A GENERALIZATION OF WINTERNITZ'S THEOREM AND ITS DISCRETE VERSION

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Abstract. Let K be a convex body in the plane. Cut K by a line passing through its centroid. It is a well-known result, due to Winternitz, that the areas of the resulting two pieces are at least $4/9$ times the area of K and at most $5/9$ times the area of K. We generalize this inequality to the case when the body is cut by a line not passing through the centroid. As an application we obtain a discrete version of Winternitz's theorem.

1. Introduction

Let K be a convex body in $\mathbb{R}^2$, i.e., a convex compact set with non-empty interior. The centroid of K is the point

$$g(K) = \frac{1}{|K|}\int_K x\,dx \in \mathrm{int}K,$$

where $|K|$ is the area of K. Consider any line passing through $g(K)$. It divides the plane into two half-planes. Let H be either of these half-planes. Then

$$\frac{4}{9} \le \frac{|K\cap H|}{|K|} \le \frac{5}{9}, \tag{1}$$

where both bounds are sharp. This inequality is due to Winternitz [3]. A generalization of this result to all dimensions was obtained by Grünbaum [8], and (1) is often referred to as the two-dimensional case of Grünbaum's inequality. For recent generalizations of Grünbaum's result to sections and projections of convex bodies, see [5], [9], [10], [12].

Now assume that the boundary of H does not contain the centroid of K. Can we obtain an inequality similar to (1) with sharp constants?

In this paper we give an affirmative answer to this question. Let K be a convex body in $\mathbb{R}^2$ whose centroid is at the origin. Let $-1 < \alpha < 2$ and let $\xi \in S^1$, where S^1 is the unit circle in $\mathbb{R}^2$. Consider the (affine) half-plane

$$H = \{x \in \mathbb{R}^2 : \langle x, \xi\rangle \ge \alpha h_K(-\xi)\},$$

where h_K is the support function of K. Then there are sharp constants $C_1(\alpha)$ and $C_2(\alpha)$ such that

$$C_1(\alpha) \le \frac{|K\cap H|}{|K|} \le C_2(\alpha). \tag{2}$$

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The values of $C_1(\alpha)$ and $C_2(\alpha)$ are given in Theorem 1, along with equality conditions. Note that in (1) it is enough to prove only one of the inequalities and the other follows immediately. In (2) one has to prove both inequalities separately.

The second goal of this paper is to discuss applications of Theorem 1. In recent years there has been a lot of interest in transferring results about convex bodies to discrete settings; see, for example, [1], [7], [11], [13]. Here we are interested in discrete versions of Winternitz's theorem.

For a bounded set $A \subset \mathbb{R}^2$, we will denote by $\#A$ the cardinality of the lattice set $A \cap \mathbb{Z}^2$. Let P be a convex polygon in $\mathbb{R}^2$ whose vertices belong to the integer lattice $\mathbb{Z}^2$. Such polygons will be called integer polygons. Let H be any closed half-plane that contains $g(P)$ on its boundary. One can ask whether there exists an absolute positive constant C such that

$$\frac{\#(P \cap H)}{\#P} \geq C.$$

The following example shows that in this formulation the question has a negative answer.

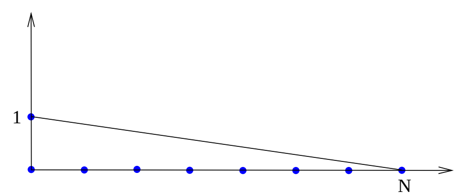

Figure 1

If we draw a horizontal line through the centroid of this triangle, then there is only one integer point from the triangle above this line, while the total number of integer points in the triangle can be made arbitrarily large.

Thus we need to modify the question. Let $t \geq 2$ be an integer and let tP be the dilate of P by a factor of t. Does there exist a positive constant $C(t)$ such that

$$\frac{\#(tP \cap H)}{\#(tP)} \geq C(t), \tag{3}$$

for every convex integer polygon P and every half-plane H containing the centroid of tP on its boundary? Moreover, in view of (1) we want $C(t)$ to approach $4/9$ as $t \to \infty$. In this paper we will show that the answer to this question is affirmative; see Theorems 2 and 3.

2. Preliminaries

Let P be a convex integer polygon in $\mathbb{R}^2$. Let i be the number of lattice points in the interior of P and b the number of lattice points on the boundary of P. The Ehrhart polynomial gives the number of lattice points in its dilates:

$$\#(tP) = |P|t^2 + \frac{b}{2}t + 1, \tag{4}$$

see [2, p. 42]. In particular, when $t = 1$, this is the famous theorem of Pick:

$$|P| = i + \frac{b}{2} - 1.$$

Combining Pick's formula and the Ehrhart polynomial, we get

$$\#(tP) = |P|t^2 + \frac{1}{2}tb + 1 = (i + \frac{1}{2}b - 1)t^2 + \frac{1}{2}tb + 1$$
$$= t^2 i + \frac{1}{2}(t^2 + t)b + 1 - t^2. \tag{5}$$

Let K be a convex body in $\mathbb{R}^2$. The *support function* of K is defined by

$$h_K(\xi) = \max_{x \in K}\langle x, \xi\rangle, \qquad \xi \in S^1.$$

A line l is called a supporting line to K if l intersects K only at boundary points of K. The support function of K gives the distances from the origin to the supporting lines of K.

