A SOLUTION TO THE LOWER DIMENSIONAL BUSEMANN-PETTY PROBLEM IN THE HYPERBOLIC SPACE

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Abstract. The lower dimensional Busemann-Petty problem asks whether origin symmetric convex bodies in $\mathbb{R}^n$ with smaller volume of all k-dimensional sections necessarily have smaller volume. As proved by Bourgain and Zhang, the answer to this question is negative if $k > 3$. The problem is still open for $k = 2, 3$. In this article we formulate and completely solve the lower dimensional Busemann-Petty problem in the hyperbolic space $\mathbb{H}^n$.

1. Introduction

The Busemann-Petty problem asks whether origin symmetric convex bodies in $\mathbb{R}^n$ with smaller hyperplane sections necessarily have smaller volume. The answer to this problem is affirmative if $n \leq 4$ and negative if $n \geq 5$ (see [GKS], [Zh2] or [K4, Chapter 5] for the solution and historical details). In [Y] the author solved the Busemann-Petty problem in hyperbolic and spherical spaces.

The lower dimensional Busemann-Petty problem (LDBP) in $\mathbb{R}^n$ asks the same question with k-dimensional subspaces in place of hyperplanes. Bourgain and Zhang [BZ] proved that this problem has a negative answer if $3 < k < n$, see [K3] for another solution. The cases $k = 2, 3$ are still open in dimensions $n > 4$.

In this paper we study the lower dimensional Busemann-Petty problem in the hyperbolic space. Namely, let $1 \leq k < n$, and K, L be origin-symmetric convex bodies in $\mathbb{H}^n$, $n \geq 3$, such that

$$\mathrm{vol}_k(K \cap H) \leq \mathrm{vol}_k(L \cap H)$$

for every k-dimensional totally geodesic plane through the origin. Does it follow that

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)?$$

For the case $k = 1$ the answer is trivially affirmative, since in all directions the radius of K does not exceed the radius of L. In this paper we prove that the answer to the hyperbolic lower dimensional Busemann-Petty problem is negative for every $2 \leq k < n$.

2. Hyperbolic geometry

It is well-known (see [DFN, §10] or [R, §4.5]) that the hyperbolic space $\mathbb{H}^n$ can be identified with the interior of the unit ball B^n in $\mathbb{R}^n$ with the metric:

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1-(x_1^2 + \cdots + x_n^2))^2}. \tag{1}$$

This is called the Poincaré model of the hyperbolic space in the ball. The geodesic lines in this model are arcs of the circles orthogonal to the boundary of the ball B^n and straight lines through the origin.

Since for any two points in the hyperbolic space there exists a unique geodesic connecting them, the definition of convexity in the hyperbolic space will be analogous to that in the Euclidean space (see [P, Chapter I, §12]). A body K (compact set with non-empty interior) is called *convex*, if for every pair of points in K, the geodesic segment joining them also belongs to the body K.

In order to distinguish between different types of convexity in the unit ball, we use the following system of notations. Let K be a body in the open unit ball B^n. The body K is called h-convex, if it is convex in the hyperbolic metric defined in the ball B^n. Similarly it is called e-convex, if it is convex in the usual Euclidean sense. Analogously, h-geodesics are the straight lines of the hyperbolic metric and e-geodesics are the usual Euclidean straight lines.

A submanifold $\mathcal{F}$ in a Riemannian space $\mathcal{R}$ is called *totally geodesic* if every geodesic in $\mathcal{F}$ is also a geodesic in the space $\mathcal{R}$. In the Euclidean space the totally geodesic submanifolds are Euclidean planes. In the Poincaré model of the hyperbolic space described above the totally geodesic submanifolds are represented by the spheres orthogonal to the boundary of the unit ball B^n and Euclidean planes through the origin. (We want to emphasize that a k-dimensional submanifold passing through the origin is totally geodesic if and only if it is a k-dimensional Euclidean plane). In a sense, totally geodesic submanifolds are analogs of Euclidean planes in Riemannian spaces. For elementary properties of totally geodesic submanifolds see [A, Chap.5, §5].

