Harmonic Analysis and Uniqueness Questions in Convex Geometry

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Abstract We discuss some open questions on unique determination of convex bodies.

1 Introduction and notation

The purpose of this note is to give a short overview of known results and open questions on unique determination of convex and star bodies. These questions are usually treated with the aid of techniques of Harmonic Analysis. We give a typical example of such a problem in the next section. The reader is referred to the books by Groemer [9] and Koldobsky [20], as well as to the articles by Falconer [5] and Schneider [31] for the use of spherical harmonics and the Fourier transform in Convex Geometry.

First we introduce some notation. For standard notions in Convex Geometry we refer the reader to the books by Gardner [8] and Schneider [33].

A convex body in $\mathbb{R}^n$ is a compact convex set with non-empty interior. The support function h_K of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max\{\langle x,y\rangle : y \in K\}, \quad x \in \mathbb{R}^n.$$

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Clearly, h_K is positively homogeneous of degree 1, and therefore is determined by its values on the unit sphere. If V is a subspace of $\mathbb{R}^n$, then we write $K|V$ for the orthogonal projection of K onto V. It is easy to see that the support function of $K|V$, as a convex body in V, is just the restriction of h_K to V.

A convex body K is of constant width if $h_K(\xi)+h_K(-\xi)=$ constant for all $\xi \in S^{n-1}$. Let $G(n,k)$ denote the Grassmanian of k-dimensional subspaces of $\mathbb{R}^n$. The kth projection function of a convex body K is the function on $G(n,k)$ that assigns $\mathrm{vol}_k(K|H)$ to every k-dimensional subspace $H \in G(n,k)$. A convex body K has constant k-brightness if $\mathrm{vol}_k(K|H)=c$ for all $H \in G(n,k)$ and some constant c. In the case $k=n-1$ we say that K has constant brightness.

Let K be a compact convex set in $\mathbb{R}^n$. Its intrinsic volumes $V_i(K)$, $1 \le i \le n$, can be defined via Steiner's formula

$$\mathrm{vol}_n(K+\varepsilon B_2^n) = \sum_{i=0}^{n} \kappa_{n-i} V_i(K) \varepsilon^{n-i},$$

where the addition is the Minkowski addition, κ_{n-i} is the volume of the $(n-i)$-dimensional Euclidean ball, and $\varepsilon \ge 0$.

In particular, if K is a convex body in $\mathbb{R}^n$, then $V_n(K)$ is its volume, and $V_{n-1}(K)$ is half the surface area. For these and other facts about intrinsic volumes we refer the reader to the book [33].

We say that K is a star body if it is compact, star-shaped at the origin, and its radial function ρ_K defined by

$$\rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}, \qquad x \in S^{n-1},$$

is positive and continuous.

2 Typical result

Harmonic Analysis is an indispensable tool in Convex Geometry. Let us demonstrate this with the following well-known result, showing that origin-symmetric star bodies are uniquely determined by the size of their central sections.

Theorem 1. *Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$ such that*

$$\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$$

for every central hyperplane H. Then $K = L$.

We will give two very similar Harmonic Analysis proofs of this theorem. The first one uses the spherical Radon transform, which is a linear operator $\mathscr{R} : C(S^{n-1}) \to C(S^{n-1})$ defined by

$$\mathscr{R} f(\xi) = \int_{S^{n-1}\cap \xi^{\perp}} f(x)dx, \qquad \xi \in S^{n-1},$$

where $\xi^\perp$ is the hyperplane passing through the origin, and orthogonal to a given direction $\xi \in S^{n-1}$,

$$\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}.$$

Proof. The spherical Radon transform arises naturally in problems about volumes of central sections of star bodies. If H has unit normal vector ξ, then passing to polar coordinates in H we obtain

$$\mathrm{vol}_{n-1}(K \cap H) = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \rho_K^{n-1}(x) dx = \frac{1}{n-1} \mathscr{R} \rho_K^{n-1}(\xi).$$

