mathrm{vol}(L)\right)^{-p/n}\left(\mathrm{vol}(K)\right)^{(n+p)/n}. \qquad (18)\end{aligned}$$

So combining (17) and (18) we get $\mathrm{vol}(L) > \mathrm{vol}(K)$.

$\square$

The result of Theorem 2.6 can be formulated as follows:

Corollary 2.7. *i) Let $-1 < p < 0$. There exist origin-symmetric convex bodies K and L in $\mathbb{R}^4$, so that $\Gamma_p^*K \subset \Gamma_p^*L$, but $\mathrm{vol}(L) > \mathrm{vol}(K)$.*

ii) Let $0 < p < 1$. There exist origin-symmetric convex bodies K and L in $\mathbb{R}^3$, so that $\Gamma_p^*K \subset \Gamma_p^*L$, but $\mathrm{vol}(L) > \mathrm{vol}(K)$.

Proof. Consider only the case $-1 < p < 0$, the other case is similar. In view of the previous theorem it is enough to construct an origin-symmetric infinitely smooth convex body $L \in \mathbb{R}^4$ for which the distribution $(\|x\|_L^p)^\wedge$ is not positive. The construction will be similar to that from [GKS].

Define $f_N(x) = (1 - x^2 - Nx^4)^{1/3}$; let $a_N > 0$ be such that $f_N(a_N) = 0$ and $f_N(x) > 0$ on the interval $(0, a_N)$. Define a body L in $\mathbb{R}^4$ by

$$L = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : \, x_4 \in [-a_N, a_N] \text{ and } \sqrt{x_1^2+x_2^2+x_3^2} \le f_N(x_4)\}.$$

The body L is strictly convex and infinitely smooth.

By the formula

$$A^{(q)}_{L,\xi}(0) = \frac{\cos\frac{\pi q}{2}}{\pi(n-q-1)}\left(\|x\|_L^{-n+q+1}\right)^\wedge(\xi)$$

from [GKS] and the definition of fractional derivatives, we get

$$\begin{aligned}\left(\|x\|_L^p\right)^\wedge(\xi) &= \frac{\pi p}{\cos\frac{\pi(3+p)}{2}} A^{(3+p)}_{L,\xi}(0)\\ &= \frac{\pi p}{\Gamma(-3-p)\cos\frac{\pi(3+p)}{2}}\int_0^\infty \frac{A_{L,\xi}(z) - A_{L,\xi}(0) - A''_{L,\xi}(0)\frac{z^2}{2}}{z^{4+p}}dz.\end{aligned}$$

Note that the coefficient in the latter formula is positive, therefore it is enough to show that the integral is negative.

The function $A_{L,\xi}$ can easily be computed:

$$A_{L,\xi}(x) = \frac{4\pi}{3}(1 - x^2 - Nx^4).$$

We have

$$\begin{aligned}&\int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2}}{z^{4+p}}dz =\\ &= \frac{4\pi}{3}\left(-\frac{1}{1+p}Na_N^{1+p} + \frac{1}{(1+p)a_N^{(1+p)}} - \frac{1}{(3+p)a_N^{3+p}}\right).\end{aligned}$$

The latter is negative for N large enough, because $N^{1/4}\cdot a_N \to 1$ as $N\to\infty$. □

3. Centroid inequalities for $p = 0$.

In this section we extend the results of the previous section to $p = 0$. First we need some preliminary results. The concept of embedding in L_0 was introduced in [KKYY]:

Definition 3.1. *We say that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_0 if there exist a finite Borel measure μ on the sphere S^{n-1} and a constant $C\in\mathbb{R}$ so that, for every $x\in\mathbb{R}^n$,*

$$\ln\|x\| = \int_{S^{n-1}} \ln|(x,\xi)|d\mu(\xi) + C. \tag{19}$$

It follows directly from the definition that μ is a probability measure, and the constant C equals

$$C = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \ln\|x\|dx - \frac{1}{2\sqrt{\pi}}\Gamma'(1/2) + \frac{1}{2}\frac{\Gamma'(n/2)}{\Gamma(n/2)}. \tag{20}$$

Also it was proved that if K is an infinitely smooth body then $(\ln\|x\|_K)^\wedge(\xi)$ is a homogeneous of degree $-n$ function on $\mathbb{R}^n\setminus\{0\}$, as seen from the following result.