It is a well-known fact due to Minkowski and Radon that for a convex body $K \subset \mathbb{R}^2$ with centroid at the origin we have

$$\frac{1}{2}h_K(\xi) \le h_K(-\xi) \le 2h_K(\xi), \tag{6}$$

for every $\xi \in S^1$; see [4, p. 58].

3. Generalization of Winternitz's theorem

We will now prove a generalization of Winternitz's theorem for the case when the body is cut by a line not passing through its centroid. In the theorem below the classical result when the line contains the centroid corresponds to $\alpha = 0$.

Theorem 1. *Let K be a convex body in $\mathbb{R}^2$ with centroid at the origin. Let $-1 < \alpha < 2$ and let $\xi \in S^1$. Consider the (affine) half-plane*

$$H = \{x \in \mathbb{R}^2 : \langle x, \xi\rangle \ge \alpha h_K(-\xi)\}.$$

Then

$$C_1(\alpha) \le \frac{|K \cap H|}{|K|} \le C_2(\alpha), \tag{7}$$

where

$$C_1(\alpha) = \begin{cases} \frac{1}{9}(2-\alpha)^2, & \alpha \in (-1,0), \\ \frac{4}{9}(1+\alpha)(1-2\alpha), & \alpha \in (0,1/2), \\ 0 & \alpha \in [1/2,2), \end{cases}$$

and

$$C_2(\alpha) = \begin{cases} 1 - \frac{4}{9}(1+\alpha)^2, & \alpha \in (-1,0], \\ \frac{5-3\alpha}{9(1+\alpha)}, & \alpha \in [0,1], \\ \frac{1}{9}(2-\alpha)^2, & \alpha \in [1,2). \end{cases}$$

The lower and upper bounds in (7) are sharp. The equality cases are discussed in the proof below.

Proof. To prove the lower and upper bounds in (7) we will first perform some transformations. Assume that the vector ξ coincides with the positive direction of the x-axis. First we will replace the body K by another convex body constructed as follows (this process is sometimes called Blaschke shaking; see [6, p. 92] for details). Take any line l perpendicular to the x-axis that intersects the body K. Replace the segment $K \cap l$ with another segment on the line l that lies in the upper half-plane and whose lower end-point belongs to the x-axis. Without loss of generality, this new body will also be denoted by K. Note that the following quantities did not change: the x-coordinate of the centroid, the support function of K in the directions of $\pm\xi$, and the areas of K and $K \cap H$.

We will shift K along the x-axis to the right so that the left vertical supporting line to K coincides with the y-axis. Let c be the x-coordinate of the centroid of the shifted body, which we still denote by K. Then the half-plane H is given by

$$H = \{(x, y) \in \mathbb{R}^2 : x \geq (1 + \alpha)c\}.$$

We will now prove the upper bound in (7). Observe that the restriction $\alpha < 2$ comes from inequality (6). Take the point of intersection of the line $x = (1 + \alpha)c$ with the boundary of K that lies above the x-axis. Draw a straight line L through this point that cuts off a region $B \subset K$ to the left of this point so that the area of B is equal to the area of the region A bounded by the line L, the positive part of the y-axis and the boundary of K; see Figure 2. If we add A and remove B from the body K, we will get another convex body whose centroid is shifted to the left.

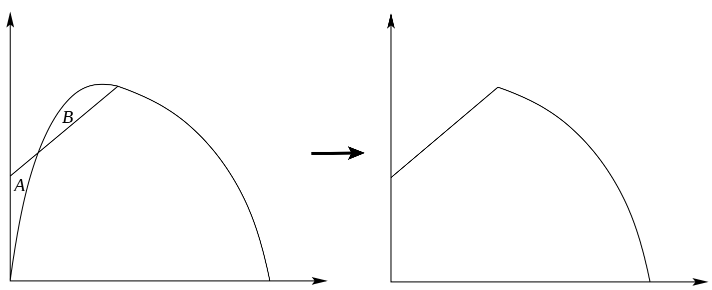

FIGURE 2. Add A and remove B

Now extend the line L and bisect the new body by a vertical line such that the area of the region C (formed by this vertical line, the line L and the boundary of K) is equal to the area of the region D (formed by the vertical line, the x-axis and the boundary of K). If we add C and remove D from the body, the centroid shifts to the left again (see Figure 3) and $|K \cap H|$ can only increase.

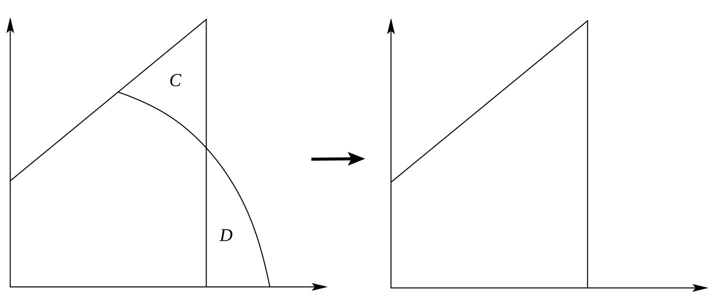

FIGURE 3. Add C and remove D

As a result, we get a trapezoid (which could possibly become a triangle). During these procedures we did not change the area of the body, but increased $|K \cap H|$. Therefore it is enough to find the maximum of the ratio $|K \cap H|/|K|$ among such trapezoids. Without loss of generality, we will assume that our trapezoid is bounded by the lines $x = 0$, $x = 1$, $y = 0$, and $y = mx + b$; see Figure 4.

