INEQUALITIES OF THE KAHANE-KHINCHIN TYPE AND SECTIONS OF L_p-BALLS.

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Abstract. We extend Kahane-Khinchin type inequalities to the case $p > -2$. As an application we verify the slicing problem for the unit balls of finite-dimensional spaces that embed in L_p, $p > -2$.

1. Introduction.

A simple version of the Kahane-Khinchin inequality states that for a convex origin symmetric body K in $\mathbb{R}^n$ with $\mathrm{vol}(K) = 1$ and $p > q > 0$, we have for all $\xi \in \mathbb{R}^n$

$$\left(\int_K |(x, \xi)|^p dx \right)^{\frac{1}{p}} \leq C(p, q) \left(\int_K |(x, \xi)|^q dx \right)^{\frac{1}{q}},$$

where $C(p, q)$ depends only on p and q, see e.g. [MP].

Latała [La] extended this result to the case $q = 0$, and later Guédon [G] showed that this inequality holds for $q > -1$. These results were extended to the quasi-convex case by Litvak [Li]. In this article we extend Kahane-Khinchin's inequality further to $q > -2$ and as an application we prove the slicing problem for the unit balls of spaces that embed in L_p, $p > -2$.

Recall that an origin-symmetric convex body (compact set with non-empty interior) $K \subset \mathbb{R}^n$ is called *isotropic* with *constant of isotropy* L_K if $\mathrm{vol}_n(K) = 1$ and

$$\int_K (x, \theta)^2 dx = L_K^2, \qquad \text{for all } \theta \in S^{n-1}.$$

For every convex origin-symmetric body K there exists a linear isomorphism T of $\mathbb{R}^n$, such that TK is isotropic, and we define the constant of isotropy of K by $L_K = L_{TK}$, see [MP] for more details.

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Recall that the slicing problem asks the following question. Does there exist a universal constant C such that, for every convex origin symmetric body K in any dimension, we have $L_K < C$?

An equivalent formulation of this problem (see [MP]) is whether there exists a universal constant C_1 such that for every origin symmetric convex body in $\mathbb{R}^n$ the following inequality holds

$$(\mathrm{vol}(K))^{(n-1)/n} \le C_1 \max_{\xi \in S^{n-1}} \mathrm{vol}(K \cap \xi^\perp), \tag{1}$$

where $\xi^\perp$ is the central hyperplane orthogonal to ξ, and S^{n-1} is the unit sphere in $\mathbb{R}^n$. In other words, does there exist a universal constant such that every convex origin symmetric body of volume one has a hyperplane section of volume greater than this universal constant?

The problem still remains open. Bourgain [Bo1] proved that $L_K \le O(n^{1/4}\log n)$, and very recently Klartag [Kl2] removed the logarithmic term in this estimate. However there are many classes of bodies for which the slicing problem holds true with a constant independent of the dimension (see e.g [Ba], [BKM], [KMP], [MP]). In particular the slicing problem is solved for the unit balls of subspaces of L_1 (and hence for subspaces of L_p, $1 \le p \le 2$) by Ball [Ba]. For the unit balls of subspaces of quotients of L_p, $1 < p < \infty$, the problem is solved by Junge [J]. In Junge's proof the bounds blow up as p approaches either 1 or ∞. E. Milman [M1] gave a simple proof of these results for the unit balls of subspaces of L_p, $0 \le p < \infty$, and for quotients of L_p, $1 < p \le \infty$. Klartag and E. Milman [KlM] showed that the isotropic constant for subspaces of quotients of L_p, $1 < p \le 2$, is bounded from above by $O(1/\sqrt{p-1})$ thus improving Junge's estimate which was of order $1/(p-1)$. We also note that for subspaces of L_p, $2 \le p < \infty$, E. Milman gave two different proofs of the fact that $L_K < O(\sqrt{p})$. Here we present yet another proof of this result, which is somewhat similar to one of E. Milman's proofs, but uses the Lewis position instead of the isotropic position.

Since the latter bound blows up as $p \to \infty$, we try a different approach, considering negative values of p. The concept of embedding in L_{-p} with $0 < p < n$ was introduced in [Ko3], and it was proved that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p} if and only if the Fourier transform of $\|\cdot\|^{-p}$ is a positive distribution in $\mathbb{R}^n$. We will call unit balls of such spaces *p-intersection bodies* or *L_{-p}-balls.* For example, L_{-1}-balls are intersection bodies and L_{-k}-balls are k-intersection bodies, see [Ko4].