The *Minkowski functional* of a star-shaped origin-symmetric body $K \subset \mathbb{R}^n$ is defined as

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}.$$

The *radial function* of K is given by $\rho_K(x) = \|x\|_K^{-1}$. If $x \in S^{n-1}$ then the radial function $\rho_K(x)$ is the Euclidean distance from the origin to the boundary of K in the direction of x.

The volume element of the metric (1) equals

$$d\mu_n = 2^n \frac{dx_1 \cdots dx_n}{(1-(x_1^2 + \cdots + x_n^2))^n} = 2^n \frac{dx}{(1-|x|^2)^n}.$$

Therefore the hyperbolic volume of a body K is given by the formula:

$$\mathrm{vol}_n(K) = \int_K d\mu_n = 2^n \int_K \frac{dx}{(1-|x|^2)^n}.$$

Note that in the polar coordinates of $\mathbb{R}^n$ the latter formula looks as follows:

$$\mathrm{vol}_n(K) = 2^n \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} \frac{r^{n-1}}{(1-r^2)^n} dr\ d\theta. \tag{2}$$

Similarly, if H is a k-dimensional hyperbolic totally geodesic plane through the origin (as mentioned above, this is just a k-dimensional Euclidean plane through the origin), then the volume element of H in the metric (1) is

$$d\mu_k = 2^k \frac{dx}{(1-|x|^2)^k},$$

therefore the hyperbolic k-volume of the section of K by H is given by the formula:

$$\mathrm{vol}_k(K\cap H) = \int_{K\cap H} d\mu_k = 2^k \int_{K\cap H} \frac{dx}{(1-|x|^2)^k},$$

or in polar coordinates:

$$\mathrm{vol}_k(K\cap H) = 2^k \int_{S^{n-1}\cap H} \int_0^{\|\theta\|_K^{-1}} \frac{r^{k-1}}{(1-r^2)^k} dr\ d\theta. \tag{3}$$

Even though our main object is hyperbolic geometry, let us briefly mention that, along with the hyperbolic and Euclidean metrics, we can define the spherical metric in the unit ball B^n:

$$ds^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1+(x_1^2+\cdots+x_n^2))^2}.$$

The geodesic lines in this model are arcs of the circles intersecting the boundary of the ball B^n in antipodal points and straight lines through the origin. Such lines will be called s-geodesics. The body K is called s-convex, if it is convex in the spherical metric defined in the ball B^n. (This notion is well-defined, since in this model every two points can be joined by a unique geodesic).

Finally, a simple observation about all introduced types of convexity is that any s-convex body containing the origin is also e-convex and any e-convex body containing the origin is h-convex. (See for example [MP]).

3. Fourier transform of distributions

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function ϕ from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$.

We say that a distribution f is *positive definite* if its Fourier transform is a positive distribution, in the sense that $\langle \hat{f}, \phi\rangle \geq 0$ for every non-negative test function ϕ.

We say that a closed bounded set K in $\mathbb{R}^n$ is a *star body* if for every $x \in K$ each point of the interval $[0, x)$ is an interior point of K, and $\|x\|_K$, the Minkowski functional of K, is a continuous function on $\mathbb{R}^n$.

Let K be a star body and $\xi \in S^{n-1}$, the *parallel section function* of K is defined as follows:

$$A_{K,\xi}(z) = \mathrm{vol}_{n-1}(K \cap \{\langle x, \xi \rangle = z\}).$$

(We also assume that $K \cap \{\langle x, \xi \rangle = z\}$ is star-shaped for small z). Recall the following fact:

Theorem 3.1. ([GKS], Theorem 1) *Let K be an origin-symmetric star body in $\mathbb{R}^n$ with C^∞ boundary, and let $k \in \mathbb{N} \setminus \{0\}$, $k \neq n-1$. Suppose that $\xi \in S^{n-1}$, and let A_ξ be the corresponding parallel section function of K.*
(a) If k is even, then

$$(\|x\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_\xi^{(k)}(0).$$

(b) If k is odd, then

$$(\|x\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{(k+1)/2}2(n-1-k)k! \times$$
$$\times \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2} - \cdots - A_\xi^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}dz,$$

where $A_\xi^{(k)}$ stands for the derivative of the order k and the Fourier transform is considered in the sense of distributions.

