UNIQUE DETERMINATION OF CONVEX LATTICE SETS

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Abstract. Let K and L be origin-symmetric convex lattice sets in $\mathbb{Z}^n$. We study a discrete analogue of the Aleksandrov theorem for the surface areas of projections. If for every $u \in \mathbb{Z}^n$, the sets $(K|u^\perp) \cap \partial(\mathrm{conv}(K)|u^\perp)$ and $(L|u^\perp) \cap \partial(\mathrm{conv}(L)|u^\perp)$ have the same number of points, is then necessarily $K = L$? We give a positive answer to this question in $\mathbb{Z}^3$. In higher dimensions, we obtain an analogous result when $\mathrm{conv}(K)$ and $\mathrm{conv}(L)$ are zonotopes.

1. Introduction and main results

Typical questions in geometric tomography are concerned with the unique determination of convex bodies from quantitative information coming from their sections or projections. There are numerous results of this nature, and Aleksandrov's theorem is, arguably, one of the most interesting and well-known among them. It asserts that if K is an origin-symmetric convex body, then K is uniquely determined by the areas of its projections; see e.g. [4, p. 115].

Discrete tomography deals with the study of finite sets in place of solid objects. Gardner, Gronchi, and Zong [5] initiated a new direction where the main idea is to transfer questions from geometric to discrete tomography in order to establish the corresponding results. In particular, they formulated a discrete analogue of the Aleksandrov theorem that reads as follows.

Problem 1.1. *Let K and L be origin-symmetric convex lattice sets in $\mathbb{Z}^n$, $n \geq 2$. If for every $u \in \mathbb{Z}^n$ we have*

$$|K|u^\perp| = |L|u^\perp|,$$

is then necessarily $K = L$?

Here, for a finite set K we denote by $|K|$ the cardinality of K, and by $\mathrm{conv}(K)$ the convex hull of K. We say that a finite set K is a convex lattice set if $K = \mathrm{conv}(K) \cap \mathbb{Z}^n$.

In [5] the authors showed that the answer to Problem 1.1 is negative if $n = 2$. They constructed two noncongruent convex lattice sets in $\mathbb{Z}^2$ with equal projection counts (they used the term "projection count" when

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referring to the cardinality of a projection of a lattice set). However, it is unknown whether there are other examples in $\mathbb{Z}^2$. Some work has been done in this direction, [9], [11]. Gardner, Gronchi, and Zong also asked if it is possible to impose an additional condition to guarantee the affirmative answer in $\mathbb{Z}^2$. A positive answer to this question has been recently obtained by N. Zhang; see [10]. If $n \geq 3$, Problem 1.1 is completely open.

Note that the Aleksandrov projection theorem is also true for other intrinsic volumes of projections. For example, if the projections of two origin-symmetric convex bodies onto all hyperplanes have equal surface areas, then the bodies coincide, [4, p. 115].

In this note we suggest to study an analogue of the latter result in discrete settings. Let K be a convex lattice set in $\mathbb{Z}^n$ and $u \in \mathbb{Z}^n$. By the discrete surface area $|\partial(K|u^\perp)|$ of the projection of K onto $u^\perp$ we will understand the number of points in $K|u^\perp$ that lie on the boundary of the convex hull of $K|u^\perp$, i.e.

$$|\partial(K|u^\perp)| = |(K|u^\perp) \cap \partial(\mathrm{conv}(K)|u^\perp)|.$$

When $K \subset \mathbb{Z}^3$, we will use the term "discrete perimeter".

We say that a finite set K in $\mathbb{R}^n$ is full-dimensional if $\mathrm{conv}(K)$ has non-empty interior. In questions below, we will only consider full-dimensional convex lattice sets.

Problem 1.2. *Let K and L be origin-symmetric full-dimensional convex lattice sets in $\mathbb{Z}^n$. If for every $u \in \mathbb{Z}^n$ we have*

$$|\partial(K|u^\perp)| = |\partial(L|u^\perp)|,$$

is then necessarily $K = L$?

Below we give a positive answer to this problem in $\mathbb{Z}^3$.

Theorem 1.3. *Let K and L be origin-symmetric full-dimensional convex lattice sets in $\mathbb{Z}^3$. If the discrete perimeters of $K|u^\perp$ and $L|u^\perp$ are equal for all $u \in \mathbb{Z}^3$, then $K = L$.*

As one can see, if we drop the assumption that the sets are full-dimensional, then Problem 1.2 in $\mathbb{Z}^3$ has a negative answer, since it reduces to Problem 1.1 in $\mathbb{Z}^2$.

