On a Uniqueness Problem in Discrete Tomography

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Abstract. Let A be a subset of the integer lattice $\mathbb{Z}^n$ contained inside a sphere S in $\mathbb{R}^n$. Suppose that for each $u \in S$ we know the number of points of A visible from u. Does this information determine the set A uniquely? In this note we answer this question affirmatively, provided the center of the sphere is not a rational point.

Geometric tomography deals with the retrieval of information about solid objects, typically convex bodies, using data from their sections or projections. The book *Geometric tomography* [**3**] by R. J. Gardner gives a comprehensive account of various results and problems in this area. The following is a celebrated theorem of Aleksandrov; see e.g., [**3**, p. 115]. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, such that

$$\mathrm{vol}_{n-1}(K|u^\perp) = \mathrm{vol}_{n-1}(L|u^\perp), \quad \text{for every } u \in S^{n-1}.$$

Then $K = L$. Here and below $K|u^\perp$ denotes the orthogonal projection of K onto the hyperplane $u^\perp = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$.

In recent years there has been a lot of interest in transferring results about convex bodies to discrete settings. Gardner, Gronchi, and Zong [**4**] suggested studying an analogue of Aleksandrov's theorem for subsets of the integer lattice. We say that a set A in $\mathbb{R}^n$ is a *lattice set* if $A \subset \mathbb{Z}^n$. A finite lattice set A is a *convex lattice set* if $A = \mathrm{conv}(A) \cap \mathbb{Z}^n$, where $\mathrm{conv}(A)$ is the convex hull of A. A set A is *origin-symmetric* if for every $x \in A$, $-x$ is also in A. Recall that an affine subspace is a translation of a linear subspace. A set A in $\mathbb{R}^n$ is *full-dimensional* if A is not contained in any proper affine subspace of $\mathbb{R}^n$. For a finite set we use $\#$ to denote its cardinality.

Problem 1. [**4**] *Let A and B be full-dimensional origin-symmetric convex lattice sets in $\mathbb{R}^n$, $n \geq 2$. If for every $u \in S^{n-1}$ we have*

$$\#(A|u^\perp) = \#(B|u^\perp),$$

is it necessarily true that $A = B$?

Gardner, Gronchi, and Zong showed in [**4**] that the answer to Problem 1 is negative if $n = 2$. The problem is still open for $n \geq 3$.

We propose to investigate the following closely related problem. Let $A \subset \mathbb{R}^n$ be a lattice set contained inside a sphere $S \subset \mathbb{R}^n$. We say that a point $v \in A$ is *visible* from $u \in S$ if the line segment connecting u and v does not contain any other points of A. Define $N_A(u)$ to be the number of points of A visible from $u \in S$. In other words,

$$N_A(u) = \#(\{\ell : \ell \text{ is a line such that } u \in \ell \text{ and } \ell \cap A \neq \emptyset\}).$$

Problem 2. *Let A and B be full-dimensional lattice sets in $\mathbb{R}^n$, $n \geq 2$. Assume that A and B are contained inside a sphere S. If*

$$N_A(u) = N_B(u)$$

for every $u \in S$, is it necessarily true that $A = B$?

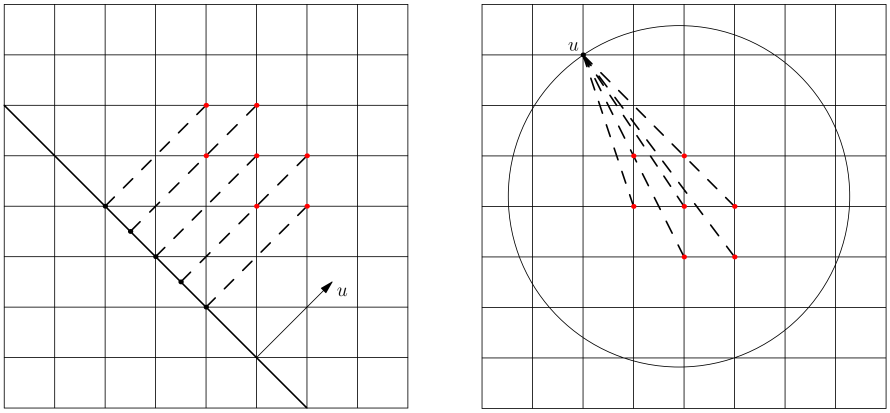

Figure 1. Tomographic data in Problem 1 (left) and Problem 2 (right).