In particular, it follows that for infinitely smooth bodies the Fourier transform of $\|x\|^{-n+k+1}$ restricted to the unit sphere is a continuous function (see also [K4, Section 3.2]). This remark explains why integration over the sphere in the next lemma makes sense. The following is Parseval's formula on the sphere proved by Koldobsky [K2].

Lemma 3.2. *If K and L are origin symmetric infinitely smooth star bodies in $\mathbb{R}^n$ and $0 < p < n$, then*

$$\int_{S^{n-1}} \left(\|x\|_K^{-p}\right)^\wedge(\xi)\left(\|x\|_L^{-n+p}\right)^\wedge(\xi)d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p}\|x\|_L^{-n+p}dx.$$

The following result was also proved in [K2].

Lemma 3.3. *Let L be an origin symmetric star body with C^∞ boundary in $\mathbb{R}^n$. Then for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$ we have*

$$(2\pi)^k \int_{S^{n-1}\cap H} \|\theta\|_L^{-n+k}d\theta = \int_{S^{n-1}\cap H^\perp} (\|x\|_L^{-n+k})^\wedge(\theta)d\theta.$$

The preceding two lemmas were formulated for Minkowski functionals, but in fact they are true for arbitrary infinitely differentiable even functions on the sphere extended to $\mathbb{R}^n \setminus \{0\}$ as homogeneous functions of corresponding degrees. (Indeed, any such function of degree $-p$ can be obtained as the difference of Minkowski functionals raised to the power $-p$).

The next lemma is a Fourier analytic version of a result of Zhang [Zh1, Lemma 2].

Lemma 3.4. *Let k be an integer, $1 \le k \le n-1$, and let f be an infinitely differentiable even function on the sphere S^{n-1}, such that $f(x/|x|)|x|^{-k}$ is not a positive definite distribution on $\mathbb{R}^n$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. Then there exists an even function $g \in C^\infty(S^{n-1})$ such that*

$$\int_{S^{n-1}} f(x)g(x)dx > 0 \tag{4}$$

and

$$\int_{S^{n-1}\cap H} g(x)dx \le 0, \tag{5}$$

for any $(n-k)$-dimensional plane H through the origin.

Proof. Since f is infinitely differentiable, by [K4, Section 3.2], $(f(x/|x|)|x|^{-k})^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$. By our assumption there exists $\xi \in S^{n-1}$ such that $(f(x/|x|)|x|^{-k})^\wedge(\xi) < 0$. By continuity of $(f(x/|x|)|x|^{-k})^\wedge$ there is a neighborhood of ξ where this function is negative. Let

$$\Omega = \{\theta \in S^{n-1} : (f(x/|x|)|x|^{-k})^\wedge(\theta) < 0\}.$$

Choose a non-positive infinitely-smooth even function v supported in Ω. Extend v to a homogeneous function $|x|^{-n+k}v(x/|x|)$ of degree $-n+k$ on $\mathbb{R}^n$. By [K4, Section 3.2], the Fourier transform of $|x|^{-n+k}v(x/|x|)$ is equal to $|x|^{-k}g(x/|x|)$ for some infinitely smooth function g on S^{n-1}.

By Parseval's formula on the sphere (Lemma 3.2) we have

$$\begin{aligned}\int_{S^{n-1}} f(x)g(x)dx &= \int_{S^{n-1}} \left(f(x/|x|)|x|^{-k}\right)\left(g(x/|x|)|x|^{-n+k}\right)dx \\ &= \frac{1}{(2\pi)^n}\int_{S^{n-1}} \left(f(x/|x|)|x|^{-k}\right)^\wedge(\theta)\left(g(x/|x|)|x|^{-n+k}\right)^\wedge(\theta)d\theta \\ &= \frac{1}{(2\pi)^n}\int_{S^{n-1}} \left(f(x/|x|)|x|^{-k}\right)^\wedge(\theta)v(\theta)d\theta > 0,\end{aligned}$$

since v is non-positive and supported in the set where $\left(f(x/|x|)|x|^{-k}\right)^\wedge$ is negative.