We would like to know whether the statement of the slicing problem is true for L_p-balls with p negative. Of course, if one could show this for $p \in (-n, -n+3]$, then one would solve the slicing problem completely, since for any convex body $K \in \mathbb{R}^n$, the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p for such values of p, see [Ko5, Section 4.2]. In this paper we employ Kahane-Khinchin type inequalities, discussed above, to show that the slicing problem is true for L_p-balls, $p > -2$.

For other results on the slicing problem we refer the reader to [Bo2], [D], [Kl1], [MP], [P].

2. Subspaces of L_p with $p > 2$.

In this section we give a different proof of the result $L_K < O(\sqrt{p})$ mentioned in the introduction. Note that if $0 \le p \le 2$ then the unit ball of a finite-dimensional subspace of L_p is an intersection body (see [Ko3] for $0 < p \le 2$ and [KKYY] for $p = 0$), and the slicing problem for such bodies follows from the positive part of the Busemann-Petty problem. This problem asks the following question. Let K and L be two origin-symmetric convex bodies in $\mathbb{R}^n$, such that $\mathrm{vol}_{n-1}(K \cap H) \le \mathrm{vol}_{n-1}(L \cap H)$ for every central hyperplane H. Is it true that $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$? The connection between intersection bodies and the Busemann-Petty problem was found by Lutwak [Lu]. The answer to the problem is affirmative if K is an intersection body and L is any origin symmetric star body. Hence, in order to prove the slicing problem for intersection bodies it is enough to take L to be the Euclidean ball of the same volume as K, see [MP, Proposition 5.5].

In view of the previous remarks it is enough to consider $p > 2$.

Let K be a convex origin-symmetric body in $\mathbb{R}^n$, denote by

$$\|x\|_K = \min\{a > 0 : x \in aK\}$$

the norm on $\mathbb{R}^n$ generated by K.

Theorem 2.1. *Let $p > 2$, there exists a constant $C(p)$ depending only on p such that $L_K \le C(p)$ for the unit ball K of any finite-dimensional subspace of L_p. Moreover, $C(p) = O(\sqrt{p})$, as $p \to \infty$.*

Proof. According to a theorem of Lewis [Le] (we formulate it in the form given in [LYZ, Theorem 8.2]), if $(\mathbb{R}^n, \|\cdot\|_K)$ is a subspace of L_p, $p \ge 1$, then there exist a position of the body K (which will again be denoted by K and will be called Lewis' position) and a finite Borel measure μ on S^{n-1} such that for all $x \in \mathbb{R}^n$

$$\|x\|_K^p = \int_{S^{n-1}} |(x,u)|^p \, d\mu(u), \tag{2}$$

and

$$|x|^2 = \int_{S^{n-1}} |(x,u)|^2 \, d\mu(u). \tag{3}$$

On the other hand, for any body K one has (see [MP, Section 1.6])

$$L_K^2 \le \frac{1}{n(\mathrm{vol}(K))^{1+2/n}} \int_K |x|^2 \, dx. \tag{4}$$

Using formula (3), applying Hölder's inequality twice and then using formula (2) we get

$$\int_K |x|^2 \, dx = \int_K \int_{S^{n-1}} |(x,u)|^2 \, d\mu(u) \, dx$$

$$\le (\mathrm{vol}(K))^{1-2/p} \int_{S^{n-1}} \left(\int_K |(x,u)|^p \, dx \right)^{2/p} d\mu(u)$$

$$\le (\mathrm{vol}(K))^{1-2/p} \left(\int_{S^{n-1}} \int_K |(x,u)|^p \, dx \, d\mu(u) \right)^{2/p} \left(\int_{S^{n-1}} d\mu(u) \right)^{1-2/p}$$

$$= (\mathrm{vol}(K))^{1-2/p} \left(\int_K \|x\|_K^p \, dx \right)^{2/p} \left(\int_{S^{n-1}} d\mu(u) \right)^{1-2/p}.$$

Passing to polar coordinates one can easily check that

$$\int_K \|x\|_K^p \, dx = \frac{n}{n+p} \mathrm{vol}(K),$$

therefore the previous computations combined with inequality (4) yield

$$\begin{aligned} L_K^2 &\le \frac{1}{n} (\mathrm{vol}(K))^{-2/n} \Big(\frac{n}{n+p} \Big)^{2/p} \left(\int_{S^{n-1}} d\mu(u) \right)^{1-2/p} \\ &\le \frac{1}{n} (\mathrm{vol}(K))^{-2/n} \left(\int_{S^{n-1}} d\mu(u) \right)^{1-2/p}. \end{aligned} \tag{5}$$