Theorem 3.2. [KKYY, Theorem 4.1] *Let K be an infinitely smooth origin-symmetric star body in $\mathbb{R}^n$. Extend $A^{(n-1)}_{K,\xi}(0)$ to a homogeneous function of degree $-n$ of the variable $\xi \in \mathbb{R}^n \setminus \{0\}$. Then*

i) if n is odd

$$(\ln \|x\|_K)^\wedge (\xi) = (-1)^{(n+1)/2} \pi A^{(n-1)}_{K,\xi}(0), \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

ii) if n is even, then for $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$(\ln \|x\|_K)^\wedge (\xi) = a_n \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2} - ... - A^{n-2}_\xi(z)\frac{z^{n-2}}{(n-2)!}}{z^n} dz,$$

where $a_n = 2(-1)^{n/2+1}(n-1)!$

In particular, for an infinitely smooth origin-symmetric star body K, $(\ln \|x\|_K)^\wedge (\xi)$ is a continuous function on S^{n-1}, and moreover the measure in Definition 3.1 equals

$$d\mu(\xi) = -\frac{1}{(2\pi)^n} (\ln \|x\|_K)^\wedge (\xi) d\xi.$$

Since μ is a probability measure, one can see that

$$\int_{S^{n-1}} (\ln \|x\|_K)^\wedge(\theta) d\theta = -(2\pi)^n \tag{21}$$

for any infinitely smooth origin-symmetric star body K (see [KKYY, Remark 3.2]).

In our next Lemma we prove that a representation similar to (19) holds for all infinitely smooth bodies, with μ being a signed measure.

Lemma 3.3. *Let K be an infinitely smooth origin-symmetric star body in $\mathbb{R}^n$. Then*

$$\ln \|x\|_K = -\frac{1}{(2\pi)^n} \int_{S^{n-1}} \ln |(x,\xi)| (\ln \|x\|_K)^\wedge (\xi) d\xi + C_K, \tag{22}$$

where C_K is the constant from (20).

Proof. Since the body K is infinitely smooth, by Theorem 3.2, $(\ln \|x\|_K)^\wedge (\xi)$ is a continuous homogeneous function of degree $-n$ on $\mathbb{R}^n \setminus \{0\}$.

Let ϕ be an even test function supported outside of the origin, then

$$\left\langle \left(\int_{S^{n-1}} \ln |(x,\xi)| (\ln \|x\|_K)^\wedge (\xi) d\xi \right)^\wedge, \phi \right\rangle$$
$$= \left\langle \int_{S^{n-1}} \ln |(x,\xi)| (\ln \|x\|_K)^\wedge (\xi) d\xi, \hat{\phi}(x) \right\rangle$$
$$= \int_{\mathbb{R}^n} \left[\int_{S^{n-1}} \ln |(x,\xi)| (\ln \|x\|_K)^\wedge (\xi) d\xi \right] \hat{\phi}(x) dx$$

$$= \int_{S^{n-1}} \left[\int_{\mathbb{R}^n} \ln|(x,\xi)|\hat{\phi}(x)dx \right] \left(\ln\|x\|_K\right)^\wedge(\xi)d\xi$$

Now compute the inner integral using Fubini's theorem and the connection between the Radon and Fourier transforms (see e.g. [K5, Lemma 2.11]):

$$\int_{\mathbb{R}^n} \ln|(x,\xi)|\hat{\phi}(x)dx = \int_{\mathbb{R}} \ln|t| \int_{(x,\xi)=t} \hat{\phi}(x)dxdt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} (\ln|t|)^\wedge(z) \left(\int_{(x,\xi)=t} \hat{\phi}(x)dx \right)^\wedge (z)dz = -\frac{1}{2}\int_{\mathbb{R}} |z|^{-1}\hat{\hat{\phi}}(z\xi)dz$$

$$= -2^{n-1}\pi^n \int_{\mathbb{R}} |z|^{-1}\phi(z\xi)dz = -(2\pi)^n \int_0^\infty z^{-1}\phi(z\xi)dz$$