Secondly, by Lemma 3.3 we have

$$\begin{aligned}(2\pi)^k\int_{S^{n-1}\cap H} g(x)dx &= (2\pi)^k\int_{S^{n-1}\cap H} g(x/|x|)|x|^{-n+k}dx \\ &= \int_{S^{n-1}\cap H^\perp}\left(g(x/|x|)|x|^{-n+k}\right)^\wedge(\theta)d\theta = \int_{S^{n-1}\cap H^\perp} v(\theta)d\theta \le 0,\end{aligned}$$

since v is non-positive.

□

4. Main results

Proposition 4.1. *Let $1 \le k \le n-2$. There exists an infinitely smooth origin symmetric strictly e-convex body L in the unit ball $B^n \subset \mathbb{R}^n$, so that*

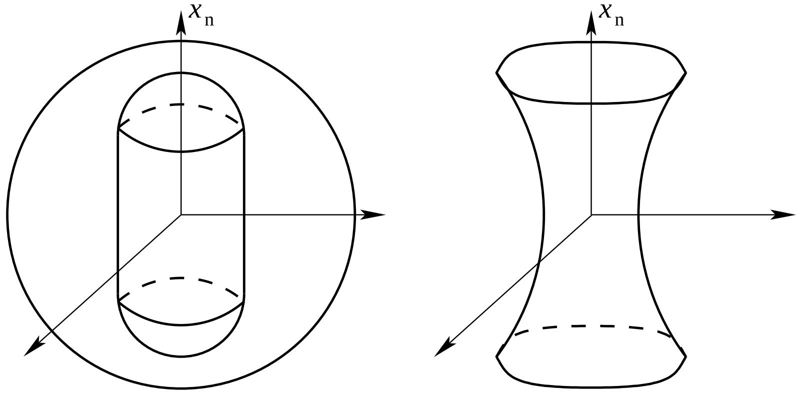
$$\frac{\|x\|_L^{-k}}{(1-(\frac{|x|}{\|x\|_L})^2)^k} \tag{6}$$

is not a positive definite distribution on $\mathbb{R}^n$.

Proof. First, we consider the cases $k = n-2$ and $n-3$. We will use a construction similar to [Y, Proposition 3.9]. Let L be a circular cylinder of radius $\sqrt{2}/2$ with x_n being its axis of revolution. To the top and bottom of the cylinder attach spherical caps, that are totally geodesic in the spherical metric. Clearly the body L constructed this way is e-convex and therefore h-convex. Using the formula

$$\|x\|_M^{-1} = \frac{\|x\|_L^{-1}}{1-(\frac{|x|}{\|x\|_L})^2} \tag{7}$$

we define a body M. (Note, that M is well-defined, since L lies entirely in the unit ball B^n and the denominator in the latter formula is never equal to zero).



L *M*

Clearly the body M is the image of L under the map:

$$(r,\theta) \mapsto \left(\frac{r}{1-r^2}, \theta\right) \tag{8}$$

It can be checked directly that the cylinder is mapped into the surface of revolution obtained by rotating the hyperbola $x_1 = \frac{1}{2}\left(\sqrt{2} + \sqrt{2+4x_n^2}\right)$ about the x_n-axis, and the top and bottom spherical caps are mapped into flat disks. The latter follows from the fact that (8) maps s-geodesics into e-geodesics. Indeed, without loss of generality we may consider a s-geodesic given by the equation: $r^2 + a\ r\cos\phi - 1 = 0$ in some 2-dimensional plane. The image of this s-geodesic under the map (8) is an e-geodesic $r = \dfrac{1}{a\cos\phi}$.