Thus, we can reduce the geometric question about sections of star bodies to the question about the injectivity properties of the spherical Radon transform. The latter is known to be injective on even functions; see [9, Section 3.4] for details. In our case, both ρ_K^{n-1} and ρ_L^{n-1} are even functions on the unit sphere, since K and L are origin-symmetric. Hence, the condition of the theorem,

$$\frac{1}{n-1} \mathscr{R} \rho_K^{n-1}(\xi) = \frac{1}{n-1} \mathscr{R} \rho_L^{n-1}(\xi) \qquad \forall \xi \in S^{n-1},$$

yields $\rho_K^{n-1}(x) = \rho_L^{n-1}(x)$, and $\rho_K(x) = \rho_L(x)$ for all $x \in S^{n-1}$. This gives the desired result. □

Another known proof of Theorem 1 is based on the Fourier transform of distributions; see [20].

Proof. The main idea is that for an even function f, homogeneous of degree $-n+1$, and continuous on $\mathbb{R}^n \setminus \{0\}$, we have

$$\mathscr{R} f(\xi) = \frac{1}{\pi} \widehat{f}(\xi), \qquad \forall \xi \in S^{n-1}.$$

Thus, the assumption of Theorem 1 can be written as

$$\widehat{\rho_K^{n-1}}(\xi) = \widehat{\rho_L^{n-1}}(\xi), \qquad \forall \xi \in S^{n-1}.$$

By homogeneity, the latter equality is true on $\mathbb{R}^n \setminus \{0\}$. Inverting the Fourier transforms, we get $\rho_K = \rho_L$. □

For bodies that are not necessarily symmetric, Theorem 1 is not true; see [8, Thm 6.2.18, Thm 6.2.19].

We would like to mention that we are not aware of any other proof of Theorem 1 that uses ideas, different from the ones we just discussed.

3 Central sections

What is the answer in Theorem 1 if we replace the $(n-1)$-volume by the surface area or other intrinsic volumes?

Question 1. Let i and k be integers with $1 \le i \le k \le n-1$, and let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$V_i(K \cap H) = V_i(L \cap H), \qquad \forall H \in G(n,k).$$

Is it true that $K = L$?

In the simplest form when $n = 3$, $k = 2$, and $i = 1$, this question was asked by Gardner in his book [8]. Namely, are two origin-symmetric convex bodies K and L in $\mathbb{R}^3$ equal if the sections $K \cap \xi^\perp$ and $L \cap \xi^\perp$ have equal perimeters for all $\xi \in S^2$? The problem is open, except for some particular cases.

Howard, Nazarov, Ryabogin and Zvavitch [16] solved the problem in the class of C^1 star bodies of revolution. Rusu [28] settled an infinitesimal version of the problem, when one of the bodies is the Euclidean ball and the other is its one-parameter analytic deformation. Yaskin [36] showed that the answer is affirmative in the class of origin-symmetric convex polytopes in $\mathbb{R}^n$, where in dimensions $n \ge 4$ the perimeter is replaced by the surface area of the sections. On the other hand, Question 1 has a negative answer in the class of general (not necessarily symmetric) convex bodies containing the origin in their interiors; see [29].

There are many interesting questions about the so-called *intersection bodies*, related to the volumes of sections of origin-symmetric star bodies. We refer the reader to the books [20], [22] for these problems.

We finish this section with several uniqueness results and questions about *congruent* sections of convex bodies. We start with the following result of Schneider [32].

Theorem 2. *Let $K \subset \mathbb{R}^n$, $n \ge 3$, be a convex body containing the origin. If for all $\xi \in S^{n-1}$ all the intersections $K \cap \xi^\perp$ are congruent, then K is a Euclidean ball.*

A similar problem about two bodies is still open even in the three-dimensional case.

Question 2. Let K and L be two convex bodies in $\mathbb{R}^n$, $n \ge 3$, containing zero in their interiors. Assume that the $(n-1)$-dimensional sections of these bodies by the hyperplanes passing through the origin are congruent. Does it follow that K and L are congruent?

What happens if we drop the convexity assumption, but require only the "parallel translation congruency" of sections? Gardner [8] asks the following.

Question 3. Let K and L be two star-shaped bodies in $\mathbb{R}^n$ containing the origin in their interiors. Assume that the $(n-1)$-dimensional sections of these bodies by the hyperplanes passing through the origin are translates of each other. Does it follow that K and L are congruent?