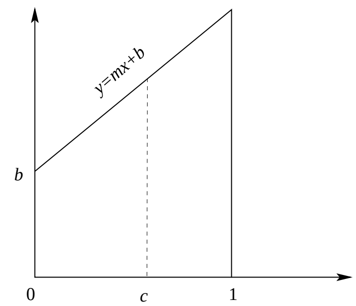

FIGURE 4

For this polygon to be well defined we need to assume that $b \geq 0$ and either (1) $m \geq 0$ or (2) $m < 0$ and $m + b \geq 0$. For now we will consider the case $m \neq 0$. We will comment on the case $m = 0$ later.

Denoting this body again by K, we see that

$$|K| = \int_0^1 (mx + b)\, dx = \frac{1}{2m}\left((m+b)^2 - b^2\right).$$

The x-coordinate of the centroid is given by

$$c = \frac{1}{|K|}\int_0^1 x(mx+b)\, dx = \frac{2m+3b}{3m+6b}. \tag{8}$$

For brevity, we will denote $\beta = 1 + \alpha$. Observe that $0 < \beta < 3$ since $-1 < \alpha < 2$. Then

$$|K \cap H| = \int_{\beta c}^1 (mx+b)\, dx = \frac{1}{2m}\left((m+b)^2 - (m\beta c + b)^2\right).$$

Let $z = b/m$. Then

$$c = \frac{2+3z}{3+6z}$$

and

$$\frac{|K \cap H|}{|K|} = \frac{(1+z)^2 - (\beta c + z)^2}{(1+z)^2 - z^2}.$$

Denote the latter function of z by f. We need to find the supremum of this function f on the domain $z \in (-\infty, -1] \cup [0, \infty)$. Differentiating f, we get the only critical number

$$z_0 = \frac{2\beta - 2}{4 - 3\beta}.$$

Then

$$f(z_0) = \frac{8 - 3\beta}{9\beta}.$$

Also, $f(0) = 1 - 4\beta^2/9$, $f(-1) = (\beta/3 - 1)^2$ and $\lim_{z\to\pm\infty} f(z) = 1 - \beta/2$.

Now we will choose the maximum of these values on corresponding intervals. Note that $z_0 \in (-\infty, -1] \cup [0, \infty)$ if and only if $\beta \in [1, 2] \setminus \{4/3\}$. For these values of β, $\sup f = (8 - 3\beta)/(9\beta)$. One can see that the latter also works for $\beta = 4/3$. If $\beta \in (0, 1)$, then $\sup f = 1 - 4\beta^2/9$. Finally, if $\beta \in [2, 3)$, then $\sup f = (\beta/3 - 1)^2$.

Replacing $\beta = 1 + \alpha$, we now see that

$$\frac{|K \cap H|}{|K|} \leq C_2(\alpha),$$

where $C_2(\alpha)$ is the function given in the statement of the theorem.

Finally we go back to the case $m = 0$. Clearly K is a rectangle and therefore

$$\frac{|K \cap H|}{|K|} = \frac{1}{2}(1-\alpha),$$

which is less than or equal to $C_2(\alpha)$.

We will now discuss the equality cases. One can see that they are achieved when K has the shape of a trapezoid (which possibly becomes a triangle). When $\alpha \in (-1, 0]$, the maximum of f is attained at $z = 0$, which means that $b = 0$ and thus the extreme shapes are triangles with a vertex at $(0, 0)$ and two vertices on the line $x = 1$. When $\alpha \in [0, 1)$, the maximum of f is attained at z_0, found above. Thus the extreme shape is a trapezoid (which becomes a rectangle when $\alpha = 1/3$). Finally, when $\alpha \in [1, 2)$, the maximum of f is attained at $z = -1$. The extreme shape is again a triangle, now with one vertex at $(1, 0)$ and two vertices on the y-axis. The evolution of extreme shapes as α changes from -1 to 2 is shown in Figure 5.

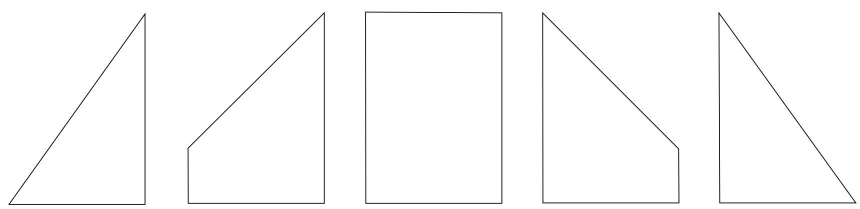

FIGURE 5. Extreme shapes for the upper bound

Now we will turn to the lower bound in (7). As in the proof of the upper bound, we can start with a body bounded by the graph of a positive concave function and the x-axis, and assume that the y-axis is the left supporting vertical line.

Take the point of intersection of the line $x = \beta c$ with the boundary of K that lies above the x-axis. Draw a straight line L through this point that cuts off a region $B \subset K$ to the right of this point so that the area of B is equal to the area of the region A bounded by the line L, the x-axis and the boundary of K; see Figure 6. If we add A and remove B from the body K, we will get another convex body whose centroid is shifted to the right.

