ON PERIMETERS OF SECTIONS OF CONVEX POLYTOPES

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Abstract. In his book "Geometric Tomography" Richard Gardner asks the following question. Let P and Q be origin-symmetric convex bodies in $\mathbb{R}^3$ whose sections by any plane through the origin have equal perimeters. Is it true that $P = Q$? We show that the answer is "Yes" in the class of origin-symmetric convex polytopes. The problem is treated in the general case of $\mathbb{R}^n$.

1. Introduction

The motivation for this article is the following problem from the book "Geometric Tomography" by R. J. Gardner, [1, Prob. 7.5, p. 258], (or Prob. 7.6, p. 289 in the 2006 edition of the book).

Let P and Q be origin-symmetric convex bodies in $\mathbb{R}^3$ such that

$$L(P \cap H) = L(Q \cap H)$$

for every plane H through the origin, where L is the length of the corresponding boundary curve. Is it true that

$$P = Q?$$

The well-known analogue of this problem for areas of sections (in place of perimeters) has an affirmative answer, see for example [1, Corollary 7.2.7]. However, for perimeters the problem is still open. In particular, it is not known whether the uniqueness holds if one of the bodies is the Euclidean ball. Some positive results were obtained by Howard, Nazarov, Ryabogin and Zvavitch [3], who solved the latter special case in the class of C^1 star bodies of revolution, and by Rusu [4], who settled an infinitesimal version of the problem, when one of the bodies is the Euclidean ball and the other is its one-parameter analytic deformation.

In this article we solve the problem for origin-symmetric convex polytopes. Instead of dealing with the 3-dimensional case, we consider its natural n-dimensional generalizations. Note that the symmetry assumption in the problem cannot be dropped, since for any body P the corresponding sections of P and $-P$ have equal perimeters.

For standard notions in geometric tomography or convex geometry the reader is referred to the books by Gardner [1], Gruber [2] and Schneider [5].

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2. Main result

Throughout the paper we use the following notations. $G(n,k)$ stands for the Grassmanian of all k-dimensional subspaces of $\mathbb{R}^n$. If ξ is a unit vector in $\mathbb{R}^n$, then $\xi^\perp$ is defined to be the hyperplane through the origin orthogonal to ξ. For a convex body B in $\mathbb{R}^k$, $S(B)$ denotes the $(k-1)$-dimensional area of the boundary surface of B. For brevity, k-dimensional planes and k-dimensional faces will be referred to as k-planes and k-faces. For a set $S \subset \mathbb{R}^n$, $\mathrm{aff}(S)$ and $\mathrm{relint}(S)$ denote correspondingly the affine hull and the relative interior of S.

Theorem. *Let $2 \leq k \leq n-1$ and suppose that P and Q are origin-symmetric convex polytopes in $\mathbb{R}^n$, $n \geq 3$, such that*

$$S(P \cap H) = S(Q \cap H)$$

for every subspace $H \in G(n,k)$. Then

$$P = Q.$$

Proof. Clearly, it is enough to prove the theorem only for $k = n-1$, i.e. in the case of sections by hyperplanes.

For the reader's convenience, let us first explain the main idea of the proof. The key observation is that the function $S(P \cap \xi^\perp)$ is able to detect vertices of the polytope. More precisely, if ξ varies on the sphere in such a way that $\xi^\perp$ does not contain vertices of the polytope P then the function $S(P \cap \xi^\perp)$ enjoys certain analytic properties, which break once the plane $\xi^\perp$ hits a vertex. Therefore, we need to exploit these conditions to show that the polytopes P and Q cannot have different vertices.

To reach a contradiction, we will assume that P and Q are different, i.e. they have different sets of vertices. We will consider separately the following two cases.

Case 1. There is a vertex u of, say, P such that the line through the origin and the vertex u does not contain any vertices of Q.

Case 2. All vertices of P and Q lie on the same lines, i.e. if a line through the origin contains a vertex of one of the polytopes, then it also contains a vertex of the other.

First we will settle case 1. Let E be any $(n-2)$-face of P adjacent to the vertex u. There exists $\xi_0 \in S^{n-1}$ such that the hyperplane $\xi_0^\perp$ satisfies the following properties:

1) $\xi_0^\perp \cap E = \{u\}$,

2) $\xi_0^\perp$ contains no vertices of either P or Q (other than u, $-u$).