We will return to analogous questions about projections in a subsequent section.

4 Maximal sections

Let K be a convex body in $\mathbb{R}^n$. The *inner section function* m_K is defined by

$$m_K(\xi) = \max_{t\in\mathbb{R}} \mathrm{vol}_{n-1}(K\cap(\xi^\perp + t\xi)),$$

for $\xi \in S^{n-1}$.

In 1969, Klee [18], [19] asked whether a convex body is uniquely determined (up to translation and reflection in the origin) by its inner section function. In [10] Gardner, Ryabogin, Yaskin, and Zvavitch answered Klee's question in the negative by constructing two convex bodies K and L, one of them origin-symmetric and the other is not centrally-symmetric, such that $m_K(\xi) = m_L(\xi)$ for all $\xi \in S^{n-1}$. Klee also asked whether a convex body in $\mathbb{R}^n$, $n \geq 3$, whose inner section function is constant, must be a ball. This question was recently answered in the negative in [27].

Since the knowledge of the inner section function is not sufficient for determining a convex body, one can try to put additional assumptions. For example, it is natural to ask the following.

Question 4. Are convex bodies uniquely determined by their inner section functions and the function $t_K(\xi)$, that gives the distance from the origin to the affine hyperplane that contains the maximal section in the direction of ξ?

Motivated by Theorem 2 one can also ask the following question about maximal sections.

Question 5. Let K be a convex body and $t_K(\xi)$ be defined as above. If for all $\xi \in S^{n-1}$ all the intersections $K\cap\{\xi^\perp + t_K(\xi)\xi\}$ are congruent, is then K a Euclidean ball?

We finish this section with the question about maximal sections which is a version of a result of Montejano (see [8] for references and related results).

Question 6. Let K be a convex body in $\mathbb{R}^3$ such that its maximal sections are of constant width. Does it follow that K is a Euclidean ball?

5 t-sections

In [2] Barker and Larman ask the following.

Question 7. Let K and L be convex bodies in $\mathbb{R}^n$ containing a sphere of radius t in their interiors. Suppose that for every hyperplane H tangent to the sphere we have $\mathrm{vol}_{n-1}(K\cap H) = \mathrm{vol}_{n-1}(L\cap H)$. Does this imply that $K = L$?

In [2] the authors obtained several partial results. They showed that in $\mathbb{R}^2$ the uniqueness holds if one of the bodies is a Euclidean disk. (The authors of [2] were apparently unaware of the paper [30] by Santaló, where he obtained an analogous result on the sphere. He then remarks that the limiting case, when the radius of the sphere tends to infinity, gives the result in the Euclidean plane). In $\mathbb{R}^n$ Barker and Larman proved that the answer to this conjecture is affirmative if hyperplanes are replaced by planes of a larger codimension. However, the answer to the original question is still unknown, even in dimension 2.

Yaskin [35] showed that the answer to the problem is affirmative if both K and L are convex polytopes in $\mathbb{R}^n$. The case $n = 2$ of this result when K and L are polygons in $\mathbb{R}^2$ was earlier settled by Xiong, Ma and Cheung [34].

Barker and Larman also suggest another generalization of the problem.

Question 8. Let K and L be convex bodies in $\mathbb{R}^n$ containing a convex body M in their interiors. Suppose that for every hyperplane H that supports M we have $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$. Does this imply that $K = L$?

Let us mention that in the case when M is just a straight line segment, the answer to the latter problem is affirmative. This is just a reformulation of the result proved independently by Falconer [6] and Gardner [8], that any convex body is uniquely determined by the volumes of hyperplane sections through any two points in the interior of the body. See also [21].

6 Slabs

Let $t > 0$ and $\xi \in S^{n-1}$. The slab of width $2t$ in the direction of ξ is defined by

$$S_t(\xi) = \{x \in \mathbb{R}^n : |\langle x, \xi \rangle| \le t\}.$$

Slabs can be thought of as “thick” sections.

In [29] the following problem was suggested.