The body L constructed above is not smooth. But we can approximate it by infinitely smooth e-convex bodies that differ from L only in a small neighborhood of the edges. Since the body M is obtained from L by (7), and the denominator in (7) is never equal to zero, the body M is also infinitely smooth.

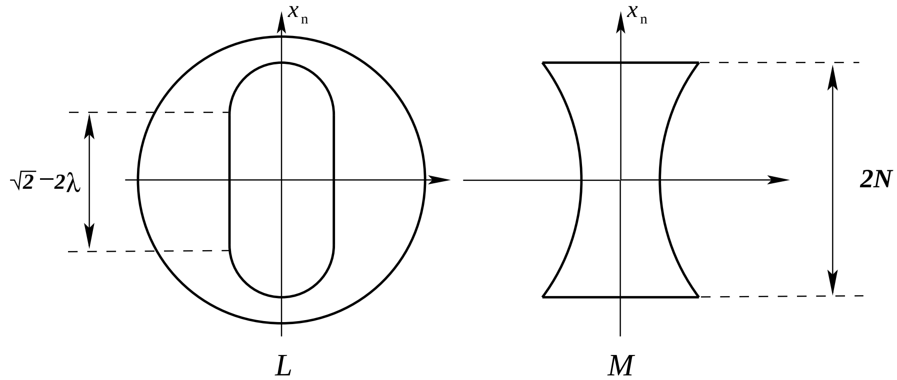
Now that we have defined the body M, we can explicitly compute its parallel section function $A_{M,\xi}$ in the direction of the x_n-axis.

$$A_{M,\xi}(t) = C_n\left(\sqrt{2} + \sqrt{2+4t^2}\right)^{n-1}.$$

Let the height of the cylindrical part of L be equal to $\sqrt{2} - 2\lambda$ and the height of its image under (8) equal to $2N$ (see the picture below). Since the radius of the cylinder equals $\sqrt{2}/2$, when λ tends to zero the top and bottom parts of the body L get closer to the sphere $x_1^2 + \cdots + x_n^2 = 1$. Recalling the definition of the radial function of M:

$$\rho_M(x) = \frac{\rho_L(x)}{1-\rho_L(x)^2}, \quad \forall x \in S^{n-1},$$

one can see that the height $2N$ of the body M approaches infinity as $\lambda \to 0$.



Since M is an infinitely smooth body, $(\|x\|_M^{-n+k+1})^\wedge$ is a function. Applying Theorem 3.1 with $k=1$ we get

$$\begin{aligned}(\|x\|_M^{-n+2})^\wedge(\xi) &= -2(n-2)\int_0^\infty \frac{A_{M,\xi}(t) - A_{M,\xi}(0)}{t^2}dt \\ &= -2(n-2)C_n\int_0^N \frac{\left(\sqrt{2}+\sqrt{2+4t^2}\right)^{n-1} - (2\sqrt{2})^{n-1}}{t^2}dt + \\ &\quad +2(n-2)C_n\int_N^\infty \frac{(2\sqrt{2})^{n-1}}{t^2}dt.\end{aligned}$$

To estimate the first integral we use the binomial theorem,

$$\begin{aligned}\left(\sqrt{2}+\sqrt{2+4t^2}\right)^{n-1} &= (\sqrt{2})^{n-1}+(n-1)(\sqrt{2})^{n-2}\sqrt{2+4t^2}+\\ &\quad+\frac{(n-1)(n-2)}{2}(\sqrt{2})^{n-3}(2+4t^2)+\cdots\\ &\geq (2\sqrt{2})^{n-1}+2(n-1)(n-2)(\sqrt{2})^{n-3}t^2,\end{aligned}$$

where the last inequality was obtained by putting $t=0$ in all the terms of the binomial expansion, except for the third term. Therefore, for some positive constants C_n' and C_n'' we have

$$(\|x\|_M^{-n+2})^\wedge(\xi) \leq -C_n'\int_0^N dt + C_n''\int_N^\infty \frac{1}{t^2}dt = -C_n'N + C_n''\frac{1}{N} < 0$$

for N large enough.