Let us estimate from below the volume of the body K. Let σ be the normalized Haar measure on the sphere.

$$\int_{S^{n-1}} \|x\|_K^p \, d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} |(x,u)|^p \, d\mu(u) \, d\sigma(x)$$

$$= \int_{S^{n-1}} |x_1|^p \, d\sigma(x) \cdot \int_{S^{n-1}} d\mu(u) \le \left(\frac{Cp}{n+p} \right)^{p/2} \int_{S^{n-1}} d\mu(u),$$

where C is an absolute constant. The latter estimate follows, for example, from [Ko5, Lemma 3.12] and Stirling's formula.

We get

$$\frac{Cp}{n+p} \left(\int_{S^{n-1}} d\mu(u) \right)^{2/p} \ge \Big(\int_{S^{n-1}} \|x\|_K^p \, d\sigma(x) \Big)^{2/p} \ge$$

$$\ge \Big(\int_{S^{n-1}} \|x\|_K^{-n} \, d\sigma(x) \Big)^{-2/n} = \big(\mathrm{vol}(K)/\mathrm{vol}(B_2^n) \big)^{-2/n} \sim \frac{1}{n} (\mathrm{vol}(K))^{-2/n},$$

since $\mathrm{vol}(B_2^n)^{1/n} \sim n^{-1/2}$, meaning that $\mathrm{vol}(B_2^n)^{1/n} n^{1/2}$ approaches a non-zero constant, as $n \to \infty$; see e.g. [Ko5, Corollary 2.20] and apply Stirling's formula.

Therefore inequality (5) implies

$$L_K^2 \le \frac{Cp}{n+p} \int_{S^{n-1}} d\mu(u), \tag{6}$$

where C is an absolute constant (possibly different from the one used above).

Finally let us compute the measure of S^{n-1} with respect to μ. Integrating equation (3) with respect to σ we get

$$1 = \int_{S^{n-1}} |x|^2 \ d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} (x,u)^2 \ d\mu(u) \ d\sigma(x)$$
$$= \int_{S^{n-1}} |x_1|^2 \ d\sigma(x) \cdot \int_{S^{n-1}} d\mu(u) = \frac{1}{n} \int_{S^{n-1}} d\mu(u).$$

This equality together with (6) implies

$$L_K \leq C\sqrt{p}.$$

□

3. Subspaces of L_p with $p < 0$.

First let us give some preliminary definitions and results to introduce the reader into the subject of Fourier analysis of distributions, which will be the main tool of this section.

Let ϕ be a function from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. We define the Fourier transform of ϕ by

$$\hat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x) e^{-i(x,\xi)} dx, \quad \xi \in \mathbb{R}^n.$$

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function $\phi \in \mathcal{S}$.

We say that a distribution is *positive definite* if its Fourier transform is a positive distribution, in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function ϕ.

Let f be an infinitely differentiable function on the sphere S^{n-1}, extend it to $\mathbb{R}^n \setminus \{0\}$ as a homogeneous function of degree $-k$, $0 < k < n$. Then the Fourier transform of the homogeneous extension is an infinitely differentiable function on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree $-n+k$. (See for example [Ko5, Section 3.3]).

We will need the following version of Parseval's formula on the sphere proved in [Ko2].

Lemma 3.1. *If K and L are origin symmetric infinitely smooth convex bodies in $\mathbb{R}^n$ and $0 < p < n$, then $(\|x\|_K^{-p})^\wedge$ and $(\|x\|_L^{-n+p})^\wedge$ are continuous functions on S^{n-1} and*

$$\int_{S^{n-1}} \left(||x||_K^{-p}\right)^\wedge (\xi) \left(||x||_L^{-n+p}\right)^\wedge (\xi) d\xi = (2\pi)^n \int_{S^{n-1}} ||x||_K^{-p} ||x||_L^{-n+p} dx.$$

A well-known result of P. Lévy, see for example [Ko5, Section 6.1], is that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p > 0$ if and only if there exists a finite Borel measure μ on the unit sphere so that, for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu(\xi). \tag{7}$$

If p is not an even integer, this condition is equivalent to the fact that $(\Gamma(-p/2)\|x\|^p)^\wedge$ is a positive distribution outside of the origin, see [Ko5, Theorem 6.10].