Let Λ be a spherical cap centered at ξ_0. We will assume that the radius of Λ is small enough to guarantee that for all $\xi \in \Lambda$ the plane $\xi^\perp$ contains no vertices of P and Q, except possibly u and $-u$.

The hyperplane $\xi_0^\perp$ divides the space into two half-spaces ξ_0^+ and ξ_0^-. We assume that they are closed. Since the face E only lies in one of these half-spaces, we will denote by ξ_0^+ the half-space that contains E and by ξ_0^- the other half-space.

Denote the edges of P that have non-empty intersection with the plane $\xi_0^\perp$ by

$$x = u_i + l_i s_i, \qquad i \in I, \tag{1}$$

where u_i is a vertex that belongs to the edge, l_i is a unit vector in the direction of the edge, s_i is a parameter, and I is a set of indices. If an edge is adjacent to the vertex u (correspondingly, $-u$) we will assume that, for this edge, $u_i = u$ (correspondingly, $u_i = -u$).

Let us write I as a union of three mutually disjoint sets

$$I = I_1 \cup I_2 \cup I_3,$$

which are defined as follows.

I_1 corresponds to those edges in (1) that are not adjacent to the vertices u or $-u$. The indices of the edges of P adjacent to either of the vertices u or $-u$ belong to $I_2 \cup I_3$. Since the plane $\xi_0^\perp$ contains u and $-u$, but no other vertex of P, none of the latter edges lies in the plane $\xi_0^\perp$ and, so, some of these edges lie in ξ_0^+ and some in ξ_0^-. Let those edges that are adjacent to u and lie in ξ_0^+, and those that are adjacent to $-u$ and lie in ξ_0^- be indexed by $i \in I_2$, and all the others by $i \in I_3$.

The edges of Q that intersect the plane $\xi_0^\perp$ we denote by

$$x = v_i + m_i t_i, \qquad i \in J,$$

where v_i, m_i, t_i are correspondingly a point on the edge, its direction and parameter along the edge, and J is an index set.

Let Λ_+ (correspondingly, Λ_-) be the subset of those vectors $\xi \in \Lambda$ for which the plane $\xi^\perp$ does not contain u and intersects the edges of P with index $i \in I_1 \cup I_2$ (correspondingly, $i \in I_1 \cup I_3$).

Denoting by p_i the points of intersection of $\xi^\perp$ and the edges of P, we get

$$p_i = u_i - l_i \frac{\langle u_i, \xi\rangle}{\langle l_i, \xi\rangle}, \tag{2}$$

where $i \in I_1 \cup I_2$, if $\xi \in \Lambda_+$, and $i \in I_1 \cup I_3$, if $\xi \in \Lambda_-$.

For $\xi \in \Lambda$, the points of intersection of the edges of Q and the plane $\xi^\perp$ are given by

$$q_i = v_i - m_i \frac{\langle v_i, \xi\rangle}{\langle m_i, \xi\rangle}, \qquad i \in J. \tag{3}$$

The $(n-2)$-dimensional surface area of $P \cap \xi^\perp$ is given by

$$S(P \cap \xi^\perp) = \sum_j \mathrm{vol}_{n-2}(F_j \cap \xi^\perp),$$

where the sum is taken over all facets F_j of P that have nonempty intersection with $\xi^\perp$.

In order to compute the latter surface area, we will fix a triangulation of each $(n-2)$-dimensional polytope $F_j \cap \xi^\perp$. The triangulation process is described below.

First of all, in each facet F_j consider an auxiliary segment constructed as follows. The segment should pass through $\mathrm{relint}(F_j)$, it should be transversal to the planes $\xi^\perp$, $\xi \in \Lambda$, that intersect $\mathrm{relint}(F_j)$, and the segment should not be parallel to E. Let z_j denote the point of intersection of the auxiliary segment with the plane $\xi^\perp$. If ω_j is a point on the segment, and ν_j is the direction of the segment, then

$$z_j = \omega_j - \nu_j \frac{\langle \omega_j, \xi\rangle}{\langle \nu_j, \xi\rangle}.$$

The procedure for the triangulation will be as follows. Consider a hyperplane $\tilde{\xi}^\perp$, $\tilde{\xi} \in \Lambda_+ \cup \Lambda_-$, and consider all $(n-2)$-faces of P that have nonempty intersection with $\tilde{\xi}^\perp$. For any such $(n-2)$-face $\bar{E}$, the intersection $\tilde{\xi}^\perp \cap \bar{E}$ is an $(n-3)$-dimensional polytope. We fix a triangulation of the latter polytope in such a way that the vertices of all the simplices in this triangulation coincide with the vertices of $\tilde{\xi}^\perp \cap \bar{E}$. For details, see [2, p.257]. In order to triangulate the $(n-2)$-dimensional polytope $F_j \cap \tilde{\xi}^\perp$, we will take the convex hulls of the point z_j and the simplices in the triangulation of the boundary of $F_j \cap \tilde{\xi}^\perp$, constructed previously.