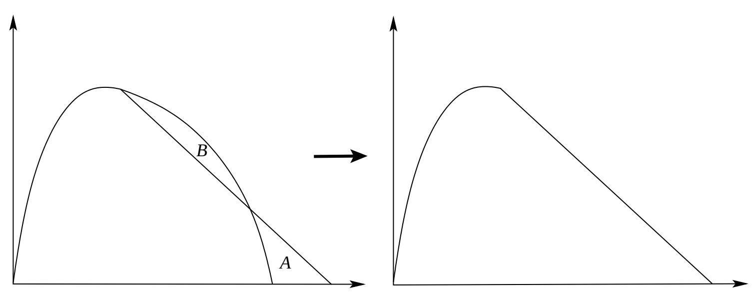

FIGURE 6. Add A and remove B

Now extend the line L and bisect the new body by a line L_1 passing through the origin so that the area of the region C (formed by the line L_1, the line L and the boundary of K) is equal to the area of the region D (formed by the line L_1 and the boundary of K). If we add C and remove D from the body, the centroid shifts to the right again; see Figure 7.

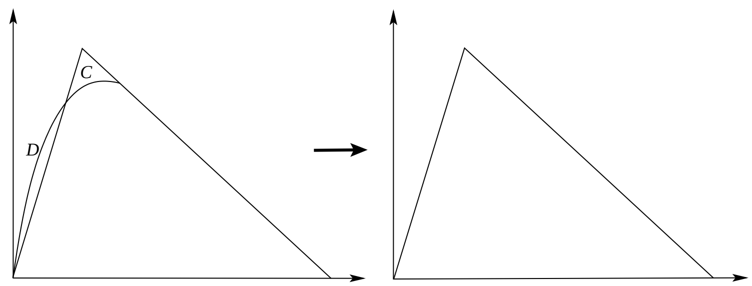

FIGURE 7. Add C and remove D

As a result, we get a triangle with its base on the x-axis. The centroid of this triangle is located to the right of the original body K, while the area and the left supporting line remain unchanged. Thus $|K \cap H|/|K|$ could only decrease. Therefore it is enough to find the minimum of the ratio $|K \cap H|/|K|$ among such triangles. Without loss of generality, we will assume that $(0,0)$, $(1,0)$, and $(s,1)$ are the vertices of our triangle, where $0 \le s \le 1$. Denoting the triangle again by K, we see that $|K| = 1/2$. One can check that the x-coordinate of its centroid equals

$$c = \frac{1}{3}(s+1).$$

Note that the interesting values of β are between 0 and 3/2: if $\beta > 3/2$, then there are triangles for which $K \cap H = \emptyset$.

We will now compute $|K \cap H|/|K|$ for the following two cases: $0 \le s \le \beta c$ and $\beta c \le s \le 1$. Note that these two conditions are equivalent to $0 \le s \le \beta/(3-\beta)$ and $\beta/(3-\beta) \le s \le 1$ respectively. Elementary calculations show that

$$\frac{|K \cap H|}{|K|} = \begin{cases} \frac{\left(1-\frac{1}{3}\beta(1+s)\right)^2}{1-s}, & 0 \le s \le \dfrac{\beta}{3-\beta}, \\ 1 - \frac{\beta^2(1+s)^2}{9s}, & \dfrac{\beta}{3-\beta} \le s \le 1. \end{cases} \tag{9}$$

To find the minimum of this function, we will find its critical numbers. Note that there are no critical numbers in the interval $(\beta/(3-\beta), 1)$ and there is a critical number in the interval $(0, \beta/(3-\beta))$ only if $\beta > 1$. This critical number is $3(\beta-1)/\beta$.

Thus to minimize the function in (9) for $0 \le \beta \le 1$, we need to compute its values at $s = 0$, $s = \beta/(3-\beta)$, and $s = 1$. These values are $(3-\beta)^2/9$, $(3-2\beta)/(3-\beta)$, and $1 - \beta^2/9$, respectively. The smallest of these is $(3-\beta)^2/9$. Thus,

$$\frac{|K \cap H|}{|K|} \ge \frac{1}{9}(3-\beta)^2, \quad \text{when} \quad 0 \le \beta \le 1.$$

When $1 \le \beta \le 3/2$ we also need to find the value of the function in (9) at the critical number $s = 3(\beta-1)/\beta$. This gives $4\beta(3-2\beta)/9$, which is smaller than $(3-\beta)^2/9$ found previously. Therefore

$$\frac{|K \cap H|}{|K|} \ge \frac{4}{9}\beta(3-2\beta), \quad \text{when} \quad 1 \le \beta \le 3/2.$$

Replacing $\beta = 1+\alpha$, we get the lower bound in (7).

Finally, let us discuss the equality cases. When $\alpha \in (-1,0)$, the minimum is attained at $s=0$. This corresponds to the triangle with the vertices $(0,0)$, $(1,0)$, and $(0,1)$. When α increases from 0 to $1/2$, s increases from 0 to 1. Thus the vertex $(s,1)$ moves from the point $(0,1)$ to $(1,1)$. The evolution of extreme shapes as α changes from -1 to $1/2$ is shown in Figure 8.