The concept of embedding in L_{-p} with $0<p<n$ was introduced in [Ko3] by extending formula (7) analytically to negative values of p. It was also proved that, as for positive p, there is a Fourier analytic characterization for such embeddings, namely a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p} if and only if the Fourier transform of $\|\cdot\|^{-p}$ is a positive distribution in $\mathbb{R}^n$. We will call unit balls of such spaces *p-intersection bodies* or *L_{-p}-balls.*

Lemma 3.2. *Let K be an infinitely smooth origin symmetric convex body in $\mathbb{R}^n$. If K is a p-intersection body, $0<p<n$, then*

$$(\mathrm{vol}(K))^{(n-p)/n} \le C(n,p) \max_{\xi\in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi),$$

where

$$C(n,p) = \frac{\Gamma(\frac{n-p}{2})}{2^p\pi^{n/2} n^{(n-p)/n}\Gamma(\frac{p}{2})}|S^{n-1}|^{(n-p)/n}.$$

Proof. Using the formula for the volume in polar coordinates and Parseval's formula

$$\mathrm{vol}(K) = \frac{1}{n}\int_{S^{n-1}} \|x\|_K^{-n} dx = \frac{1}{n}\int_{S^{n-1}} \|x\|_K^{-p}\|x\|_K^{-n+p} dx$$
$$= \frac{1}{(2\pi)^n n}\int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\xi)(\|x\|_K^{-n+p})^\wedge(\xi) d\xi.$$

If K is a $p-$intersection body, then $(\|x\|_K^{-p})^\wedge(\xi)\ge 0$, therefore

$$\mathrm{vol}(K) \le \frac{1}{(2\pi)^n n}\int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\xi) d\xi \cdot \max_{\xi\in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi).$$

Using that (see [GS, p.192]):

$$(|x|_2^{-n+p})^\wedge(\xi) = 2^p\pi^{n/2}\frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{n-p}{2})}|\xi|_2^{-p},$$

and applying Parseval's formula again, we get

$$\mathrm{vol}(K) \le \frac{2^{-p}\pi^{-n/2}}{(2\pi)^n n}\frac{\Gamma(\frac{n-p}{2})}{\Gamma(\frac{p}{2})}\int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\xi)(|x|_2^{-n+p})^\wedge(\xi) d\xi \times$$
$$\times \max_{\xi\in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi)$$
$$= \frac{2^{-p}\pi^{-n/2}}{n}\frac{\Gamma(\frac{n-p}{2})}{\Gamma(\frac{p}{2})}\int_{S^{n-1}} \|x\|_K^{-p} dx \cdot \max_{\xi\in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi)$$
$$\le \frac{2^{-p}\pi^{-n/2}}{n}\frac{\Gamma(\frac{n-p}{2})}{\Gamma(\frac{p}{2})}\left(\int_{S^{n-1}} \|x\|_K^{-n} dx\right)^{p/n} \cdot |S^{n-1}|^{(n-p)/n} \cdot \max_{\xi\in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi)$$

$$= C(n,p)(\mathrm{vol}(K))^{p/n} \cdot \max_{\xi \in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi).$$

$\square$

From the result of Lemma 3.2 it follows that one can obtain inequalities of type (1) by finding a good upper estimate for $(\|x\|_K^{-n+p})^\wedge(\xi)$ in terms of the central section. Our next Lemma gives an answer to this question for certain values of p. The proof of the Lemma will be given in Section 5.

Lemma 3.3. *Let K be an origin symmetric convex infinitely smooth body in $\mathbb{R}^n$. Then*

i) for $p \in (0,1)$ we have

$$\left(\|x\|_K^{-n+p}\right)^\wedge(\xi) \le \frac{2^{p-1}\pi(n-p)}{\Gamma(2-p)\sin(\pi p/2)} \left(\mathrm{vol}_{n-1}(K \cap \xi^\perp)\right)^p (\mathrm{vol}(K))^{1-p},$$

ii) for $p \in (1,2)$ we have

$$\left(\|x\|_K^{-n+p}\right)^\wedge(\xi) \le \frac{2^{p-1}\pi(n-p)}{\sin(\pi p/2)} \left(\mathrm{vol}_{n-1}(K \cap \xi^\perp)\right)^p (\mathrm{vol}(K))^{1-p}.$$

We remark that these inequalities become equalities in the case $p = 1$, since $(\|x\|_K^{-n+1})^\wedge(\xi) = \pi(n-1)\mathrm{vol}_{n-1}(K \cap \xi^\perp)$, see e.g. [Ko5, Theorem 3.8].

Now we are ready to state our main result.