If an $(n-2)$-face $\bar{E}$ does not contain u, then a given triangulation of $\bar{E} \cap \tilde{\xi}^\perp$ for some $\tilde{\xi} \in \Lambda$ will induce similar triangulations of $\bar{E} \cap \xi^\perp$ for all other vectors $\xi \in \Lambda$, since there is a one-to-one correspondence between the vertices of $\bar{E} \cap \xi^\perp$ for different vectors $\xi \in \Lambda$. This correspondence is given by the incidence to the same edge of P. If $\bar{E}$ contains u, then the same holds either for Λ_- or Λ_+.

After we have fixed a triangulation of $F_j \cap \xi^\perp$, we write its $(n-2)$-dimensional area as the sum of the areas of simplices in its triangulation. If a simplex in this triangulation has vertices $z_j, p_{i_1}, \dots, p_{i_{n-2}}$, then its area is equal to the determinant

$$\frac{1}{(n-2)!\sqrt{1-\langle n_j, \xi\rangle^2}} \cdot |p_{i_1} - z_j, p_{i_2} - z_j, \dots, p_{i_{n-2}} - z_j, n_j, \xi|. \tag{4}$$

In this formula, n_j is the unit outward normal to the facet F_j. The vectors in the determinant are assumed to be ordered in such a way that the determinant is positive.

Similarly we triangulate the boundary of $Q \cap \xi^\perp$ and compute its surface area.

If $\xi \in \Lambda_+$ (respectively, Λ_-), then we will write

$$S(P \cap \xi^\perp) = S_+(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp),$$

(respectively, $S(P \cap \xi^\perp) = S_-(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp)$) where S_+ (respectively, S_-) is the total area of the simplices in the boundary of $P \cap \xi^\perp$ that have at least one vertex p_i with index $i \in I_2$ (respectively, I_3), and $\tilde{S}$ is the total area of all other simplices. Note that $\tilde{S}(P \cap \xi^\perp)$ has the same formula for both Λ_+ and Λ_-, since the vertices of the simplices in this sum belong to the $(n-2)$-faces of P that are not adjacent to u.

Since $S(P \cap \xi^\perp) = S(Q \cap \xi^\perp)$ for all $\xi \in \Lambda$, we have

$$S_+(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp) = S(Q \cap \xi^\perp) \tag{5}$$

for $\xi \in \Lambda_+$, and

$$S_-(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp) = S(Q \cap \xi^\perp) \tag{6}$$

for $\xi \in \Lambda_-$.

Now let us forget about the geometric meaning of the latter two equations. After clearing the denominators and transferring all the terms to one side, these equations become $f(\xi) = 0$ for $\xi \in \Lambda_+$, and $g(\xi) = 0$ for $\xi \in \Lambda_-$. Since the functions f and g are equal to sums of products of certain scalar products and functions $\sqrt{1 - \langle n_j, \xi\rangle^2}$, see equation (4), we can formally define these functions for all $\xi \in S^{n-1}$. Our next lemma shows that these extensions of f and g vanish on the entire sphere.