We also solve Problem 1.2 in $\mathbb{Z}^n$, $n \geq 4$, in the class of convex lattice sets whose convex hulls are zonotopes. Recall that a zonotope is a finite Minkowski sum of closed line segments, [4, p. 146].

Theorem 1.4. *Let K and L be origin-symmetric full-dimensional convex lattice sets in $\mathbb{Z}^n$, $n \geq 4$, such that* $\mathrm{conv}\,(K)$ *and* $\mathrm{conv}\,(L)$ *are zonotopes. If*

$$|\partial(K|u^\perp)| = |\partial(L|u^\perp)|$$

for all $u \in \mathbb{Z}^n$, then $K = L$.

Let us briefly mention some facts and concepts that are used in this paper. The following is the well-known Theorem of Pick; see [1, p. 90] or [2, p. 38].

Let K be a full-dimensional convex lattice set in $\mathbb{Z}^2$. Then the area of $\mathrm{conv}(K)$ can be computed as follows:

$$\mathrm{vol}_2(\mathrm{conv}(K)) = I + \frac{1}{2}B - 1,$$

where I is the number of points of K in the interior of $\mathrm{conv}(K)$ and B is the number of points of K on the boundary of $\mathrm{conv}(K)$.

Let $u_1, \dots, u_m$ be linearly independent vectors in $\mathbb{Z}^n$, with $m \le n$. The set

$$\Lambda = \left\{ \sum_{i=1}^{m} a_i u_i : a_i \in \mathbb{Z}, \text{ for } 1 \le i \le m \right\}$$

is called a sublattice of $\mathbb{Z}^n$ of rank m. The vectors $u_1, \dots, u_m$ form a basis of Λ.

The set

$$\Pi = \left\{ \sum_{i=1}^{m} b_i u_i : 0 \le b_i < 1, \text{ for } 1 \le i \le m \right\}$$

is called the fundamental parallelepiped of the basis $u_1, \dots, u_m$. The m-dimensional volume of the fundamental parallelepiped does not depend on the choice of the basis of Λ; it is called the determinant of Λ and denoted $|\Lambda|$. For these and other related results, the reader is referred to the books by Barvinok [1] and Gruber [6].

We will also need the Minkowski uniqueness theorem saying that a convex polytope in $\mathbb{R}^n$ is uniquely determined (up to translation) by the areas of its facets and the normal vectors to the facets; see [7, p. 397].

2. Proofs of the main results

Proof of Theorem 1.3. The idea is to show that for every facet F_K of $\mathrm{conv}(K)$, there is a facet F_L of $\mathrm{conv}(L)$ that is parallel to F_K (and vice versa), and

$$|\partial F_K \cap \mathbb{Z}^3| - 2|F_K \cap \mathbb{Z}^3| = |\partial F_L \cap \mathbb{Z}^3| - 2|F_L \cap \mathbb{Z}^3|. \tag{1}$$

Using (1) and Pick's theorem, we will conclude that for every pair of parallel facets, $\mathrm{vol}_2(F_K) = \mathrm{vol}_2(F_L)$, and will use the Minkowski uniqueness theorem to finish the proof. Below we provide the details.

First we claim that for every facet F_K of $\mathrm{conv}(K)$, there is a facet F_L of $\mathrm{conv}(L)$ that is parallel to F_K, and vice versa. Indeed, assume that there exists a facet F_K such that no facet of $\mathrm{conv}(L)$ is parallel to F_K. Note that $\{\theta \in S^2 : \theta = |u|^{-1}u, \text{ where } u \in \mathbb{Z}^n\}$ is a dense subset of S^2. One can see that in the statement of the theorem we can take vectors from the sphere S^2. Choose a direction $\xi \in S^2$ that is parallel to F_K (and the opposite facet, since K is origin-symmetric) and not parallel to any other facets of either $\mathrm{conv}(K)$ or $\mathrm{conv}(L)$. Then, the boundary of $\mathrm{conv}(K|\xi^\perp)$ consists of the edges e and $-e$ that are the projections of F_K and $-F_K$, as well as other edges that are the projections of some edges of $\mathrm{conv}(K)$. The boundary of $\mathrm{conv}(L|\xi^\perp)$ solely consists of the projections of some edges of $\mathrm{conv}(L)$.