Therefore the body M, corresponding to this N, is not a $(n-2)$-intersection body in the Euclidean sense, which implies that

$$\frac{\|x\|_L^{-n+2}}{(1-(\frac{|x|}{\|x\|_L})^2)^{n-2}} = \|x\|_M^{-n+2} \tag{9}$$

is not a positive definite distribution.

Similarly we can show that M is not a $(n-3)$-intersection body. Indeed, if $k=2$ Theorem 3.1 implies

$$(\|x\|_M^{-n+3})^\wedge(\xi) = -\pi(n-3)A_{M,\xi}''(0) < 0,$$

since the second derivative of the function $A_{M,\xi}$ equals:

$$A_{M,\xi}''(0) = C_n(n-1)(2\sqrt{2})^{n-1} > 0.$$

Next we handle the case when $1\leq k<n-3$. For this we use a different construction. Let M be an infinitely smooth origin symmetric e-convex body in $\mathbb{R}^n$, for which $\|x\|_M^{-k}$ is not positive definite. (For example, the unit ball of the space ℓ_4^n, see [K1]). Dilate this body M, if needed, to make sure that it lies in the unit Euclidean ball. Let $\rho_M(x)$ be the radial function of this body. Define a body L as follows:

$$\rho_L(x) = \frac{-1+\sqrt{1+4(\rho_M(x))^2}}{2\rho_M(x)}, \qquad \text{for } x\in S^{n-1}.$$

One can check that

$$\rho_M(x) = \frac{\rho_L(x)}{1-\rho_L(x)^2}, \qquad \text{for } x\in S^{n-1}.$$

Clearly, M is the image of L under the transformation (8). Since (8) maps s-geodesics into e-geodesics, L is a s-convex body, and therefore e-convex.

Thus we have proved that for $1\leq k<n-3$,

$$\frac{\|x\|_L^{-k}}{\left(1-(\frac{|x|}{\|x\|_L})^2\right)^k} = \|x\|_M^{-k}$$

is not positive definite.

To finish the proof, note that in our construction L is not necessarily strictly e-convex. But one can replace L with L_ϵ, defined by

$$\|\theta\|_{L_\epsilon}^{-1} = \|\theta\|_L^{-1} + \epsilon|\theta|^{-1}.$$

One can choose $\epsilon > 0$ small enough, so that L_ϵ is strictly e-convex, and so that $\frac{\|x\|_{L_\epsilon}^{-k}}{\left(1-(\frac{|x|}{\|x\|_{L_\epsilon}})^2\right)^k}$ is still not positive definite (see, for example, the approximation argument in [K4, Lemma 4.10]). $\square$

Theorem 4.2. *Let $1 \le k < n-1$. There are origin-symmetric convex bodies K and L in $\mathbb{H}^n$, $n \ge 3$, such that*

$$\mathrm{vol}_{n-k}(K \cap H) \le \mathrm{vol}_{n-k}(L \cap H)$$

for every $(n-k)$-dimensional totally geodesic plane through the origin, but

$$\mathrm{vol}_n(K) > \mathrm{vol}_n(L).$$

Proof. Let L be an infinitely smooth origin symmetric e-convex body from Proposition 4.1, for which $\frac{||x||_L^{-k}}{\left(1-(\frac{|x|}{||x||_L})^2\right)^k}$ is not positive definite.

By Lemma 3.4 there exists an even function $g \in C^\infty(S^{n-1})$ such that

$$\int_{S^{n-1}} \frac{||x||_L^{-k}}{\left(1-(\frac{|x|}{||x||_L})^2\right)^k} g(x)dx > 0 \tag{10}$$

and

$$\int_{S^{n-1}\cap H} g(x)dx \le 0, \qquad \text{for all } H. \tag{11}$$

Now apply a standard argument to construct another body K which along with the body L provides a counterexample to the hyperbolic LDBP problem (cf. [K2], Theorem 2 or [Zv], Theorem 2). Define a new body K as follows:

$$\int_0^{\|\theta\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}}dr = \int_0^{\|\theta\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}}dr + \epsilon g(\theta) \tag{12}$$