Theorem 3.4. *Let $0 < p < 2$, if K is a convex p-intersection body, then*

$$(\mathrm{vol}(K))^{(n-1)/n} \le C(p) \max_{\xi \in S^{n-1}} \mathrm{vol}_{n-1}(K \cap \xi^\perp),$$

where

$$C(p) = \begin{cases} \left(\frac{\pi^{1-p/2}}{\Gamma(p/2)\Gamma(2-p)\sin(\pi p/2)}\right)^{1/p}, & \text{if } 0 < p \le 1, \\ \left(\frac{\pi^{1-p/2}}{\Gamma(p/2)\sin(\pi p/2)}\right)^{1/p}, & \text{if } 1 < p < 2. \end{cases}$$

Proof. For infinitely smooth bodies the theorem is a consequence of Lemma 3.2, Lemma 3.3 and Lemma 7.1 from the Appendix. For non-smooth bodies the theorem follows from the fact that every L_p-ball can be approximated in the radial metric by infinitely smooth L_p-balls, see [M2, Lemma 3.11]. $\square$

Remark. As remarked by E. Milman [M1, Remark 4.3], a uniform bound on the isotropic constant for subspaces of L_q with $-1+\epsilon < q \le 0$ follows from his argument and Guédon's extension of the Kahane-Khinchin inequality to the case $q > -1$. The novelty of the previous theorem for L_q with $-1 < q < 0$ is that the bound does not blow up as q approaches -1. (p from the previous theorem and q are related by $p = -q$). Unfortunately, the bound does blow up as p tends to 2.

Next sections will be devoted to the proof of Lemma 3.3.

4. Reduction to the section function.

Let K be an infinitely smooth origin symmetric convex body. For $\xi \in S^{n-1}$, consider the parallel section function $A_{K,\xi}$ on $\mathbb{R}$ defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}\left(K \cap \{(x,\xi) = t\}\right).$$

The fractional derivative of $A_{K,\xi}$ of order q at zero is defined as the action of the distribution $t_+^{-1-q}/\Gamma(-q)$ on this function, where $t_+ = \max\{t, 0\}$. That is

$$A^{(q)}_{K,\xi}(0) = \left\langle \frac{1}{\Gamma(-q)}\ t_+^{-1-q}, A_{K,\xi}(t) \right\rangle.$$

In particular, see [GKS], it follows that for $0 < p < 1$

$$A^{(-1+p)}_{K,\xi}(0) = \frac{1}{\Gamma(1-p)} \int_0^\infty t^{-p} A_{K,\xi}(t) dt \tag{8}$$

and for $1 < p < 2$

$$A^{(-1+p)}_{K,\xi}(0) = \frac{1}{\Gamma(1-p)} \int_0^\infty t^{-p} (A_{K,\xi}(t) - A_{K,\xi}(0)) dt. \tag{9}$$

Also note

$$\mathrm{vol}_{n-1}(K \cap \xi^\perp) = A_{K,\xi}(0) = \lim_{\epsilon \to 0^+} \frac{1}{\Gamma(\epsilon)} \int_0^\infty t^{-1+\epsilon} A_{K,\xi}(t) dt = \lim_{\epsilon \to 0^+} A^{(-\epsilon)}_{K,\xi}(0).$$

It was shown in [GKS] that if K has an infinitely smooth boundary then the fractional derivatives of the function $A_{K,\xi}$ can be computed in terms of the Fourier transform of the Minkowski functional raised to certain powers. Namely, for $p > 0$, $p \neq n$ we have

$$A^{(-1+p)}_{K,\xi}(0) = \frac{\sin(\pi p/2)}{\pi(n-p)} \left(\|x\|_K^{-n+p} \right)^\wedge (\xi). \tag{10}$$

Therefore the inequalities from Lemma 3.3 can now be written as follows:

$$A^{(-1+p)}_{K,\xi}(0) \leq C(p) \left(\mathrm{vol}(K)\right)^{(1-p)} \left(A_{K,\xi}(0)\right)^p,$$

for an appropriate constant $C(p)$. Or equivalently (if we assume for simplicity that $\mathrm{vol}(K) = 1$)

$$\left\langle \frac{1}{\Gamma(1-p)}\ t_+^{-p}, A_{K,\xi}(t) \right\rangle^{1/p} \leq c(p) \lim_{\epsilon \to 0} \left\langle \frac{1}{\Gamma(\epsilon)}\ t_+^{-1+\epsilon}, A_{K,\xi}(t) \right\rangle. \tag{11}$$