Furthermore, we can assume that $|K|\xi^\perp| = |K|$ and $|L|\xi^\perp| = |L|$, since there are only finitely many directions that do not satisfy these equalities. For ϕ small enough, consider the vectors $\zeta = \cos\phi\,\xi + \sin\phi\,\eta$ and $\theta = \cos\phi\,\xi - \sin\phi\,\eta$, where η is the unit outward normal vector to F_K. Note that the number of points in K that are projected to $\mathrm{conv}(K|\xi^\perp)$, $\mathrm{conv}(K|\zeta^\perp)$, and $\mathrm{conv}(K|\theta^\perp)$, and that do not come from the facets F_K and $-F_K$, is the same. On the other hand, at least one of the points of F_K belongs to the interior of either $\mathrm{conv}(K|\zeta^\perp)$ or $\mathrm{conv}(K|\theta^\perp)$. Thus at least one of the two inequalities holds:

$$|\partial(K|\zeta^\perp)| < |\partial(K|\xi^\perp)| \text{ or } |\partial(K|\theta^\perp)| < |\partial(K|\xi^\perp)|.$$

However,

$$|\partial(L|\zeta^\perp)| = |\partial(L|\xi^\perp)| \text{ and } |\partial(L|\theta^\perp)| = |\partial(L|\xi^\perp)|.$$

We get a contradiction. Thus, every facet of $\mathrm{conv}(K)$ is parallel to a facet of $\mathrm{conv}(L)$ and vice versa.

To prove (1), we will use the following formula:

$$|\partial(K|\zeta^\perp)| + |\partial(K|\theta^\perp)| - 2|\partial(K|\xi^\perp)| = 2|\partial F_K \cap \mathbb{Z}^3| - 4|F_K \cap \mathbb{Z}^3| + 4. \quad (2)$$

Let us explain the validity of this equality. First of all, observe that the left-hand side only sees the points that are projected from F_K and $-F_K$. (The contribution of the rest of the boundary of K is annihilated, since the number of points in K that are projected to $\mathrm{conv}(K|\xi^\perp)$, $\mathrm{conv}(K|\zeta^\perp)$, and $\mathrm{conv}(K|\theta^\perp)$, and that do not come from the facets F_K and $-F_K$, is the same). Next we see that $\partial(K|\zeta^\perp)$ gets points from one side of $\partial F_K \cap \mathbb{Z}^3$ (and its reflection about the origin), and $\partial(K|\theta^\perp)$ gets points from the other side of $\partial F_K \cap \mathbb{Z}^3$ (and its reflection about the origin). There are two points on each F_K and $-F_K$ that are projected into both $\mathrm{conv}(K|\zeta^\perp)$ and $\mathrm{conv}(K|\theta^\perp)$, which yields the constant term equal to 4 in (2). Since all points from F_K and $-F_K$ are projected into different points in $\partial(K|\xi^\perp)$, the latter set has exactly $2|F_K \cap \mathbb{Z}^3|$ points coming from those facets. Formula (2) follows.

Now equality (2) together with the assumption of the theorem yields (1) for every pair of parallel facets of $\mathrm{conv}(K)$ and $\mathrm{conv}(L)$.

Let H be the 2-dimensional subspace that is parallel to the facets F_K and F_L. Then, $\Lambda = H \cap \mathbb{Z}^3$ is a lattice of rank 2; see e.g. [8, Chap. I, §2]. Let $|\Lambda|$ be the determinant of the lattice Λ. By Pick's theorem and equality (1),

$$\begin{aligned} \mathrm{vol}_2(F_K) &= |\Lambda|(|F_K \cap \mathbb{Z}^3| - \frac{1}{2}|\partial F_K \cap \mathbb{Z}^3| - 1) \\ &= |\Lambda|(|F_L \cap \mathbb{Z}^3| - \frac{1}{2}|\partial F_L \cap \mathbb{Z}^3| - 1) \\ &= \mathrm{vol}_2(F_L). \end{aligned}$$

Thus we have proved that for each facet F_K in $\mathrm{conv}(K)$, there is a facet F_L in $\mathrm{conv}(L)$ (and vice versa), such that F_K and F_L are parallel and

