7 Conjugate points and lines

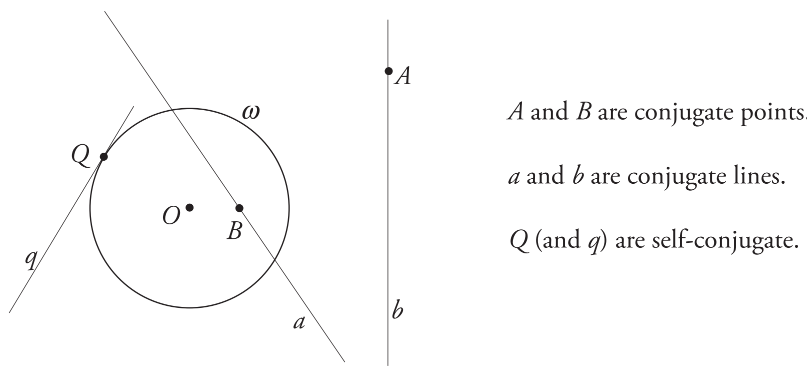
Let ω be a fixed circle with centre O.

Two points A and B are said to be ***conjugate points*** with respect to ω if each lie on the polar of the other (that is, A lies on b and B lies on a).

Two lines a and b are ***conjugate lines*** with respect to ω if each passes through the pole of the other.

A point or line which is conjugate to itself is said to be ***self-conjugate***.

Examples:

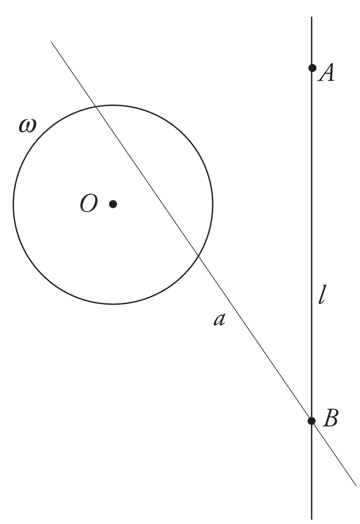


Remarks:

1. A and B are conjugate points $\iff$ a and b are conjugate lines.
2. B is conjugate to A $\iff$ B lies on a.
3. b conjugate to a $\iff$ b passes through A.
4. The set of lines conjugate to a is the pencil of lines through A.
5. The set of points conjugate to B is the range of points on b.
6. A is self-conjugate $\iff$ A is on ω $\iff$ a is tangent to ω.

Example 7.1. *Each point on a line has a conjugate point on that line.*

Proof. Let A be on l. Let $B = a \cap l$ (B is an ideal point if l and a are parallel). Then B is on a and, by the basic reciprocation theorem, A is on b. □



Example 7.2. *Each line through a point A has a conjugate line through A.*

Proof. This is the dual of the previous example.

An alternate direct proof is as follows:

Let b be a line through A. Then $\overleftrightarrow{BA}$ is a line conjugate to b. (Remember that the pencil of lines at B is the set of all lines conjugate to b.) □

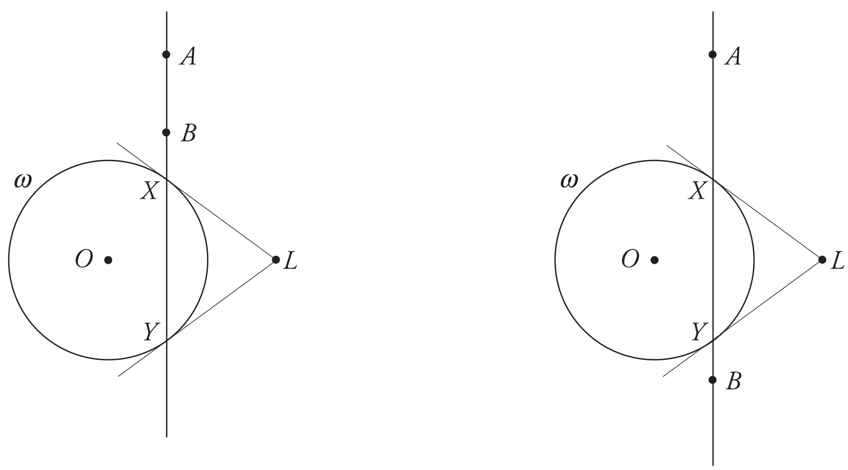
Example 7.3. *Of two distinct conjugate point on a line that cuts the circle of reciprocation, one point is inside or on the circle, the other point is outside the circle.*

Proof. Let ω be the circle of reciprocation, and suppose that A and B are conjugate points on the line l that cuts ω.

If A is inside ω then a misses ω and since B is on a, then B must be outside ω.

If A is on ω then a is tangent to ω at A, and since B is on a and is different than A, then B must be outside ω.

If A is outside ω then a cuts ω. Suppose for a contradiction that B is also outside ω. Let L be the pole of l. Since it is given that l cuts ω, then L is outside ω. The situation is as depicted in one of the two diagrams below.