Figure 1 illustrates the difference between the tomographic data in Problem 1 and Problem 2.

Remark 3. Observe that the answer to Problem 2 in dimension 2 is negative if we do not require the sets to be full-dimensional. Indeed, let $A = \{(0,0),(1,0)\}$, $B = \{(0,0),(2,0)\}$, and S be any circle containing the sets A and B in its interior. Let u_1 and u_2 be the points of intersection of S with the x-axis. Then $N_A(u) = N_B(u) = 2$ for all $u \in S$ different from u_1 and u_2, and $N_A(u_1) = N_B(u_1) = N_A(u_2) = N_B(u_2) = 1$; see Figure 2. This counterexample works in all dimensions. More generally, if $n \geq 3$ and at least one of the sets A or B is contained in an affine subspace H, then the other set is also contained in H and the problem is reduced to the lower-dimensional case of finite subsets of the lattice $\mathbb{Z}^n \cap H$ contained inside the sphere $S \cap H$.

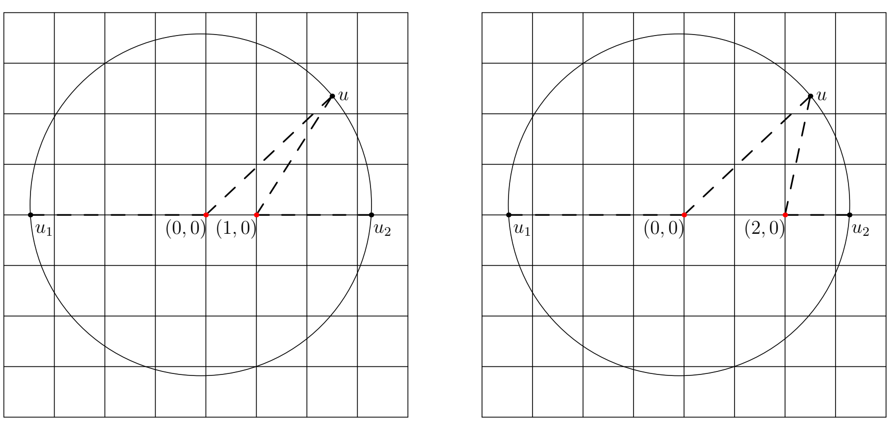

Figure 2. Sets A (left) and B (right) in Remark 3.

We say that a point in $\mathbb{R}^n$ is rational if all its coordinates are rational numbers. In this note we solve Problem 2 affirmatively in the case when the center of the sphere S is not a rational point. First we will need the following lemma.

Lemma 4. *Let S be a sphere in $\mathbb{R}^n$ whose center has at least one irrational coordinate. Then the set of all rational points on S is not full-dimensional.*

Proof. To reach a contradiction, let us assume that there are $n+1$ rational points $x_0, x_1, \ldots x_n$ on S that do not lie in any affine hyperplane. These points form a simplex inscribed in S. Thus the system of vectors $\{w_i\}_{i=1}^n$, where $w_i = x_i - x_0$, $i = 1, \ldots, n$, is linearly independent. Note that the vectors w_i have rational coordinates, and the midpoints $\bar{x}_i$ of the segments $[x_0, x_i]$ also have rational coordinates. Denote by Π_i, $i = 1, \ldots, n$, the hyperplane passing through $\bar{x}_i$ and orthogonal to w_i. Its equation is $\langle w_i, x\rangle = \langle w_i, \bar{x}_i\rangle$. Observe that the center of S is the point of intersection of the hyperplanes Π_i, $i = 1, \ldots, n$, which is the unique solution of the linear system $Wx = b$, where W is the matrix with rows w_i and b is the vector with components $\langle w_i, \bar{x}_i\rangle$. Since W is a matrix with rational entries, its inverse W^{-1} is also a matrix with rational entries. Therefore, the center of the sphere, given by $W^{-1}b$ is a rational point, which is a contradiction.