Here we used the formula for the Fourier transform of $\ln|t|$ (see [GS, p.362])

$$\left(\ln|z|\right)^\wedge(t) = -\pi|t|^{-1} \tag{23}$$

outside of the origin. Therefore, passing from polar to Euclidean coordinates and recalling from Theorem 3.2, that $(\ln\|x\|_K)^\wedge$ is a homogeneous function of degree $-n$ on $\mathbb{R}^n \setminus \{0\}$, we get

$$\langle \left(\int_{S^{n-1}} \ln|(x,\xi)| \left(\ln\|x\|_K\right)^\wedge(\xi)d\xi \right)^\wedge, \phi\rangle$$

$$= -(2\pi)^n \int_{S^{n-1}} \left[\int_0^\infty z^{-1}\phi(z\xi)dz \right] \left(\ln\|x\|_K\right)^\wedge(\xi)d\xi$$

$$= -(2\pi)^n \int_{\mathbb{R}^n} \phi(y) \left(\ln\|x\|_K\right)^\wedge(y)dy = -(2\pi)^n \langle \left(\ln\|x\|_K\right)^\wedge, \phi\rangle.$$

It follows that

$$\left(\int_{S^{n-1}} \ln|(x,\xi)| \left(\ln\|x\|_K\right)^\wedge(\xi)d\xi \right)^\wedge = -(2\pi)^n \left(\ln\|x\|_K\right)^\wedge$$

as distributions outside of the origin. Hence, the functions $-(2\pi)^n \ln\|x\|_K$ and $\int_{S^{n-1}} \ln|(x,\xi)| \left(\ln\|x\|_K\right)^\wedge(\xi)d\xi$ may differ only by a polynomial. But

$$\frac{1}{(2\pi)^n}\int_{S^{n-1}} \ln|(x,\xi)| \left(\ln\|x\|_K\right)^\wedge(\xi)d\xi + \ln\|x\|_K$$

is a homogeneous function of degree zero, therefore this polynomial is some constant C, which is exactly the constant from Definition 3.1, as computed in [KKYY].

□

Now we need a version of Parseval's formula for L_0. How does the formula of Lemma 2.4 look if we pass to the limit as $p \to 0$? The answer to this question is given in our next Lemma. Even though in the proof we use an argument based on Lemma 3.3, one can obtain the following Lemma by taking the limit in Parseval's formula.

Lemma 3.4. *Let K and L be infinitely smooth origin-symmetric star bodies in $\mathbb{R}^n$. Then*

$$-\frac{1}{(2\pi)^n}\int_{S^{n-1}}\left[\int_L \ln|(x,\xi)|dx\right](\ln\|x\|_K)^\wedge(\xi)d\xi = \int_L(\ln\|x\|_K - C_K)dx.$$

Proof. By Lemma 3.3 we have

$$-\frac{1}{(2\pi)^n}\int_{S^{n-1}}\ln|(x,\xi)|\,(\ln\|x\|_K)^\wedge(\xi)d\xi = \ln\|x\|_K - C_K.$$

Integrating this equality over the body L we get the statement of the Lemma. □

Now we prove the main result of this section.

Theorem 3.5. *Let K and L be two origin-symmetric star bodies in $\mathbb{R}^n$ such that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 and*

$$\Gamma_0^* K \subset \Gamma_0^* L \tag{24}$$

for every $\xi \in S^{n-1}$. Then

$$\mathrm{vol}(L) \le \mathrm{vol}(K).$$

Proof. Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0, there exist a probability measure μ_K on S^{n-1} (which is the restriction of the Fourier transform of $\ln\|x\|_K$ to the unit sphere) and a constant C_K from Definition 3.1.

Rewrite inequality (24) as follows:

$$\frac{\int_L \ln|(x,\xi)|dx}{\mathrm{vol}(L)} \le \frac{\int_K \ln|(x,\xi)|dx}{\mathrm{vol}(K)},$$

and integrate it over S^{n-1} with respect to μ_K to get

$$\int_{S^{n-1}}\frac{\int_L \ln|(x,\xi)|dx}{\mathrm{vol}(L)}d\mu_K(\xi) \le \int_{S^{n-1}}\frac{\int_K \ln|(x,\xi)|dx}{\mathrm{vol}(K)}d\mu_K(\xi).$$

Using the Fubini theorem and the definition of embedding in L_0, we get

$$\frac{1}{\mathrm{vol}(L)}\int_L(\ln\|x\|_K - C_K)dx \le \frac{1}{\mathrm{vol}(K)}\int_K(\ln\|x\|_K - C_K)dx.$$

Therefore

$$\frac{1}{\mathrm{vol}(L)}\int_L \ln\|x\|_K dx \le \frac{1}{\mathrm{vol}(K)}\int_K \ln\|x\|_K dx = -\frac{1}{n},$$

where the latter equality follows from the formula

$$\frac{1}{\mathrm{vol}(K)}\int_K \|x\|_K^p dx = \frac{n}{n+p},$$

that we had earlier, after differentiating and letting $p = 0$.