Now, A on l implies L on a, and B conjugate to a implies B on a. Together these imply that $a = \overleftrightarrow{LB}$.

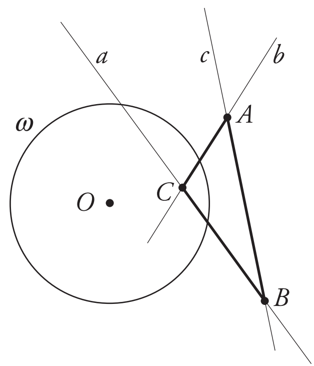
Let X and Y be the points of tangency from L to ω. Then B is outside the segment XY, and LB misses ω. But this contradicts the fact that a cuts ω. So we must conclude that B is on or inside ω. □

The following is the dual of the previous example. (You should try to prove it directly as well.)

Example 7.4. *Of two distinct conjugate lines that intersect outside a circle ω one cuts the circle (or is tangent to it) and the other misses the circle.*

Self-polar triangles

A triangle is ***self-polar*** if each vertex is the pole of the opposite side (sides considered as lines).



a, b, c are polars of *A, B, C,* respectively

Remarks: For a self-polar triangle

- Any two vertices are conjugate points.
- Any two sides are conjugate lines.
- Given any two conjugate points, $A \neq B$, they are the vertices of some self-polar triangle. The third vertex is $C = a \cap b$, that is $c = \overleftrightarrow{AB}$.

Theorem 7.5. *Every non-degenerate self-polar triangle is obtuse, with the obtuse angle inside the circle of reciprocation, ω.*

Proof. Let ABC be self-polar.

Then exactly one vertex must be inside ω. Here are the reasons:

(1) Suppose one vertex, say A, is inside ω. Then a misses ω. But both other vertices are on a, so B and C are outside ω. This shows that there is at most one vertex inside ω.

(2) It is impossible for A to be on ω, for in this case B and C would have to be on a, in which case all three of A, B, C would be on a. That is, ABC would be degenerate.

(3) Suppose A, B, C are all outside ω. Then a cuts ω, and B and C are on a. Then, by Example 7.3, one of B or C is inside ω, the other is outside.

This proves that exactly one vertex is inside ω. Supposing that A is inside ω, it remains to show that $\angle BAC$ is obtuse.

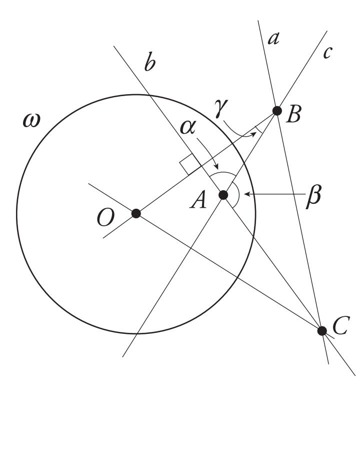
A is on b. Join OB, then b is the line through A perpendicular to OB. Note that C is on b and B is on c (because C and B are conjugates).

A is on c. Join OC, then c is the line through A perpendicular to OC.

Referring to the diagram, for angles α, β and γ we have

$$\beta = 180 - \alpha = 180 - (90 - \gamma) = 90 + \gamma,$$

showing that β is obtuse.

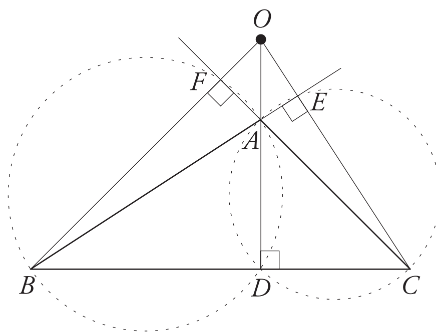


□

Theorem 7.6. *Every obtuse triangle ABC is self-polar with respect to a unique circle ω.*

(This circle is called the ***polar circle*** for the triangle.)

Proof. Let O be the orthocentre of $\triangle ABC$.



Referring to the diagram, $\angle AEC = 90° = \angle ADC$, so $\square AECD$ is cyclic. By the power of the point O with respect to the circle circumscribing $\square AECD$, we have

$$OA \cdot OD = OC \cdot OE\,. \tag{1}$$

Similarly $\square ADBF$ is cyclic, and by the power of the point O we get

$$OA \cdot OD = OF \cdot OB\,. \tag{2}$$

Let $OA \cdot OD = k^2$, so that $OC \cdot OE = k^2$ and $OF \cdot OB = k^2$ by (1) and (2). Let ω be the circle with centre O and radius k. Then ω is the polar circle for triangle ABC.

Note that this works because:

$OA \cdot OD = k^2$ means that $D = A'$, so $BC = a$ (with respect to ω),
$OC \cdot OE = k^2$ means that $E = C'$, so $AB = c$, and
$OF \cdot OB = k^2$ means that $F = B'$, so $AC = b$. □