■

We will now present the solution of Problem 2 when the center of the sphere S is not a rational point. Our proof provides an algorithm for reconstructing a lattice set A from the knowledge of the function N_A. Note that, unlike in Problem 1, A is neither required to be a convex lattice set nor to be symmetric.

Theorem 5. *Let S be a sphere in $\mathbb{R}^n$ whose center is not a rational point. Any full-dimensional lattice set A contained inside the sphere S is uniquely determined by the values of $N_A(u)$, $u \in S$.*

Proof. Let $\mathcal{L}$ be the set of lines that pass through at least two lattice points inside S. Since $\mathcal{L}$ is finite, $S_{\mathcal{L}} = \{u \in S : u \in \ell \text{ for some } \ell \in \mathcal{L}\}$ is also finite. It is easy to see that $N_A(u) = \#(A)$ for all $u \in S \setminus S_{\mathcal{L}}$. This allows us to determine $\#(A)$ as the value attained by N_A on $S \setminus S_{\mathcal{L}}$, or alternatively as the maximum value of N_A on the sphere.

Let H be the affine hull (possibly empty) of all rational points of S. By Lemma 4, H is not full-dimensional. We claim that for every $u \in S_{\mathcal{L}} \setminus H$ there is a unique line $\ell \in \mathcal{L}$ containing u. Indeed, if u belongs to two distinct lines ℓ_1, $\ell_2 \in \mathcal{L}$, then u is rational as a point of intersection of two lines with rational coefficients, and therefore u must be in H.

Let $\mathcal{L}'$ be the set of those lines $\ell \in \mathcal{L}$, not contained in H, that pass through at least two points of A. Let us show how to determine whether a given line ℓ belongs to $\mathcal{L}'$. If $\ell \in \mathcal{L}$ is not contained in H, then at least one of the points of intersection of ℓ with S is not in H. Denote this point by u. Since $u \in S_{\mathcal{L}} \setminus H$, it follows that ℓ is the only line from $\mathcal{L}$ that contains u. Therefore the set of points of A visible from u consists of one point of $\ell \cap A$ together with all the points of A disjoint from ℓ, and thus, $N_A(u) = 1 + \#(A) - \#(\ell \cap A)$. We obtain the relation

$$\#(\ell \cap A) = \#(A) - N_A(u) + 1. \tag{1}$$

Note that formula (1) uniquely determines the set $\mathcal{L}'$. Indeed, if ℓ is not contained in H, then $\ell \in \mathcal{L}'$ if and only if the right-hand side of (1) is greater than 1. Moreover, by formula (1) we know how many points of A belong to each line $\ell \in \mathcal{L}'$.

Let y be a lattice point that lies inside S and not in the affine subspace H. To determine if $y \in A$, we will use the following procedure. Let $\mathcal{L}'_y = \{\ell \in \mathcal{L}' : y \in \ell\}$. If $y \in A$, then every point of $A \setminus \{y\}$ belongs to exactly one line from $\mathcal{L}'_y$, while y belongs to all lines from $\mathcal{L}'_y$. On the other hand, if $y \notin A$, then each point of A belongs

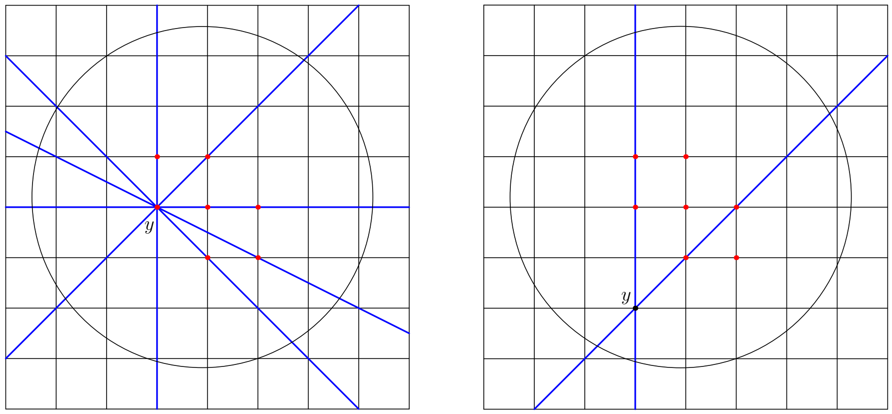

Figure 3. Formula (2) when $y \in A$ (left) and $y \notin A$ (right).