Lemma 2.1. *$f(\xi) = 0$ and $g(\xi) = 0$ for all $\xi \in S^{n-1}$.*

Proof. Let us consider only the function f, since the argument for g is absolutely the same. To reach a contradiction, we will assume that there is a point on the sphere where f is not zero. Consider a two-dimensional subspace V of $\mathbb{R}^n$ that contains this point and a point from the interior of the set Λ_+. Perturbing V if necessary, we may assume that this subspace does not contain any of the vectors n_j. Let $\tilde{f}$ be the restriction of f to the circle $V \cap S^{n-1}$. We will think of $\tilde{f}$ as a function of one variable $\phi \in [0, 2\pi]$, which is given by a sum of products that involve $\cos\phi$, $\sin\phi$, and roots of the form $\sqrt{1 - \langle n_j, a\cos\phi + b\sin\phi\rangle^2}$. Here, a and b are fixed unit vectors that span V. Since $\tilde{f}$ is equal to zero in an open subset of $[0, 2\pi]$, but not identically zero on $[0, 2\pi]$, there is a point ϕ_0, such that $\tilde{f}(\phi) = 0$ for $\phi \in [\phi_0 - \delta_1, \phi_0]$, for some $\delta_1 > 0$, and $\tilde{f}$ is not identically zero in any of the intervals $(\phi_0, \phi_0 + \delta_2)$ for all sufficiently small $\delta_2 > 0$. But this is impossible since $\tilde{f}(\phi)$ is an analytic function at ϕ_0. Contradiction. Therefore, f is identically equal to zero on the sphere.

□

In conjunction with the previous lemma and the fact that $S(Q \cap \xi^\perp)$ is given by the same formula for both $\xi \in \Lambda_+$ and $\xi \in \Lambda_-$, equations (5) and (6) imply that

$$S_+(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp) = S_-(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp),$$

or simply

$$S_+(P \cap \xi^\perp) = S_-(P \cap \xi^\perp)$$

for all $\xi \in S^{n-1}$ except finitely many great subspheres and finitely many points, i.e. except the set where the denominators vanish.

Let F_1 and F_2 be the facets of P such that $F_1 \cap F_2 = E$. Let n_1 and n_2 be the outward unit normal vectors to F_1 and F_2 correspondingly. Define

$$\eta = \alpha n_1 + \beta n_2,$$

where $\alpha,\ \beta > 0,\ \alpha^2 + \beta^2 = 1$, are chosen in such a way that η is not perpendicular to all other $(n-2)$-faces of P that are adjacent to u. This is possible since n_1 and n_2 span the normal space to E, and none of the other $(n-2)$-faces adjacent to u is parallel to E. In particular, η is not parallel to those n_j that are normal vectors to the facets adjacent to u. Similarly, we can assume that η is not perpendicular to any auxiliary segments. We will also observe that η is not perpendicular to u, since

$$\langle u, \eta \rangle = \alpha \langle u, n_1 \rangle + \beta \langle u, n_2 \rangle > 0.$$

Let λ be a vector that is not perpendicular to any of the edges of P adjacent to u, and such that $\langle \lambda, l_{i_k} \rangle > 0$ for all edges of E that are adjacent to u. The existence of such λ can be seen from the following argument. Consider a support hyperplane to the face E at u (considered as an $(n-2)$-dimensional polytope in $\mathbb{R}^n$). Let λ be its normal vector. Perturbing λ, if needed, we can assume that λ is not perpendicular to any vector l_i (direction vectors of edges of P). If λ belongs to the same half-space with respect to the support plane as E, then $\langle \lambda, l_{i_k} \rangle > 0$ for all edges of E that are adjacent to u.

Consider the following curve on the sphere:

$$\xi(\epsilon) = \frac{\eta + \epsilon \lambda}{|\eta + \epsilon \lambda|},$$

for small enough ϵ.

Now put $\xi(\epsilon)$ into the equality

$$S_+(P \cap \xi^\perp) = S_-(P \cap \xi^\perp) \tag{7}$$

multiply both sides by ϵ^{n-2}, and send $\epsilon \to 0$.

Let us find the limit, as $\epsilon \to 0$, of the terms

$$\frac{\epsilon^{n-2}}{\sqrt{1 - \langle n_j, \xi \rangle^2}} \cdot |p_{i_1} - z_j, p_{i_2} - z_j, \ldots, p_{i_{n-2}} - z_j, n_j, \xi|$$

that occur in (7).

One can see that

$$\lim_{\epsilon \to 0} \epsilon\ p_{i_k} = \lim_{\epsilon \to 0} \epsilon \left(u_{i_k} - l_{i_k} \frac{\langle u_{i_k}, \eta + \epsilon \lambda \rangle}{\langle l_{i_k}, \eta + \epsilon \lambda \rangle} \right) = \begin{cases} 0, & \text{if } \ \langle l_{i_k}, \eta \rangle \neq 0, \\ -l_{i_k} \frac{\langle u_{i_k}, \eta \rangle}{\langle l_{i_k}, \lambda \rangle}, & \text{if } \ \langle l_{i_k}, \eta \rangle = 0. \end{cases}$$

Similarly,

$$\lim_{\epsilon \to 0} \epsilon\ z_j = \lim_{\epsilon \to 0} \epsilon \left(\omega_j - \nu_j \frac{\langle \omega_j, \eta + \epsilon \lambda \rangle}{\langle \nu_j, \eta + \epsilon \lambda \rangle} \right) = 0,$$

since $\langle \nu_j, \eta \rangle \neq 0$ by the choice of η.