for $\theta \in S^{n-1}$ and some $\epsilon > 0$ small enough (to guarantee that K is still convex in hyperbolic sense). Indeed, define a function $\alpha_\epsilon(\theta)$ such that

$$\int_0^{\|\theta\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}}dr + \epsilon v(\theta) = \int_0^{\|\theta\|_L^{-1}+\alpha_\epsilon(\theta)} \frac{r^{n-k-1}}{(1-r^2)^{n-k}}dr,$$

then

$$\|\theta\|_K^{-1} = \|\theta\|_L^{-1} + \alpha_\epsilon(\theta).$$

The function $\alpha_\epsilon(\theta)$ and its first and second derivatives converge uniformly to zero as $\epsilon \to 0$ (cf. [Zv, Proposition 2]), therefore since L is strictly e-convex, there exists ϵ small enough, so that K is also strictly e-convex, and hence h-convex.

Let H be an $(n-k)$-plane through the origin. Integrating (12) over $S^{n-1} \cap H$ and using inequality (11), we get

$$\int_{S^{n-1}\cap H} \int_0^{\|\theta\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr d\theta \leq \int_{S^{n-1}\cap H} \int_0^{\|\theta\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr d\theta,$$

which, by formula (3), is equivalent to

$$\mathrm{vol}_{n-k}(K \cap H) \leq \mathrm{vol}_{n-k}(L \cap H).$$

On the other hand, multiplying both sides of (12) by $\left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k$ and integrating over the sphere S^{n-1} we get

$$\int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_0^{\|x\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx =$$
$$= \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_0^{\|x\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx +$$
$$+\epsilon \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k g(x) dx.$$

From (10) it follows that

$$\int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_0^{\|x\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx >$$
$$> \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_0^{\|x\|_L^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx$$

Therefore,

$$0 < \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_{\|x\|_L^{-1}}^{\|x\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr dx \tag{13}$$

Next we need the following elementary inequality (cf. Zvavitch, [Zv]). For any $a, b \in (0,1)$

$$\frac{a^k}{(1-a^2)^k} \int_a^b \frac{r^{n-k-1}}{(1-r^2)^{n-k}} dr \leq \int_a^b \frac{r^{n-1}}{(1-r^2)^n} dr.$$

Indeed, since the function $\frac{r^k}{(1-r^2)^k}$ is increasing on the interval $(0,1)$ we have the following

$$\begin{aligned} \frac{a^k}{(1-a^2)^k}\int_a^b \frac{r^{n-k-1}}{(1-r^2)^{n-k}}dr &= \int_a^b \frac{r^{n-1}}{(1-r^2)^n}\frac{a^k}{(1-a^2)^k}\left(\frac{r^k}{(1-r^2)^k}\right)^{-1}dr \\ &\le \int_a^b \frac{r^{n-1}}{(1-r^2)^n}dr. \end{aligned}$$

Note that in the latter inequality it does not matter whether $a \le b$ or $a \ge b$.

Applying the elementary inequality to (13) with $a = \|x\|_L^{-1}$ and $b = \|x\|_K^{-1}$, we get

$$\begin{aligned} 0 &< \int_{S^{n-1}} \left(\frac{\|x\|_L^{-1}}{1-\|x\|_L^{-2}}\right)^k \int_{\|x\|_L^{-1}}^{\|x\|_K^{-1}} \frac{r^{n-k-1}}{(1-r^2)^{n-k}}drdx \\ &\le \int_{S^{n-1}}\int_{\|x\|_L^{-1}}^{\|x\|_K^{-1}} \frac{r^{n-1}}{(1-r^2)^n}drdx. \end{aligned}$$

Hence

$$\int_{S^{n-1}}\int_0^{\|x\|_L^{-1}} \frac{r^{n-1}}{(1-r^2)^n}drdx < \int_{S^{n-1}}\int_0^{\|x\|_K^{-1}} \frac{r^{n-1}}{(1-r^2)^n}drdx,$$

that is

$$\mathrm{vol}_n(L) < \mathrm{vol}_n(K).$$

□

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