to at most one line from $\mathcal{L}'_y$; see Figure 3. Thus we have

$$\sum_{\ell \in \mathcal{L}'_y} \#(\ell \cap A) \begin{cases} = \#(A) - 1 + \#(\mathcal{L}'_y), & \text{if } y \in A, \\ \leq \#(A), & \text{if } y \notin A. \end{cases} \tag{2}$$

Since the set A is full-dimensional, we have $\#(L'_y) > 1$ for any $y \in A \setminus H$. Therefore, from formula (2) we get

$$\sum_{\ell \in \mathcal{L}'_y} \#(\ell \cap A) \begin{cases} > \#(A), & \text{if } y \in A, \\ \leq \#(A), & \text{if } y \notin A, \end{cases}$$

which allows us to determine whether a given lattice point, not contained in H, does or does not belong to A.

To identify the points of $A \cap H$, first observe that we already know $\#(A)$ and $\#(A \setminus H)$, and therefore we know $\#(A \cap H)$. Now we will obtain a formula similar to (2). Let $y \in H$ be a lattice point inside the sphere S. If $y \in A$, then every point of $A \setminus H$ belongs to exactly one line from $\mathcal{L}'_y$; no point of $A \cap H$, except for y, belongs to any line from $\mathcal{L}'_y$; and y belongs to all lines from $\mathcal{L}'_y$. On the other hand, if $y \notin A$, then each point of $A \setminus H$ belongs to at most one line from $\mathcal{L}'_y$; and no points of $A \cap H$ belong to lines from $\mathcal{L}'_y$. Therefore,

$$\sum_{\ell \in \mathcal{L}'_y} \#(\ell \cap A) \begin{cases} = \#(A) - \#(A \cap H) + \#(\mathcal{L}'_y), & \text{if } y \in A, \\ \leq \#(A) - \#(A \cap H), & \text{if } y \notin A. \end{cases}$$

Since A is full-dimensional, the number $\#(\mathcal{L}'_y)$ above is strictly positive when $y \in A \cap H$, and so we get

$$\sum_{\ell \in \mathcal{L}'_y} \#(\ell \cap A) \begin{cases} > \#(A) - \#(A \cap H), & \text{if } y \in A, \\ \leq \#(A) - \#(A \cap H), & \text{if } y \notin A, \end{cases}$$

which allows us to distinguish between the sets $A \cap H$ and $(\mathbb{Z}^n \setminus A) \cap H$.

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Remark 6. The proof above also works in the case when the center of S is rational, but S does not contain any rational points, e.g., $S = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 10\pi\}$. However, we do not know how to approach the situation when the set of rational points of S is full-dimensional, e.g., $S = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 100\}$. It would also be interesting to see what happens if we use other surfaces instead of spheres.

Remark 7. The algorithm for reconstructing the set A in the proof of Theorem 5 is purely theoretical. In practice one would face various obstacles. For example, knowing that the affine subspace H, introduced above, exists does not mean that we know how to find it. Similarly, verifying whether a given point lies inside the sphere S may be an issue from the practical point of view.

Remark 8. In this article we were interested in visible points only in the context of discrete tomography. However, there is a vast literature on other aspects of visibility. For example, one of the classical questions is what proportion of the integer lattice $\mathbb{Z}^n$ is visible from the origin. Interestingly, the answer is $1/\zeta(n)$, where ζ is the Riemann zeta function; see [**1**, Section 3.8], [**5**]. For recent generalizations of lattice-point visibility the reader is referred to [**2**] and the references contained therein.

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