Question 9. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ that contain the Euclidean ball of radius t in their interiors. Suppose that for some i $(1 \le i \le n)$

$$V_i(K \cap S_t(\xi)) = V_i(L \cap S_t(\xi)), \qquad \forall \xi \in S^{n-1}.$$

Is it true that $K = L$?

Note that without the symmetry assumption this question has a negative answer; see [29].

7 Projections

The well-known Aleksandrov’s projection theorem states that origin-symmetric convex bodies are uniquely determined by the sizes of their projections.

Theorem 3. *Let $1 \le i \le k \le n$ and let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. If*

$$V_i(K|H) = V_i(L|H), \qquad \forall H \in Gr(n,k),$$

then $K = L$.

This result fails in the absence of symmetry; see [8, Thm 3.3.17, Thm 3.3.18] or [12]. In fact, they prove more.

Theorem 4. *There are noncongruent convex bodies K and L in $\mathbb{R}^n$ such that for all i and k with $1 \le i \le k \le n$ we have*

$$V_i(K|H) = V_i(L|H), \qquad \forall H \in Gr(n,k).$$

Moreover, the pair of bodies can be chosen to be C_+^∞ bodies of revolution, or polytopes.

As we can see, even the knowledge of all projection functions does not allow to determine a convex body uniquely. Suppose that some projection function of convex body is known to be constant, is then the body a ball? No, since there are non-spherical bodies of constant *k*-brightness; see [7].

The following is an old question of Bonnesen [4].

Question 10. Let $K \subset \mathbb{R}^n$, $n \ge 3$, be a convex body whose inner section function and brightness function are constant. Does it follow that K is a Euclidean ball?

If the inner section function and the brightness function of a body K are both equal to the same constant, then the answer is known to be affirmative. A simple proof of this result was communicated to us by Nazarov.

Assume now that *two* projection functions are constant. Is then the body a ball? The question of whether a convex body in $\mathbb{R}^3$ of constant width and constant brightness must be a ball, is known as the Nakajima problem. Back in 1926, Nakajima [26] gave an affirmative answer to the problem under the additional assumption that the boundary of the body is of class C_+^2. The general case was resolved by Howard [13].

Theorem 5. *Let K be a convex body in $\mathbb{R}^3$ of constant width and constant brightness. Then K is a Euclidean ball.*

Generalizations were obtained by Howard and Hug [14], [15] and by Hug [17].

Theorem 6. *Let K be a convex body in $\mathbb{R}^n$. Let $1 \le i < j \le n-2$ and $(i,j) \ne (1,n-2)$. Assume that K has constant i-brightness and constant j-brightness. Then K is a Euclidean ball.*

However, the following, for example, is still unknown, even in the smooth case.

Question 11. Let K be a convex body in $\mathbb{R}^n$, $n \ge 4$, with constant $(n-1)$-brightness and constant $(n-2)$-brightness. Is K a Euclidean ball? What about bodies of constant width and $(n-1)$-brightness?

There are questions where one is interested in the shape of projections, rather than their size. The following problem is a "dual" version of Question 2.

Question 12. Let K and L be two convex bodies in $\mathbb{R}^n$, $n \geq 3$. Assume that the projections of these bodies (on corresponding hyperplanes) are congruent. Does it follow that K and L are congruent?

A beautiful Fourier analytic lemma was obtained by Golubyatnikov [11], who showed that in the three-dimensional case the corresponding projections could only be of three types: translations, reflections, and two-dimensional bodies of constant width. Using this lemma, Golubyatnikov, in particular, proved the following.

Theorem 7. *Let K and L be convex bodies in $\mathbb{R}^3$ such that for all $\xi \in S^2$ the projections $K|\xi^\perp$ and $L|\xi^\perp$ are $SO(2)$-congruent and one of the following is true.*
i) The projections are discs,
ii) the projections are not of constant width,
iii) the projections have no $SO(2)$ symmetries.
Then K is congruent to L.

This result is a generalization of the so-called Süss' Lemma (an analogue of Theorem 7 when the projections $K|\xi^\perp$ and $L|\xi^\perp$ are translates of each other). We would also like to mention that a beautiful and elementary proof of Süss' Lemma was obtained by Lieberman [1].

There is no doubt that the reader is now able to come up with other questions of mixed nature, involving sections and projections. We would like to add one more suggested to us by Gardner.