Therefore, the only surviving determinants will be those of the form

$$\frac{1}{\sqrt{1-\langle n_j,\eta\rangle^2}}\cdot|-l_{i_1}\frac{\langle u_{i_1},\eta\rangle}{\langle l_{i_1},\lambda\rangle},\ldots,-l_{i_{n-2}}\frac{\langle u_{i_{n-2}},\eta\rangle}{\langle l_{i_{n-2}},\lambda\rangle},n_j,\eta|,$$

where l_{i_1},..., $l_{i_{n-2}}$ are all perpendicular to η.

Since all vectors l_{i_k} in the latter determinant must be linearly independent, it follows that they span an $(n-2)$-plane that contains an $(n-2)$-face of P. Moreover, this face has to be perpendicular to η, since all vectors l_{i_k} are perpendicular to η, and must contain either u or $-u$. There are only two such $(n-2)$-faces, these are E and $-E$. We will ignore $-E$; it gives the same contribution as E, since the body is symmetric.

Since the face E only belongs to the facets F_1 and F_2, and since the vectors l_{i_k} emanate from u, the nonzero terms in the limiting case of equality (7) will be of the form

$$\frac{(-1)^{n-2}\langle u,\eta\rangle^{n-2}}{\langle l_{i_1},\lambda\rangle\cdots\langle l_{i_{n-2}},\lambda\rangle\sqrt{1-\langle n_1,\eta\rangle^2}}\left|l_{i_1},\ldots,l_{i_{n-2}},n_1,\beta n_2\right| \tag{8}$$

and

$$\frac{(-1)^{n-2}\langle u,\eta\rangle^{n-2}}{\langle l_{j_1},\lambda\rangle\cdots\langle l_{j_{n-2}},\lambda\rangle\sqrt{1-\langle n_2,\eta\rangle^2}}\left|l_{j_1},\ldots,l_{j_{n-2}},n_2,\alpha n_1\right|, \tag{9}$$

where $(j_1,...,j_{n-2})$ is a permutation of the indices $(i_1,...,i_{n-2})$.

Lemma 2.2. *The determinants*

$$\left|l_{i_1},\ldots,l_{i_{n-2}},n_1,n_2\right|$$

in (8) have the same sign for all combinations of indices $(i_1,...,i_{n-2})$ that correspond to the simplices in the triangulation of $E\cap\xi^\perp$.

The same is true about the determinants in (9).

Proof. We will discuss only the determinants in (8), the other case is similar.

Recall that all the determinants

$$|p_{i_1}-z_1,p_{i_2}-z_1,\ldots,p_{i_{n-2}}-z_1,n_1,\xi|, \tag{10}$$

with $\xi\in\Lambda_+$, are positive for all combinations of indices $(i_1,...,i_{n-2})$. Therefore, in order to prove the lemma it is enough to show that the determinants (10) and

$$\left|l_{i_1},\ldots,l_{i_{n-2}},n_1,n_2\right|$$

have either the same sign for all combinations of indices $(i_1,...,i_{n-2})$ or opposite sign, again for all combinations.

To see this, let us denote by $\hat{\xi}$ and $\hat{n}_2$ the orthogonal projections of ξ and n_2, correspondingly, onto the hyperplane $n_1^\perp$. Note that $\hat{\xi}$ and $\hat{n}_2$ are nonzero vectors. Consider a rigid motion A in $\mathbb{R}^n$ that leaves fixed both the vector n_1 and the affine hull of $E\cap\xi^\perp$, and such that A maps the vector $\hat{n}_2$ into a vector collinear to $\hat{\xi}$. Note that $\hat{n}_2$ is a normal vector to the affine hull of E, and $\hat{\xi}$ is a normal vector to the affine hull of $F_1\cap\xi^\perp$. Therefore, under the transformation A all the points $p_{i_1}, p_{i_2}, \dots, p_{i_{n-2}}$ remain fixed, and the vertex u gets mapped into the affine hull of $F_1 \cap \xi^\perp$.