5. Kahane-Khinchin type inequalities.

Assume that $\mathrm{vol}(K) = 1$ (K is not necessarily convex) and $0 < p < q$. Then for all $\xi \in S^{n-1}$ the following holds by virtue of Hölder's inequality.

$$\left(\int_K |(x,\xi)|^p dx \right)^{\frac{1}{p}} \leq \left(\int_K |(x,\xi)|^q dx \right)^{\frac{1}{q}}.$$

However, if K is convex and origin-symmetric, then this inequality can be reversed. Namely, there is a constant $C(p,q)$ depending on p and q only, such that

$$\left(\int_K |(x,\xi)|^q dx\right)^{\frac{1}{q}} \le C(q,p)\left(\int_K |(x,\xi)|^p dx\right)^{\frac{1}{p}}, \quad \forall \xi \in S^{n-1}.$$

The latter is called the Kahane-Khinchin inequality for linear functionals; see [Ka], [Bor], [MP].

Note that

$$\int_K |(x,\xi)|^q dx = \int_{\mathbb{R}^n} |(x,\xi)|^q \chi_K(x) dx = \int_{\mathbb{R}} |t|^q \int_{(x,\xi)=t} \chi_K(x) dx\ dt$$
$$= \int_{\mathbb{R}} |t|^q A_{K,\xi}(t)\ dt = 2\left\langle t_+^q, A_{K,\xi}(t)\right\rangle.$$

Therefore the Kahane-Khinchin inequality can be written as

$$\left\langle \frac{1}{\Gamma(q)} t_+^q, A_{K,\xi}(t)\right\rangle^{1/q} \le \tilde{c}(p,q)\left\langle \frac{1}{\Gamma(p)} t_+^p, A_{K,\xi}(t)\right\rangle^{1/p},$$

which resembles Lemma 3.3 in the form of inequality (11). Hence in order to prove Lemma 3.3, we need to extend the Kahane-Khinchin inequality to negative values of p and q.

Proof of Lemma 3.3: case $0 < p < 1$.

From [MP, p.76] it follows that

$$F(q) = \left((q+1)\int_0^\infty t^q \frac{A_{K,\xi}(t)}{A_{K,\xi}(0)} dt\right)^{1/(1+q)}$$

is an increasing function of q on $(-1,\infty)$.

Therefore, taking $q = -p$ with $0 < p < 1$ and using $F(-p) \le F(0)$ we get

$$\left((1-p)\int_0^\infty t^{-p} \frac{A_{K,\xi}(t)}{A_{K,\xi}(0)} dt\right)^{1/(1-p)} \le \int_0^\infty \frac{A_{K,\xi}(t)}{A_{K,\xi}(0)} dt = \frac{\mathrm{vol}(K)}{2\ A_{K,\xi}(0)}.$$

Using formulas (10), (8) and applying the previous inequality , we get

$$\left(\|x\|_K^{-n+p}\right)^\wedge(\xi) = \frac{\pi(n-p)}{\sin(\pi p/2)} A_{K,\xi}^{(-1+p)}(0)$$
$$= \frac{\pi(n-p)}{\Gamma(1-p)\sin(\pi p/2)} \int_0^\infty t^{-p} A_{K,\xi}(t) dt$$
$$\le \frac{2^{p-1}\pi(n-p)}{(1-p)\Gamma(1-p)\sin(\pi p/2)} \left(\mathrm{vol}(K)\right)^{(1-p)} \left(A_{K,\xi}(0)\right)^p$$
$$= \frac{2^{p-1}\pi(n-p)}{\Gamma(2-p)\sin(\pi p/2)} \left(\mathrm{vol}(K)\right)^{(1-p)} \left(A_{K,\xi}(0)\right)^p.$$

□

Proof of Lemma 3.3: case $1 < p < 2$.

What follows is similar to [MP, Section 2.6]. Consider the function

$$G(p) = \left(\frac{\int_0^\infty t^{-p} \frac{A_{K,\xi}(0) - A_{K,\xi}(t)}{A_{K,\xi}(0)} dt}{\int_0^\infty t^{-p}(1-e^{-t}) dt} \right)^{\frac{1}{1-p}}.$$

We want to show that it is increasing on $(1,2)$.