$\mathrm{vol}_2(F_K) = \mathrm{vol}_2(F_L)$. Minkowski's uniqueness theorem then implies that $\mathrm{conv}(K) = \mathrm{conv}(L)$, or equivalently, $K = L$. □

Before we present the proof of Theorem 1.4, let us introduce the following notation. If P is a convex body in $\mathbb{R}^n$, we define the upper boundary $\mathcal{U}_\xi(P)$ of P in the direction $\xi \in S^{n-1}$ to be

$$\mathcal{U}_\xi(P) := \{x \in P : (x + \epsilon\xi) \cap P = \emptyset,\ \forall \epsilon > 0\},$$

and the lower boundary $\mathcal{L}_\xi(P)$ of P in the direction ξ to be

$$\mathcal{L}_\xi(P) := \{x \in P : (x - \epsilon\xi) \cap P = \emptyset,\ \forall \epsilon > 0\}.$$

If P is a polytope, then $\mathcal{U}_\xi(P)$ is the union of the facets F_i of P whose outer normal vectors n_i satisfy the inequality $\langle n_i, \xi\rangle > 0$. Similarly, $\mathcal{L}_\xi(P)$ is the union of the facets F_i of P whose outer normal vectors n_i satisfy the inequality $\langle n_i, \xi\rangle < 0$.

We will need the following lemma that will be used as an analogue of Pick's Theorem.

Lemma 2.1. *Let Z be a zonotope with vertices in the lattice $\Lambda \subset \mathbb{R}^n$. Let $\xi \in S^{n-1}$ be a direction that is not parallel to any of the facets of K. Then*

$$\mathrm{vol}_n(Z) = |\Lambda|(|Z \cap \Lambda| - |\mathcal{U}_\xi(Z) \cap \Lambda|).$$

This formula is discussed in [3, Section 2.3.2], but, for the sake of completeness, we outline a sketch of our proof below.

First of all, without loss of generality, we can assume that $\Lambda = \mathbb{Z}^n$. Next we proceed by induction on the number of summands of Z. The base case is when Z is the sum of n segments. If Z is a box with facets parallel to the coordinate planes, the formula is obvious. Furthemore, it is not hard to show that it is true for all parallelotopes. The inductive step is as follows. Assume that the formula is true for zonotopes that are the sum of N segments. If Z is the sum of $N+1$ segments, it can be written as the sum of a segment and a zonotope with N summands. If the latter is full-dimensional, its facets are zonotopes with at most $N-1$ summands. (If it is not full-dimensional, write it as sum of a segment and a zonotope with $N-1$ summands). Thus Z can be written as the union of zonotopes (with disjoint interiors) that are sums of no more than N segments. Next use the induction hypothesis.

We will now present a solution of Problem 1.2 in the class of zonotopes in $\mathbb{Z}^n$.

Proof of Theorem 1.4. The proof is similar to that of Theorem 1.3, but some additional considerations will be needed. Again, we can assume that the hypothesis of the theorem is true for all directions u from S^{n-1}. Let us denote $Z_K = \mathrm{conv}(K)$ and $Z_L = \mathrm{conv}(L)$. As above, one can show that for every facet F_K of Z_K, there is a facet F_L of Z_L that is parallel to F_K, and vice versa.

Let $\xi \in S^{n-1}$ be a vector that is parallel to a facet F_K (and the opposite facet $-F_K$) of Z_K, but not parallel to any other facet of Z_K. Furthermore, we can assume that $|K|\xi^\perp| = |K|$ and $|L|\xi^\perp| = |L|$, since there are only finitely many directions that do not satisfy these equalities.

Observe that

$$|\partial(K|\xi^\perp)| = 2|(F_K \cap \mathbb{Z}^n)|\xi^\perp| + R(\xi) = 2|F_K \cap \mathbb{Z}^n| + R(\xi), \tag{3}$$

where $R(\xi)$ counts those points on the boundary of $(Z_K \cap \mathbb{Z}^n)|\xi^\perp$ that did not come from the facets F_K or $-F_K$.

Let η be the unit outward normal vector to F_K. For $\phi > 0$ small enough consider the vector $\zeta = \cos\phi\,\xi + \sin\phi\,\eta$. We claim that

$$|\partial(K|\zeta^\perp)| = 2|\mathcal{L}_\xi(F_K) \cap \mathbb{Z}^n| + R(\zeta), \tag{4}$$

where $R(\zeta)$ counts those points on the boundary of $K|\zeta^\perp$ that did not come from the facets F_K or $-F_K$.