Now use the following inequality from Milman and Pajor [MP, Section 2.2]:

$$\frac{1}{\mathrm{vol}(L)}\int_L \ln\|x\|_K dx \ge -\frac{1}{n} + \frac{1}{n}[\ln(\mathrm{vol}(L)) - \ln(\mathrm{vol}(K))]. \tag{25}$$

Therefore

$$\mathrm{vol}(L) \leq \mathrm{vol}(K).$$

$\square$

Remark 3.6. *Since every three dimensional normed space embeds in L_0 (see* [KKYY, Corollary 4.3]*), the previous theorem holds for all convex bodies in $\mathbb{R}^3$.*

To prove our next Theorem we need the following Lemma.

Lemma 3.7. *Let K be an origin-symmetric star body in $\mathbb{R}^n$, then the Fourier transform of $\|x\|_K^{-n}$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$ and equals*

$$\begin{aligned}(\|x\|_K^{-n})^\wedge(\xi) = & - & n\int_K \ln|(x,\xi)|dx + \\ & + & (n\Gamma'(1)-1)\mathrm{vol}(K) - \int_{S^{n-1}} \|\theta\|_K^{-n} \ln\|\theta\|_K d\theta.\end{aligned}$$

Proof. Let ϕ be an even test function. Using the definition of the action of a homogeneous function of degree $-n$ (see [GS, p.303]) we get

$$\begin{aligned}
&\langle (\|x\|_K^{-n})^\wedge, \phi\rangle = \langle \|x\|_K^{-n}, \hat{\phi}(x)\rangle \\
&= \int_{B_1(0)} \|x\|_K^{-n}(\hat{\phi}(x)-\hat{\phi}(0))dx + \int_{\mathbb{R}^n\setminus B_1(0)} \|x\|_K^{-n}\hat{\phi}(x)dx \\
&= \int_{S^{n-1}}\int_0^1 r^{-1}\|\theta\|_K^{-n}(\hat{\phi}(r\theta)-\hat{\phi}(0))drd\theta + \int_{S^{n-1}}\int_1^\infty r^{-1}\|\theta\|_K^{-n}\hat{\phi}(r\theta)drd\theta \\
&= \int_{S^{n-1}} \|\theta\|_K^{-n}\left(\int_0^1 r^{-1}(\hat{\phi}(r\theta)-\hat{\phi}(0))dr + \int_1^\infty r^{-1}\hat{\phi}(r\theta)dr\right)d\theta \\
&= \frac{1}{2}\int_{S^{n-1}} \|\theta\|_K^{-n}\langle |r|^{-1}, \hat{\phi}(r\theta)\rangle d\theta \\
&= \frac{1}{2}\int_{S^{n-1}} \|\theta\|_K^{-n}\langle 2\Gamma'(1) - 2\ln|t|, \int_{(\theta,\xi)=t}\phi(\xi)d\xi\rangle d\theta \\
&= \langle \int_{S^{n-1}} \|\theta\|_K^{-n}\left(\Gamma'(1)-\ln|(\theta,\xi)|\right)d\theta, \phi(\xi)\rangle.
\end{aligned}$$

Here we used the formula for the Fourier transform of $|r|^{-1}$ from [GS, p.361]:

$$(|r|^{-1})^\wedge(t) = 2\Gamma'(1) - 2\ln|t|.$$

Thus we have proved that

$$(\|x\|_K^{-n})^\wedge(\xi) \quad = \quad \int_{S^{n-1}} \|\theta\|_K^{-n}\Big(\Gamma'(1) - \ln|(\theta,\xi)|\Big)d\theta. \tag{26}$$