Let $\Phi(t) = \log A_{K,\xi}(0) - \log A_{K,\xi}(t)$. By Brunn's theorem (see e.g. [Ko5, Theorem 2.3]), $\Phi(t) \geq 0$ and it is increasing and convex on the support of $A_{K,\xi}(t)$. Now

$$G(p) = \left(\frac{\int_0^\infty t^{-p}(1-e^{-\Phi(t)}) dt}{\int_0^\infty t^{-p}(1-e^{-t}) dt} \right)^{\frac{1}{1-p}}.$$

Let $\alpha = 1/G(p)$, then it is not hard to check that

$$\int_0^\infty t^{-p}(1-e^{-\alpha t}) dt = \int_0^\infty t^{-p}(1-e^{-\Phi(t)}) dt.$$

Consider the function

$$H(x) = \int_x^\infty t^{-p}(e^{-\Phi(t)} - e^{-\alpha t}) dt.$$

We want to show that $H(x) \leq 0$ for $x \in [0,\infty)$. Since $H(0) = H(\infty) = 0$, it suffices to show that $H(x)$ is first decreasing and then increasing.

Indeed,

$$H'(x) = -x^{-p}(e^{-\Phi(x)} - e^{-\alpha x}).$$

Since $\Phi(x)$ is increasing and convex, there is a point x_0, such that $\Phi(x) \leq \alpha x$ for $0 < x < x_0$ and $\Phi(x) \geq \alpha x$ for $x > x_0$. Therefore $H'(x) \leq 0$ if $0 < x < x_0$ and $H'(x) \geq 0$ if $x > x_0$. So, we have proved that $H(x) \leq 0$, which means that for every $x > 0$

$$\int_x^\infty t^{-p}(1-e^{-\Phi(t)}) dt \geq \int_x^\infty t^{-p}(1-e^{-\alpha t}) dt.$$

Now let $1 < q < p < 2$, we have

$$\int_0^\infty t^{-q}(1-e^{-\Phi(t)}) dt = (p-q) \int_0^\infty x^{p-q-1} \int_x^\infty t^{-p}(1-e^{-\Phi(t)}) dt$$

$$\geq (p-q) \int_0^\infty x^{p-q-1} \int_x^\infty t^{-p}(1-e^{-\Phi(t)}) dt = \int_0^\infty t^{-q}(1-e^{-\alpha t}) dt$$

$$= \alpha^{q-1} \int_0^\infty t^{-q}(1-e^{-t}) dt.$$

Therefore, using the definition of α, we get

$$\frac{\int_0^\infty t^{-q}(1-e^{-\Phi(t)}) dt}{\int_0^\infty t^{-q}(1-e^{-t}) dt} \geq G(p)^{1-q}$$

or

$$G(q) \leq G(p).$$

So, $G(p)$ is increasing on $(1,2)$.

If we extend the function $G(p)$ to $p \in (0,1)$ by the formula

$$G(p) = \left(\frac{\int_0^\infty t^{-p} e^{-\Phi(t)} dt}{\int_0^\infty t^{-p} e^{-t} dt} \right)^{\frac{1}{1-p}},$$

then according to [MP, p.81], this function is increasing on $(0,1)$.

Note that on both intervals $(0,1)$ and $(1,2)$ the function can be written as

$$G(p) = \left(\frac{A_{K,\xi}^{(-1+p)}(0)}{A_{K,\xi}(0)} \right)^{\frac{1}{1-p}}.$$

Moreover, since $A_{K,\xi}^{(-1+p)}(0)$ is an analytic function of $p \in \mathbb{C}$ (see [Ko5, p.37]), we have

$$\lim_{p\to 1^+} G(p) = \lim_{p\to 1^-} G(p) = \exp\left(-\frac{\frac{d}{dp} A_{K,\xi}^{(-1+p)}(0)|_{p=0}}{A_{K,\xi}(0)} \right).$$

Consequently, for $p \in (1,2)$ we get

$$G(p) \geq G(0),$$

and therefore

$$\left(\frac{\int_0^\infty t^{-p} \frac{A_{K,\xi}(0)-A_{K,\xi}(t)}{A_{K,\xi}(0)} dt}{\int_0^\infty t^{-p}(1-e^{-t}) dt} \right)^{\frac{1}{1-p}} \geq \frac{\mathrm{vol}(K)}{2A_{K,\xi}(0)},$$

or

$$\frac{1}{\Gamma(1-p)} \int_0^\infty t^{-p} \frac{A_{K,\xi}(t) - A_{K,\xi}(0)}{A_{K,\xi}(0)} dt \leq \left(\frac{\mathrm{vol}(K)}{2A_{K,\xi}(0)} \right)^{1-p}.$$