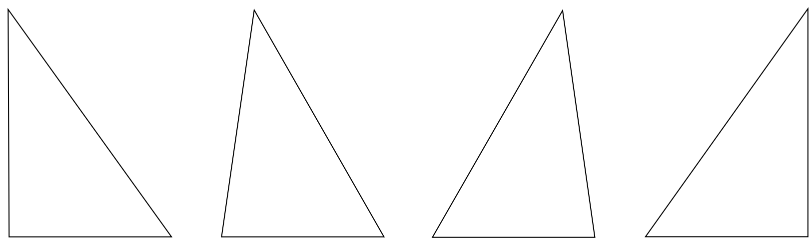

Figure 8. Extreme shapes for the lower bound

□

4. Discrete versions of Winternitz's theorem

In this section we will discuss discrete versions of Winternitz's theorem. Our goal is to obtain a bound of the form (3). When $t=3$ we will obtain such a bound using Theorem 1.

Theorem 2. *Let P be an integer polygon and $3P$ its dilate by a factor of 3. Let H be any half-plane that contains the centroid of $3P$ on its boundary. Then*

$$\frac{\#(3P\cap H)}{\#(3P)} > \frac{1}{6},$$

where $1/6$ is the best possible constant.

Proof. Consider the supporting line to the polygon P that is parallel to the boundary line of H and is contained in H. Let O be any integer point from P located on this supporting line. We will translate the origin to O and consider dilates with respect to this point.

Let $\bar{H}$ be the closed half-plane complementing H. Let Q be the largest integer polygon contained in $(2P)\cap \bar{H}$. First assume that Q is a non-degenerate polygon, i.e., it contains interior points.

We have

$$\#(3P\cap H) \geq \#(2P\cap H) \geq \#(2P) - \#Q.$$

Let i be the number of integer points in the interior of P and let b be the number of integer points on the boundary of P. Using formula (5), we get

$$\#(2P) = 4i + 3b - 3.$$

On the other hand, if we denote by i_Q and b_Q the number of integer points in the interior and on the boundary of Q respectively, then

$$\#Q = i_Q + b_Q \leq 2\left(i_Q + \frac{1}{2}b_Q\right) = 2|Q| + 2.$$

Since the distance from O to the centroid of $2P$ is twice the distance between the centroids of $2P$ and $3P$, we can use the upper bound from (7) with $\alpha = 1/2$ to get

$$|Q| \le |(2P) \cap \bar{H}| \le \frac{7}{27}|2P| = \frac{28}{27}|P|.$$

Therefore,

$$\#Q \le \frac{56}{27}|P| + 2 = \frac{56}{27}\left(i + \frac{b}{2} - 1\right) + 2 = \frac{56}{27}i + \frac{28}{27}b - \frac{2}{27},$$

and so

$$\#(3P \cap H) \ge \#(2P) - \#Q \ge \frac{52}{27}i + \frac{53}{27}b - \frac{79}{27}.$$

By formula (5),

$$\#(3P) = 9i + 6b - 8.$$

Hence,

$$\frac{\#(3P \cap H)}{\#(3P)} \ge \frac{\frac{52}{27}i + \frac{53}{27}b - \frac{79}{27}}{9i + 6b - 8} \ge \frac{52}{243},$$

where the last inequality can be checked directly using $b \ge 3$.

Now assume that Q is a degenerate polygon; i.e., all its points lie on a line. Observe that these points can only lie on the boundary of $2P$, or more precisely, on one edge of $2P$. The number of integer points on the boundary of $2P$ is $2b$. Also note that each edge of $2P$ contains two integer vertices and at least one integer point between them. Since there is a vertex of $2P$ that does not belong to Q, we also conclude that the two adjacent integer points on the boundary of $2P$ do not belong to Q as well. Therefore,

$$\#Q \le 2b - 3.$$

Thus,

$$\#(3P \cap H) \ge \#(2P) - \#Q \ge 4i + 3b - 3 - (2b - 3) = 4i + b,$$

and therefore,

$$\frac{\#(3P \cap H)}{\#(3P)} \ge \frac{4i + b}{9i + 6b - 8} > \frac{1}{6}.$$

Since $\frac{52}{243} > \frac{1}{6}$, we obtain the statement of the theorem.

The optimality of the constant $1/6$ follows from the following example. Consider the trapezoid with vertices at the points $(0,0)$, $(0,1)$, $(1,1)$, and $(N,0)$, where N is a large positive integer; see Figure 9. We will denote this trapezoid by T.

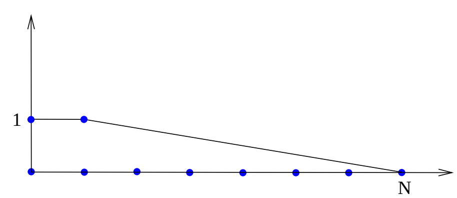

FIGURE 9

Note that for the trapezoid shown in Figure 4 with $b > 0$ and $m > 0$, the x-coordinate of its centroid is strictly less than $2/3$, which can be seen from formula (8). Therefore, the centroid of T lies strictly above the line $y = 1/3$. Now dilate the trapezoid T by a factor of 3. There will be $3N + 1$ integer points of $3T$ on the line

$y = 0$, $2N+2$ points on the line $y = 1$, $N+3$ points on the line $y = 2$, and 4 points on the line $y = 3$. The centroid of $3T$ lies strictly above the line $y = 1$. Above this line we have $N+7$ integer points of $3T$, while the total number of integer points in $3T$ is $6N + 10$. The optimality of the constant $1/6$ follows by taking N arbitrarily large.