Next, let us compute the following:

$$\begin{aligned}
\int_K \ln|(x,\xi)|dx &= \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} r^{n-1} \ln|(r\theta,\xi)|drd\theta \\
&= \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} r^{n-1} \ln r dr d\theta + \int_{S^{n-1}} \ln|(\theta,\xi)| \int_0^{\|\theta\|_K^{-1}} r^{n-1} dr d\theta \\
&= -\frac{1}{n} \int_{S^{n-1}} \left(\|\theta\|_K^{-n} \ln\|\theta\|_K + \frac{1}{n}\|\theta\|_K^{-n}\right) d\theta + \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} \ln|(\theta,\xi)|d\theta.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{S^{n-1}} \|\theta\|_K^{-n} \ln|(\theta,\xi)|d\theta = \\
&\qquad = n\int_K \ln|(x,\xi)|dx + \int_{S^{n-1}} \left(\|\theta\|_K^{-n} \ln\|\theta\|_K + \frac{1}{n}\|\theta\|_K^{-n}\right) d\theta.
\end{aligned}$$

Combining this formula with the formula (26), we get

$$\begin{aligned}
(\|x\|_K^{-n})^\wedge(\xi) = &- n\int_K \ln|(x,\xi)|dx + \\
&+ (n\Gamma'(1)-1)\mathrm{vol}(K) - \int_{S^{n-1}} \|\theta\|_K^{-n} \ln\|\theta\|_K d\theta.
\end{aligned}$$

□

Theorem 3.8. *There are convex bodies K and L in $\mathbb{R}^n$, $n \geq 4$ such that*

$$\Gamma_0^* K \subset \Gamma_0^* L$$

for every $\xi \in S^{n-1}$, but

$$\mathrm{vol}(K) < \mathrm{vol}(L).$$

Proof. Let L be a strictly convex infinitely smooth body in $\mathbb{R}^n$, $n \geq 4$, for which $-(\ln\|x\|_L)^\wedge$ is not positive everywhere. (See [KKYY, Theorem 4.4] for an explicit construction of such a body.)

Let $\xi \in S^{n-1}$ be such that $-(\ln\|x\|_L)^\wedge(\xi) < 0$. By continuity of the function $(\ln\|x\|_L)^\wedge(\theta)$ on the sphere there is a neighborhood of ξ where this function is negative. Let

$$\Omega = \{\theta \in S^{n-1} : -(\ln\|x\|_L)^\wedge(\theta) < 0\}.$$

Choose an infinitely smooth body D whose Minkowski norm $\|x\|_D$ is equal to 1 outside of Ω and $\|x\|_D < 1$ for $x \in \Omega$. Let v be a homogeneous function of degree 0 on $\mathbb{R}^n \setminus \{0\}$, defined as follows:

$$v(x) = \ln\|x\|_D - \ln|x|_2.$$

Clearly $v(x) < 0$ if $x \in \Omega$ and $v(x) = 0$ if $x \in S^{n-1} \setminus \Omega$.

In view of Theorem 3.2, the Fourier transforms of $\ln\|x\|_D$ and $\ln|x|_2$ outside of the origin are some homogeneous functions of degree $-n$, therefore the Fourier transform of $v(x)$ outside of the origin is equal to $|x|_2^{-n} g(x/|x|_2)$ for some infinitely smooth function g on S^{n-1}. Since by (21)

$$\int_{S^{n-1}} (\ln \|x\|_D)^\wedge(\theta) d\theta = \int_{S^{n-1}} (\ln |x|_2)^\wedge(\theta) d\theta = -(2\pi)^n,$$

we have

$$\int_{S^{n-1}} g(\theta) d\theta = 0. \tag{27}$$

Define a body K by the formula:

$$\frac{\|x\|_K^{-n}}{\mathrm{vol}(K)} = \frac{\|x\|_L^{-n}}{\mathrm{vol}(L)} + n(2\pi)^{-n}\epsilon |x|_2^{-n} g(x/|x|_2). \tag{28}$$

Note that formula (27) validates this definition, since integrating the last equality over the unit sphere we get the same quantity in both sides. Also, since L is strictly convex, there is an ϵ small enough, so that K is also convex (see e.g. the perturbation argument from [K5, p.96]). From now on we fix such an ϵ.