Using formulas (10), (9) and applying the previous inequality , we get

$$\begin{aligned} \left(\|x\|_K^{-n+p} \right)^\wedge (\xi) &= \frac{\pi(n-p)}{\sin(\pi p/2)} A_{K,\xi}^{(-1+p)}(0) \\ &= \frac{\pi(n-p)}{\sin(\pi p/2)\Gamma(1-p)} \int_0^\infty t^{-p} (A_{K,\xi}(t) - A_{K,\xi}(0)) dt \\ &\leq \frac{2^{p-1}\pi(n-p)}{\sin(\pi p/2)} A_{K,\xi}(0)^p \left(\mathrm{vol}(K)\right)^{1-p}. \end{aligned}$$

□

6. Higher order derivatives.

In this section we show that inequalities similar to those from Lemma 3.3 exist for bigger values of p, however in this case we need to pay the price of averaging the Fourier transform over a sphere.

Let $H \in G(n, n-2)$ and let ξ_1, ξ_2 be an orthonormal basis in $H^\perp$. Define:

$$A_{K,H}(u) = \mathrm{vol}_{n-2}(K \cap \{H + u_1\xi_1 + u_2\xi_2\}), \qquad u \in \mathbb{R}^2.$$

Lemma 6.1. *Let K be an origin symmetric infinitely smooth convex body in the isotropic position. Then for $q \in (0,1)$ we have*

$$\int_{S^{n-1}\cap H^\perp} \left(\|x\|^{-n+2+q}\right)^\wedge (\theta) d\theta \leq C(q) L_K^{-2-q},$$

where L_K is the constant of isotropy of K and $C(q)$ is a constant depending only on q.

Proof. From the proof of Theorem 2 in [Ko4] we know that

$$\int_{S^{n-1}\cap H^\perp} \left(\|x\|^{-n+2+q}\right)^\wedge (\theta) = C(q) \left\langle |u|^{-2-q}, A_{K,H}(u)\right\rangle$$
$$= C_1(q) \int_{\mathbb{R}^2} |u|^{-2-q}\Big(A_{K,H}(u) - A_{K,H}(0)\Big) du,$$

and passing to polar coordinates, we get

$$= C_1(q) \int_{S^{n-1}\cap H^\perp} \int_0^\infty r^{-1-q}\Big(A_{K,H}(r\theta) - A_{K,H}(0)\Big) dr d\theta.$$

Since $A_{K,H}$ is log-concave, we can apply a Kahane-Khinchin type inequality (part (ii) of Lemma 3.3) to the inner integral.

$$\int_0^\infty r^{-1-q}\Big(A_{K,H}(0) - A_{K,H}(r\theta)\Big) dr$$
$$\leq C_2(q)(A_{K,H}(0))^{1+q} \mathrm{vol}_{n-1}(K \cap \ \mathrm{span}\{H,\theta\})^{-q}.$$

Since for isotropic bodies central sections of codimension 1 and 2 are equivalent to L_K^{-1} and L_K^{-2} correspondingly (see e.g. [MP, p.96]) we get

$$\int_0^\infty r^{-1-q}\Big(A_{K,H}(0) - A_{K,H}(r\theta)\Big) dr \leq C(q) L_K^{-2-2q} L_K^q = C(q) L_K^{-2-q}.$$

□

7. Appendix.

Here we prove a result used in one of the previous sections.

Lemma 7.1. *Let $0 \leq p < n$ and $C(n,p)$ as in Lemma 3.2, then*

$$C(n,p)\cdot(n-p) \leq \frac{2^{1-p}\ \pi^{-p/2}}{\Gamma(\frac{p}{2})}.$$

Proof. We need to show that

$$\frac{(n-p)}{n^{(n-p)/n}} \frac{\Gamma(\frac{n-p}{2})}{2\pi^{(n-p)/2}} |S^{n-1}|^{(n-p)/n} \leq 1.$$

The left-hand side is equal to

$$\frac{(n-p)}{n^{(n-p)/n}} \frac{\Gamma(\frac{n-p}{2})}{2\pi^{(n-p)/2}} \left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\right)^{(n-p)/n} = \frac{\Gamma(\frac{n-p}{2}+1)}{(\Gamma(n/2+1))^{(n-p)/n}}.$$

Since the function $\log(\Gamma(x))$ is convex [Ko5, p.30], we have

$$\frac{\log(\Gamma(n/2+1))-\log(\Gamma(1))}{n/2} \geq \frac{\log(\Gamma((n-p)/2+1))-\log(\Gamma(1))}{(n-p)/2},$$

therefore

$$(\Gamma(n/2+1))^{n/2} \geq (\Gamma((n-p)/2+1))^{(n-p)/2}.$$

$\square$

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