□

The method from the previous theorem can be used for any dilation factor t, but unfortunately it does not give a good bound for large values of t. For large t we will use a different approach.

Theorem 3. *Let P be an integer polygon and tP its dilation by a factor of t. Let H be any half-plane that contains the centroid of tP on its boundary. Then*

$$\frac{\#(tP \cap H)}{\#(tP)} \geq \frac{\frac{4}{9}t^2 - 2t - 3}{t^2 + 3t + 2}.$$

The bound is meaningful for $t \geq 6$.

Proof. Let l be the line that is parallel to the boundary line of H and that passes through the centroid of P. Let q_1 and q_2 be the points of intersection of l and the boundary of P; see Figure 10. Let q_3 (resp. q_4) be the neighboring integer point of

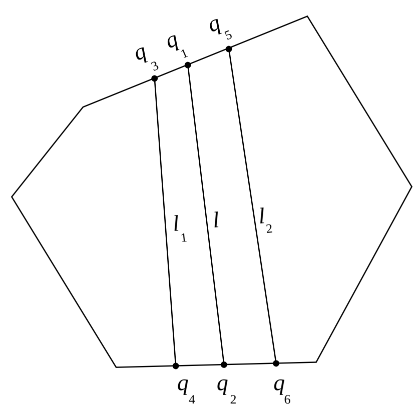

FIGURE 10

q_1 (resp. q_2) on the boundary of P that is contained in H. Similarly, let q_5 (resp. q_6) be the neighboring integer point of q_1 (resp. q_2) on the boundary of P that is contained in $\mathbb{R}^2 \setminus H$. If q_1 (resp. q_2) is an integer point, then we will assume $q_5 = q_3 = q_1$ (resp. $q_6 = q_4 = q_2$). Let l_1 (resp. l_2) be the line through q_3 and q_4 (resp. q_5 and q_6). Denote by R_1 the region of P bounded by the lines l_1 and l_2. Observe that R_1 is a quadrilateral (which may degenerate into a triangle or a segment) with vertices q_4, q_3, q_5, q_6. We will now compute the area of R_1.

Let $u = q_2 - q_1$, $v = q_3 - q_5$, and $w = q_4 - q_6$. Let $\lambda \in (0, 1)$ be such that $q_3 - q_1 = \lambda v$ and $q_5 - q_1 = -(1 - \lambda)v$. Similarly, let $\mu \in (0, 1)$ be such that $q_4 - q_2 = \mu w$ and $q_6 - q_2 = -(1 - \mu)w$. Then we have $q_4 - q_1 = u + \mu w$ and $q_6 - q_1 = u - (1 - \mu)w$.

For two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2$, let

$$[a, b] = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

denote the determinant of the matrix with their coordinates.

Thus if we consider a triangle whose two sides are given by the vectors a and b, then the area of this triangle is $\pm\frac{1}{2}[a,b]$, where the sign depends on the orientation of the frame a, b.

Splitting R_1 into four triangles $q_1q_3q_4$, $q_1q_4q_2$, $q_1q_2q_6$, $q_1q_6q_5$ and computing their areas using the operation introduced above, we see that the area of R_1 equals

$$\begin{aligned}|R_1| &= \frac{1}{2}\left([\lambda v, u+\mu w] + [u+\mu w, u] + [u, u-(1-\mu)w] + [u-(1-\mu)w, -(1-\lambda)v]\right) \\ &= \frac{1}{2}\left([v,u] + [w,u] + (\lambda+\mu-1)[v,w]\right). \end{aligned} \tag{10}$$

Note that the formula remains valid even if R_1 becomes a triangle or a segment.

Now we will repeat exactly the same procedure for tP. Let $l(t)$ be the boundary line of H. Recall that $l(t)$ contains the centroid of tP. The points where $l(t)$ intersects the boundary of tP are denoted by $q_1(t)$ and $q_2(t)$. The corresponding neighboring integer points on the boundary of tP are $q_3(t)$, $q_4(t)$, $q_5(t)$, $q_6(t)$. The convex hull of these points is denoted by R_t. Let $v_t = q_3(t) - q_5(t)$ and $w_t = q_4(t) - q_6(t)$. Note that $v_t = v$ and $w_t = w$ with the same vectors v and w as for R_1, provided $q_1(t)$ and $q_2(t)$ are not integer points. If $q_1(t)$ (resp. $q_2(t)$) is an integer point, then v_t (resp. w_t) becomes a zero vector. Also observe that $q_2(t) - q_1(t) = tu$, where u is the vector coming from the construction for R_1.

Now we will compute the area of R_t as above by splitting it into triangles. We have

$$\begin{aligned}|R_t| &= \frac{1}{2}\left(t[v_t,u] + t[w_t,u] + (\lambda_t+\mu_t-1)[v_t,w_t]\right) \\ &\le \frac{1}{2}\left(t[v,u] + t[w,u] + (\lambda_t+\mu_t-1)[v_t,w_t]\right),\end{aligned}$$

for some λ_t and μ_t from the interval $(0,1)$. The inequality sign above accounts for the cases when v_t or w_t become zero vectors.