Since for all edges adjacent to u we have

$$l_{i_k} = \frac{p_{i_k} - u}{|p_{i_k} - u|},$$

it follows that

$$\begin{aligned}\big|l_{i_1}, \dots, l_{i_{n-2}}, n_1, n_2\big| &= C_1 |p_{i_1} - u, \dots, p_{i_{n-2}} - u, n_1, \hat{n}_2| \\ &= C_2 |p_{i_1} - Au, \dots, p_{i_{n-2}} - Au, n_1, \hat{\xi}|,\end{aligned}$$

for some positive quantities C_1 and C_2.

The latter determinant is similar to (10), except that it has Au instead of z_1. But z_1 and Au are points in the $(n-2)$-plane $\mathrm{aff}(F_1 \cap \xi^\perp)$. Therefore, the sign of the corresponding determinant depends on the relative position of these two points with respect to the $(n-3)$-plane $\mathrm{aff}(E \cap \xi^\perp)$. Therefore, either for all combinations of indices the determinants have the same sign, or the opposite.

□

In view of the previous lemma, the limit of (7) equals

$$\left(\pm \frac{\beta}{\sqrt{1 - \langle n_1, \eta \rangle^2}} \pm \frac{\alpha}{\sqrt{1 - \langle n_2, \eta \rangle^2}} \right) \times \sum_I \frac{\langle u, \eta \rangle^{n-2}}{\langle l_{i_1}, \lambda \rangle \cdots \langle l_{i_{n-2}}, \lambda \rangle} \big|l_{i_1}, \dots, l_{i_{n-2}}, n_1, n_2\big| = 0 \quad (11)$$

where the sum is taken over all $(n-2)$-tuples of indices $(i_1, ..., i_{n-2})$ that correspond to simplices in the triangulation of $E \cap \xi^\perp$, $\xi \in \Lambda_+$.

Note that $\langle l_{i_k}, \lambda \rangle > 0$ due to the choice of λ, and $\langle u, \eta \rangle \neq 0$ due to the choice of η. Furthermore, as we saw above, all the determinants in (11) have the same sign. If we choose $\alpha \neq \beta$, then the left-hand side of (11) is nonzero. Contradiction.

Case 2. All vertices of P and Q come in pairs, that is if a line through the origin contains a vertex of one of the polytopes, then it also contains a vertex of the other. Under this assumption the following holds.

Lemma 2.3. *There exist a vertex u of P, a corresponding vertex v of Q lying on the same line and on the same side with respect to the origin, and an $(n-2)$-face of, say, P adjacent to u that is not parallel to any $(n-2)$-face of Q adjacent to v.*

Proof. Suppose the claim is false. Then for every vertex u and every $(n-2)$-face E of P adjacent to u, there is an $(n-2)$-face $\tilde{E}$ of Q adjacent to the corresponding vertex v and parallel to E, and vice versa. Since we are assuming that P and Q are different, there is a vertex u_1 of, say, P that lies farther from the origin than the corresponding vertex v_1 of Q. Let u_2 be another vertex of P that belongs to the face E. Consider the hyperplane H

through the origin that contains the face E, and therefore, also $\tilde{E}$. Consider the $(n-1)$-dimensional polytopes $P \cap H$ and $Q \cap H$. They have parallel facets E and $\tilde{E}$. The $(n-2)$-planes aff(E) and aff$(\tilde{E})$ do not intersect (since u_1 and v_1 do not coincide). Therefore, v_2, the vertex of Q corresponding to u_2, lies closer to the origin than u_2. Applying the same reasoning to all the vertices of P, we see that $P \supset Q$. This means that the same inclusion holds for all their sections. But larger bodies have larger projections, and therefore, by Cauchy's projection formula [1, p.361], larger surface areas. This is impossible since all sections of P and Q have equal surface areas.

□

Let E be an $(n-2)$-face of P adjacent to a vertex u that is not parallel to any $(n-2)$-face of Q at the corresponding vertex v. There is a hyperplane $\xi_0^\perp$ with the following properties:

1) $\xi_0^\perp \cap E = \{u\}$,

2) $\xi_0^\perp$ contains no vertices of either P or Q (other than u, $-u$, v, $-v$),

Let Λ be a spherical cap around the point ξ_0 such that for all $\xi \in \Lambda$ the plane $\xi^\perp$ contains no vertices of P and Q, except possibly u, $-u$, v, $-v$.