Question 13. Let K in $\mathbb{R}^3$ be a convex body containing the origin in its interior. If all one-dimensional central sections of K have constant length and K is of constant brightness, does it follow that K is a ball?

We finish our note with several questions about symmetry of convex bodies.

8 Symmetry

Let K be a convex body in $\mathbb{R}^n$. The parallel section function of K in the direction of $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap (\xi^\perp + t\xi)), \qquad t \in \mathbb{R}.$$

It is a consequence of the Brunn-Minkowski inequality that the maximal sections of an origin-symmetric convex body pass through the origin. Makai, Martini, Ódor [25] has shown that the converse statement is also true.

Theorem 8. *Let K be a convex body in $\mathbb{R}^n$ such that $A_{K,\xi}(0) \geq A_{K,\xi}(t)$ for all $\xi \in S^{n-1}$ and all $t \in \mathbb{R}$, then K is symmetric with respect to the origin.*

They also posed a similar question for lower intrinsic volumes of sections (see also [23] for other results in this direction).

Question 14. Let K be a convex body in $\mathbb{R}^n$ such that $V_i(K\cap\xi^\perp)\geq V_i(K\cap\{\xi^\perp+t\xi\})$ for all $\xi\in S^{n-1}$ and all $t\in\mathbb{R}$. Is K an origin-symmetric body?

The problem is open unless K is a smooth perturbation of the Euclidean ball. In the latter case, Makai and Martini [24] have shown that the answer is affirmative.

We suggest a question in the same spirit.

Question 15. Let $t_K(\xi)$ be such a function on the sphere that $A_{K,\xi}(t_K(\xi)) = \max_{t\in\mathbb{R}} A_{K,\xi}(t)$. Assume that for every $\xi\in S^{n-1}$ the hyperplane $\{\xi^\perp+t_K(\xi)\xi\}$ divides the surface of K into two parts of equal area. Is K centrally symmetric?

What are other criteria that allow to determine the symmetry of a given body? In [3] Bianchi and Gruber ask the following.

Question 16. Let K be a convex body and t a continuous function on the sphere S^{n-1}. Assume that for every $\xi\in S^{n-1}$ the $(n-1)$-dimensional body $K\cap\{x\in\mathbb{R}^n: \langle x,\xi\rangle = t(\xi)\}$ has a center of symmetry. Is then K centrally symmetric?

In particular, is the following true?

Question 17. Let $t_K(\xi)$ be such that $A_{K,\xi}(t_K(\xi)) = \max_{t\in\mathbb{R}} A_{K,\xi}(t)$. If for every $\xi\in S^{n-1}$ the $(n-1)$-dimensional body $K\cap\{x\in\mathbb{R}^n:\langle x,\xi\rangle = t_K(\xi)\}$ is centrally symmetric, does it follow that K is centrally symmetric?

Such questions are directly related to questions about t-sections, mentioned before.

Question 18. Let K be convex body in $\mathbb{R}^n$ containing the Euclidean ball of radius $t>0$ in its interior. Suppose that the sections $K\cap\{\xi^\perp+t\xi\}$ are $(n-1)$-dimensional centrally symmetric bodies for all $\xi\in S^{n-1}$. Is it true that K is origin symmetric? Is it an ellipsoid?

Question 19. Let K be convex body in $\mathbb{R}^n$ containing the Euclidean ball of radius $t>0$ in its interior. Suppose that $A_{K,\xi}(t)=A_{K,\xi}(-t)$ for every $\xi\in S^{n-1}$. Is K origin symmetric?

As one can check, the latter question is equivalent to the following question about slabs.

Question 20. Let K be a convex body in $\mathbb{R}^n$ containing the Euclidean ball of radius $t>0$ in its interior. Suppose that

$$\mathrm{vol}_n(K\cap\{x:|\langle x,\xi\rangle|\leq t\}) = \max_{a\in\mathbb{R}} \mathrm{vol}_n(K\cap\{x: -t+a\leq\langle x,\xi\rangle\leq t+a\})$$

for all $\xi\in S^{n-1}$. Does this imply that K is origin symmetric?

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