Now we will show that K together with L constructed above satisfy the assumptions of the theorem. Apply the Fourier transform to both sides of (28). Note, that the Fourier transform of $|x|_2^{-n} g(x/|x|_2)$ is equal to $(2\pi)^n v$ on test functions, whose Fourier transform is supported outside of the origin. Such distributions can differ only by a polynomial, which must be a constant in this case, since both functions cannot grow faster than a logarithm (see Lemma 3.7). So

$$\left(|x|_2^{-n} g(x/|x|_2)\right)^\wedge = (2\pi)^n (v + \alpha),$$

for some constant α whose value has no significance for us. Hence, by Lemma 3.7, the Fourier transform of (28) looks as follows:

$$-\frac{n \int_K \ln |(x,\xi)| dx}{\mathrm{vol}(K)} = -\frac{n \int_L \ln |(x,\xi)| dx}{\mathrm{vol}(L)} + n\epsilon \cdot v(\xi) + C, \tag{29}$$

where the constant C equals

$$C = \frac{\int_K \|\theta\|_K^{-n} \ln \|\theta\|_K d\theta dx}{\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln \|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha.$$

Since the bodies L and D are fixed, dilating the body K we can make this constant equal to zero. Indeed, multiply the Minkowski functional of K by a positive constant λ, then

$$\begin{aligned} C &= \frac{\int_K (\lambda \|\theta\|_K)^{-n} \ln \lambda \|\theta\|_K d\theta dx}{\lambda^{-n} \mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln \|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha \\ &= \frac{\int_K \|\theta\|_K^{-n} \left[\ln \lambda + \ln \|\theta\|_K\right] d\theta dx}{\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln \|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha \\ &= n \ln \lambda + \frac{\int_K \|\theta\|_K^{-n} \ln \|\theta\|_K d\theta dx}{\mathrm{vol}(K)} - \frac{\int_L \|\theta\|_L^{-n} \ln \|\theta\|_L d\theta dx}{\mathrm{vol}(L)} + n\epsilon \cdot \alpha. \end{aligned}$$

One can choose a $\lambda > 0$ so that $C = 0$. Therefore from (29) we get

$$\frac{\int_K \ln|\langle x,\xi\rangle|dx}{\mathrm{vol}(K)} = \frac{\int_L \ln|\langle x,\xi\rangle|dx}{\mathrm{vol}(L)} - \epsilon\ v(\xi) \geq \frac{\int_L \ln|\langle x,\xi\rangle|dx}{\mathrm{vol}(L)}, \tag{30}$$

since v is non-positive. Therefore

$$\Gamma_0^* K \subset \Gamma_0^* L.$$

Now using Parseval's formula and inequality (30) we get

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(K)} \int_K (\ln\|x\|_L - C_L)dx = \\
&= -\frac{1}{(2\pi)^n}\frac{1}{\mathrm{vol}(K)}\int_{S^{n-1}} \left[\int_K \ln|\langle x,\xi\rangle|dx\right] (\ln\|x\|_L)^\wedge(\xi)d\xi \\
&= -\frac{1}{(2\pi)^n}\int_{S^{n-1}} \left[\frac{1}{\mathrm{vol}(L)}\int_L \ln|\langle x,\xi\rangle|dx - \epsilon v(\xi)\right] (\ln\|x\|_L)^\wedge(\xi)d\xi \\
&= -\frac{1}{(2\pi)^n}\frac{1}{\mathrm{vol}(L)}\int_{S^{n-1}} \left[\int_L \ln|\langle x,\xi\rangle|dx\right] (\ln\|x\|_L)^\wedge(\xi)d\xi \\
&\quad +\frac{1}{(2\pi)^n}\frac{1}{\mathrm{vol}(L)}\int_{S^{n-1}} \epsilon v(\xi)(\ln\|x\|_L)^\wedge(\xi)d\xi \\
&< -\frac{1}{(2\pi)^n}\frac{1}{\mathrm{vol}(L)}\int_{S^{n-1}} \left[\int_L \ln|\langle x,\xi\rangle|dx\right] (\ln\|x\|_L)^\wedge(\xi)d\xi \\
&= \frac{1}{\mathrm{vol}(L)}\int_L (\ln\|x\|_L - C_L)dx,
\end{aligned}$$

where the inequality follows from the fact that v is non-positive and supported on the set where $-(\ln\|x\|_L)^\wedge(\xi) < 0$.

Recalling the inequality (25)

$$-\frac{1}{n} \geq \frac{1}{\mathrm{vol}(K)}\int_K \ln\|x\|_L dx \geq -\frac{1}{n} + \frac{1}{n}[\ln(\mathrm{vol}(K)) - \ln(\mathrm{vol}(L))],$$

we get

$$\mathrm{vol}(K) < \mathrm{vol}(L).$$

□

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