Without loss of generality, assume that $[v,w] \le 0$. Then to maximize $|R_t|$ we need to take $\lambda_t = \mu_t = 0$ and thus

$$|R_t| \le \frac{1}{2}\left(t[v,u] + t[w,u] - [v,w]\right). \tag{11}$$

We will now show that

$$|R_t| \le (2t+1)|R_1|. \tag{12}$$

This inequality is clearly true if $[v,w] = 0$, since in this case

$$|R_t| \le \frac{1}{2}\left(t[v,u] + t[w,u]\right)$$

and

$$|R_1| = \frac{1}{2}\left([v,u] + [w,u]\right).$$

Now we will assume that $[v,w] < 0$. Extend the vectors v and w to straight lines and let q be the point of intersection of the two lines. We will consider three different cases.

Case 1. Assume $|q - q_1| \ge |v|$ and $|q - q_2| \ge |w|$.

Let $\bar{q}_3 = q_1 + v$ and $\bar{q}_4 = q_2 + w$. Consider the quadrilateral $\bar{q}_3q_1q_2\bar{q}_4$; see Figure 11. Its area is $\frac{1}{2}\left([v,u] + [w,u] + [v,w]\right)$, which does not exceed the area of R_1 (just take $\lambda = \mu = 1$ in formula (10)).

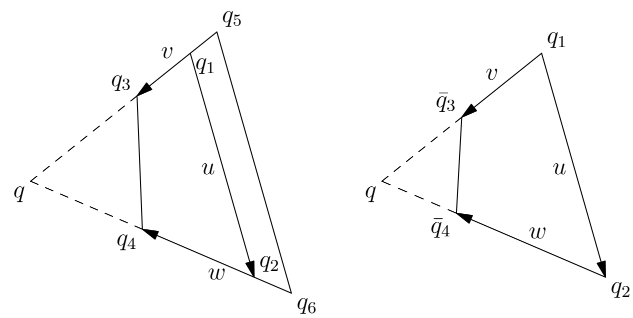

FIGURE 11. Case 1.

So, using this together with inequality (11), we get

$$\begin{aligned}\frac{|R_t|}{|R_1|} &\leq \frac{t[v,u]+t[w,u]-[v,w]}{[v,u]+[w,u]+[v,w]}\\ &= \frac{t[v,u]+t[w,u]+t[v,w]-t[v,w]-[v,w]}{[v,u]+[w,u]+[v,w]}\\ &= t+(t+1)\frac{-[v,w]}{[v,u]+[w,u]+[v,w]} \leq 2t+1,\end{aligned}$$

provided we can show that

$$\frac{-[v,w]}{[v,u]+[w,u]+[v,w]} \leq 1,$$

that is,

$$2[w,v] \leq [v,u]+[w,u].$$

Indeed, there exist $\alpha \geq 1$ and $\beta \geq 1$ such that $u = \alpha v - \beta w$. Therefore,

$$[v,u]+[w,u] = \beta[w,v]+\alpha[w,v] \geq 2[w,v].$$

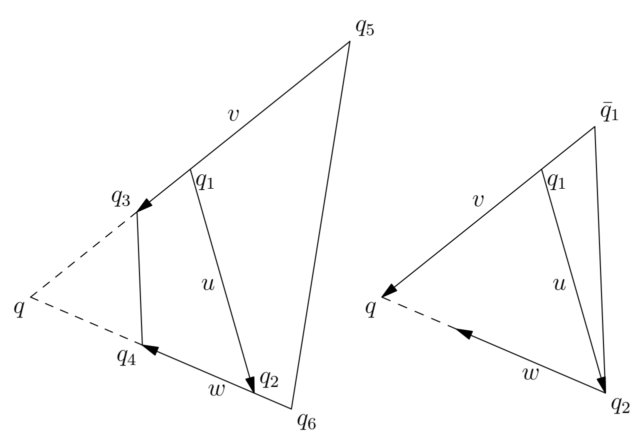

FIGURE 12. Case 2.

Case 2. Assume that $|q-q_1| < |v|$ and $|q-q_2| \geq |w|$. (The case $|q-q_1| \geq |v|$ and $|q-q_2| < |w|$ is similar).

Let $\bar{q}_1 = q - v$. Consider the triangle $q\bar{q}_1 q_2$; see Figure 12. Replacing λ and μ in formula (10) by the (larger) numbers $|q - q_1|/|v|$ and 1 respectively, we see that the area of the triangle $q\bar{q}_1 q_2$ is less than or equal to the area of the quadrilateral $q_3q_5q_6q_4$. Also observe that the area of the triangle $q\bar{q}_1 q_2$ equals $\frac{1}{2}[v, u]$. In addition, $[w, v] < [v, u]$ and $[w, u] < [v, u]$.

Therefore,

$$\frac{|R_t|}{|R_1|} \le \frac{t[v, u] + t[w, u] - [v, w]}{[v, u]} \le 2t + 1.$$

Case 3. $|q - q_3| < |v|$ and $|q - q_4| < |w|$.

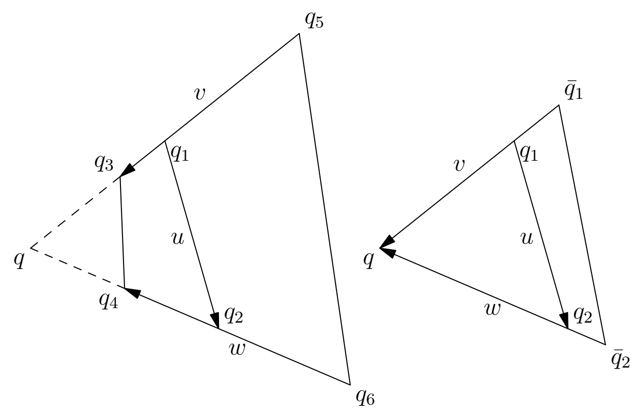

FIGURE 13. Case 3.