We will use the same indexing of the edges of P as in case 1. For Q the edges will be labeled similarly. Namely, the edges of Q that are not adjacent to the vertices v or $-v$ are indexed by $i \in J_1$. The edges of Q adjacent to either of the vertices v or $-v$ are indexed by $i \in J_2 \cup J_3$. Let those edges that are adjacent to v and lie in ξ_0^+, and those that are adjacent to $-v$ and lie in ξ_0^- be indexed by $i \in J_2$, and all the others by $i \in J_3$.

Triangulations of the boundaries of $P \cap \xi^\perp$ and $Q \cap \xi^\perp$ are done in the same way as in case 1.

For $\xi \in \Lambda_+$ (respectively, Λ_-), we will write $S(P \cap \xi^\perp) = S_+(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp)$, (respectively, $S(P \cap \xi^\perp) = S_-(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp)$) with S_+, S_-, $\tilde{S}$ having the same meaning as in case 1. Similarly, for Q. If $\xi \in \Lambda_+$ (respectively, Λ_-), then we will write

$$S(Q \cap \xi^\perp) = S_+(Q \cap \xi^\perp) + \tilde{S}(Q \cap \xi^\perp),$$

(respectively, $S(Q \cap \xi^\perp) = S_-(Q \cap \xi^\perp) + \tilde{S}(Q \cap \xi^\perp)$) where S_+ (respectively, S_-) is the total area of the simplices in the boundary of $Q \cap \xi^\perp$ that have at least one vertex q_i with index $i \in J_2$ (respectively, $i \in J_3$), and $\tilde{S}$ is the total area of all other simplices. Note that $\tilde{S}(Q \cap \xi^\perp)$ has the same formula for both Λ_+ and Λ_-.

We now use that $S(P \cap \xi^\perp) = S(Q \cap \xi^\perp)$ to get

$$S_+(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp) = S_+(Q \cap \xi^\perp) + \tilde{S}(Q \cap \xi^\perp), \quad \text{for } \xi \in \Lambda_+$$

and

$$S_-(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp) = S_-(Q \cap \xi^\perp) + \tilde{S}(Q \cap \xi^\perp), \quad \text{for } \xi \in \Lambda_-$$

Moreover, we may assume that the latter two equalities hold for all $\xi \in S^{n-1}$ except finitely many great subspheres and finitely many points.

Since $\tilde{S}(P\cap\xi^\perp)-\tilde{S}(Q\cap\xi^\perp)$ is given by the same formula for both Λ_+ and Λ_-, we have

$$S_+(P\cap\xi^\perp)-S_+(Q\cap\xi^\perp)=S_-(P\cap\xi^\perp)-S_-(Q\cap\xi^\perp) \tag{12}$$

for all $\xi\in S^{n-1}$ except finitely many great subspheres.

Note that, unlike in case 1, Q plays a stronger role in these formulas. This is due to the fact that $\xi_0^\perp$ contains a vertex of Q and therefore $S_+(Q\cap\xi^\perp)$ and $S_-(Q\cap\xi^\perp)$ are no longer the same functions. However, Lemma 2.3 will allow us to eliminate the contribution of Q.

As before, let F_1 and F_2 be the facets of P such that $F_1\cap F_2=E$, and n_1 and n_2 their outward unit normal vectors. Choose

$$\eta=\alpha n_1+\beta n_2,$$

in such a way that η is not perpendicular to all other $(n-2)$-faces of P that are adjacent to u, and all $(n-2)$-faces of Q that are adjacent to v (here we use Lemma 2.3). We also assume that η is not perpendicular to any auxiliary segments, and not parallel to the normal vectors to the facets of P and Q adjacent to u and v.

Now choose $\xi(\epsilon)$ as in case 1 (and make sure that λ is not perpendicular to any of the edges of Q adjacent to u), substitute it into equation (12), multiply by ϵ^{n-2} and send $\epsilon\to 0$. Due to the choice of η, we see that the only terms that survive in the limit correspond to the face E of P. Therefore, we are in the same situation, as in case 1. Arguing as above, we see that one side of equation (12) is zero and the other is not. Contradiction. This finishes the proof of the theorem.

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