Assume for the moment that the claim is proved. Then, by the hypothesis of the theorem, we have

$$|\partial(K|\xi^\perp)| - |\partial(K|\zeta^\perp)| = |\partial(L|\xi^\perp)| - |\partial(L|\zeta^\perp)|.$$

Subtracting formulas (3) and (4), and using the fact that $R(\xi) = R(\zeta)$, and

$$|\mathcal{U}_\xi(F_K) \cap \mathbb{Z}^n| = |\mathcal{L}_\xi(F_K) \cap \mathbb{Z}^n|,$$

we get

$$|F_K \cap \mathbb{Z}^n| - |\mathcal{U}_\xi(\partial F_K) \cap \mathbb{Z}^n| = |F_L \cap \mathbb{Z}^n| - |\mathcal{U}_\xi(\partial F_L) \cap \mathbb{Z}^n|.$$

Since any facet of a zonotope is a zonotope, we can apply Lemma 2.1 to the facets F_K, F_L and (a shift of) the sublattice $\Lambda = H \cap \mathbb{Z}^n$, where H is the subspace parallel to F_K and F_L. Thus, we obtain that $\mathrm{vol}_{n-1}(F_K) = \mathrm{vol}_{n-1}(F_L)$, and we can use Minkowski's uniqueness theorem to conclude that $\mathrm{conv}(K) = \mathrm{conv}(L)$, or equivalently, $K = L$.

It remains to prove (4). We will use the boundary structure of Z_K and its projection $Z_K|\xi^\perp$. One can see that

$$\partial Z_K = F_K \cup (-F_K) \cup \mathcal{U}_\xi(Z_K) \cup \mathcal{L}_\xi(Z_K)$$

and

$$Z_K|\xi^\perp = \mathcal{U}_\xi(Z_K)|\xi^\perp = \mathcal{L}_\xi(Z_K)|\xi^\perp.$$

Note that

$$\partial(Z_K|\xi^\perp) = \partial(\mathcal{U}_\xi(Z_K))|\xi^\perp = \partial(\mathcal{L}_\xi(Z_K))|\xi^\perp,$$

and for $\zeta = \cos\phi\,\xi + \sin\phi\,\eta$ we have

$$\partial(Z_K|\zeta^\perp) = (\mathcal{U}_\zeta(Z_K) \cap \mathcal{L}_\zeta(Z_K))|\zeta^\perp. \tag{5}$$

We see that if $x \in \partial(Z_K|\xi^\perp)$ and $x \notin (F_K \cup (-F_K))|\xi^\perp$, then $x \in (\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K))|\xi^\perp$. Therefore, $R(\xi)$ counts the number of lattice points on $(\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \setminus (F_K \cup (-F_K))$. Note that the latter number does not change if we replace ξ by another vector ζ that is close enough. In particular,

$$R(\zeta) = |(\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \setminus (F_K \cup (-F_K)) \cap \mathbb{Z}^n|. \tag{6}$$

Now observe that

$$\mathcal{U}_\zeta(Z_K) = \mathcal{U}_\xi(Z_K) \cup F_K \qquad \text{and} \qquad \mathcal{L}_\zeta(Z_K) = \mathcal{L}_\xi(Z_K) \cup (-F_K).$$

Hence,

$$\mathcal{U}_\zeta(Z_K) \cap \mathcal{L}_\zeta(Z_K) = (\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \cup (\mathcal{U}_\xi(Z_K) \cap (-F_K)) \cup (\mathcal{L}_\xi(Z_K) \cap F_K),$$

that is

$$\mathcal{U}_\zeta(Z_K) \cap \mathcal{L}_\zeta(Z_K) = (\mathcal{U}_\xi(Z_K) \cap \mathcal{L}_\xi(Z_K)) \cup \mathcal{U}_\xi(-F_K) \cup \mathcal{L}_\xi(F_K).$$

In view of the latter formula, and (5), (6), we get

$$|\partial(K|\zeta^\perp)| - R(\zeta) = |\mathcal{U}_\xi(-F_K) \cap \mathbb{Z}^n| + |\mathcal{L}_\xi(F_K) \cap \mathbb{Z}^n| = 2|\mathcal{L}_\xi(F_K) \cap \mathbb{Z}^n|.$$

Thus, formula (4) is proved. This finishes the proof of the theorem. □

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