Let $\bar{q}_1 = q - v$ and $\bar{q}_2 = q - w$. Consider the triangle $q\bar{q}_1\bar{q}_2$; see Figure 13. Replacing λ and μ in formula (10) by the (larger) numbers $|q-q_1|/|v|$ and $|q-q_2|/|w|$ respectively, we see that the area of the triangle $q\bar{q}_1\bar{q}_2$ is less than or equal to the area of the quadrilateral $q_3q_5q_6q_4$. Also observe that the area of the triangle $q\bar{q}_1\bar{q}_2$ equals $\frac{1}{2}[w, v]$. In addition, $[v, u] < [w, v]$ and $[w, u] < [w, v]$.

Therefore,

$$\frac{|R_t|}{|R_1|} \le \frac{t[v, u] + t[w, u] - [v, w]}{[w, v]} \le 2t + 1.$$

The proof of inequality (12) is now completed.

Let $\bar{H}$ be the closed half-plane complementing H. Let Q be the largest integer polygon contained in $(tP) \cap \bar{H}$. Note that Q is a full-dimensional polygon. By Winternitz's theorem we have

$$|Q| \le |(tP) \cap \bar{H}| \le \frac{5}{9}|tP| = \frac{5}{9}t^2|P|.$$

Let b_Q be the number of integer points on the boundary of Q. Pick's formula gives

$$|Q| = \#Q - \frac{1}{2}b_Q - 1.$$

Therefore,

$$\#Q \le \frac{5}{9}t^2|P| + \frac{1}{2}b_Q + 1.$$

Assume for now that R_t does not degenerate into a segment. Note that the set of integer points on the boundary of Q consists of two subsets: those points that lie in the region R_t and those that belong to $(\partial(tP) \cap \bar{H}) \setminus R_t$. The number of integer points in $(\partial(tP)\cap\bar{H})\setminus R_t$ is at most tb. (It is possible to get a better estimate, but we will not do that since we are not claiming that our final bound is sharp). The number of integer points in R_t can be estimated using Pick's formula:

$$|R_t| = \frac{1}{2}\#R_t + \frac{1}{2}i_{R_t} - 1,$$

where i_{R_t} is the number of integer points in the interior of R_t. Thus

$$\#R_t \le 2|R_t| + 2.$$

Using these bounds and inequality (12), we obtain

$$b_Q \le tb + 2|R_t| + 2 \le tb + (4t+2)|R_1| + 2 \le tb + (4t+2)|P| + 2.$$

Thus,

$$\#Q \le \frac{5}{9}t^2|P| + \frac{1}{2}\left(tb + (4t+2)|P| + 2\right) + 1 = \left(\frac{5}{9}t^2 + 2t + 1\right)|P| + \frac{1}{2}tb + 2.$$

This yields

$$\begin{aligned}\#((tP)\cap H) &\ge \#(tP) - \#Q \\ &\ge t^2|P| + \frac{1}{2}tb + 1 - \left(\frac{5}{9}t^2 + 2t + 1\right)|P| - \frac{1}{2}tb - 2 \\ &\ge \left(\frac{4}{9}t^2 - 2t - 1\right)|P| - 1. \qquad (13)\end{aligned}$$

If R_t degenerates into a segment, then $b_Q \le tb + \#R_t$, and thus,

$$\#Q \le \frac{5}{9}t^2|P| + \frac{1}{2}\left(tb + \#R_t\right) + 1.$$

This yields a bound similar to (13):

$$\begin{aligned}\#((tP)\cap H) &= \#(tP) - \#Q + \#R_t \\ &\ge t^2|P| + \frac{1}{2}tb + 1 - \frac{5}{9}t^2|P| - \frac{1}{2}\left(tb + \#R_t\right) - 1 + \#R_t \\ &\ge \frac{4}{9}t^2|P|.\end{aligned}$$

Thus, regardless of whether or not R_t is a segment, we can use bound (13). By means of formula (4) and the inequality $\frac{1}{2}b \le |P| + 1$, we obtain

$$\begin{aligned}\frac{\#((tP)\cap H)}{\#(tP)} &\ge \frac{\left(\frac{4}{9}t^2 - 2t - 1\right)|P| - 1}{|P|t^2 + \frac{1}{2}tb + 1} \\ &\ge \frac{\left(\frac{4}{9}t^2 - 2t - 1\right)|P| - 1}{(t^2+t)|P| + t + 1} \\ &\ge \frac{\frac{4}{9}t^2 - 2t - 3}{t^2 + 3t + 2}.\end{aligned}$$

The last inequality can be checked directly using $|P| \ge 1/2$.

In conclusion we will remark on the sharpness of this bound. We believe that the optimal bound may be given by the example from Figure 9. For this trapezoid the ratio $\frac{\#((tP)\cap H)}{\#(tP)}$ is about

$$\frac{\frac{4}{9}t^2 - \frac{2}{3}t}{t^2 + t} \sim \frac{4}{9} - \frac{10}{9}\,t^{-1},$$

while the bound in Theorem 3 is of the order

$$\frac{4}{9} - \frac{10}{3}\, t^{-1}.$$

□

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