5 Harmonic conjugates

Harmonic conjugates (Ogilvy p. 13–14, Eves p. 82–84)

If $A$ and $B$ are two points on a line, any pair of points $C$ and $D$ on the line for which

\[
\frac{AC}{CB} = \frac{AD}{DB}
\]

are said to divide $AB$ harmonically. The points $C$ and $D$ are then said to be harmonic conjugates with respect to $A$ and $B$.

Given ordinary points $A$ and $B$, and given a positive number $k$, $k \neq 1$, there are two ordinary points $C$ and $D$ such that $\frac{AC}{CB} = \frac{AD}{DB} = k$. One of the points $C$ and $D$ is between $A$ and $B$, the other is exterior to the segment $AB$. The midpoint $C$ of $AB$ satisfies $\frac{AC}{CB} = 1$, and we will adopt the convention that $\frac{AI}{IB} = 1$ (where $I$ is the ideal point in the inversive plane).

Using this convention, given two ordinary points $A$ and $B$, for every positive $k$ there are harmonic conjugates $C$ and $D$ for which

\[
\frac{AC}{CB} = \frac{AD}{DB} = k.
\]

**Theorem 5.1.** Given four ordinary points, $A$, $B$, $C$, and $D$, if $AB$ is divided harmonically by $C$ and $D$, then $CD$ is divided harmonically by $A$ and $B$.

The reason for the terminology is explained by the following:

**Theorem 5.2.** Suppose that $P$, $Q$, $R$, and $S$ are consecutive ordinary points on a line and that $QS$ divides $PR$ harmonically. Then the sequence of distances $PQ$, $PR$, $PS$ forms a harmonic progression.

**Proof.** The hypothesis says that

\[
\frac{RQ}{QP} = \frac{RS}{SP}.
\]

(1)

We want to show that $\frac{1}{PQ}$, $\frac{1}{PR}$, $\frac{1}{PS}$ are in arithmetic progression, that is, that

\[
\frac{1}{PQ} - \frac{1}{PR} = \frac{1}{PR} - \frac{1}{PS}.
\]

(2)

From (1) we get

\[
\frac{RQ}{QP \cdot PR} = \frac{RS}{SP \cdot PR}
\]

\[
\Rightarrow \frac{PR - PQ}{PQ \cdot PR} = \frac{PS - PR}{PR \cdot PS}
\]

\[
\Rightarrow \frac{1}{PQ} - \frac{1}{PR} = \frac{1}{PR} - \frac{1}{PS}.
\]

showing that (2) holds. □
The Circle of Apollonius (Ogilvy, p. 14-17)

If we are given points \( A \) and \( B \) and a positive number \( k \neq 1 \), we can find precisely two ordinary points \( X \) on the line \( AB \) such that \( \frac{AX}{XB} = k \). There are also points \( X \) not on the line \( AB \) for which \( \frac{AX}{XB} = k \).

**Theorem 5.3.** Given two ordinary points \( A \) and \( B \), and a positive number \( k \neq 1 \), the set of all points \( X \) in the plane for which \( \frac{AX}{XB} = k \) forms a circle.

**Remark:** The circle referred to in the theorem is called the **Circle of Apollonius for \( A \), \( B \), and \( k \).**

**Proof.** Let \( C \) and \( D \) be the two points on \( AB \) for which \( \frac{AC}{CB} = \frac{AD}{DB} = k \), and let \( \xi \) be the circle with diameter \( CD \).

There are two things to show:

(1) Every point \( X \) for which \( \frac{AX}{XB} = k \) is on \( \xi \).

(2) Every point \( X \) on \( \xi \) satisfies \( \frac{AX}{XB} = k \).

(1) Let \( X \) be a point such that \( \frac{AX}{XB} = k \). Since \( \frac{AC}{CB} = k \) and \( \frac{AD}{DB} = k \), we know from the Angle Bisector Theorem that \( XC \) and \( XD \) are internal and external bisectors of angle \( AXB \). Referring to the figure, we see that \( \alpha + \beta = 90^\circ \), that is, \( \angle CXD \) is a right angle, so by the converse to Thales’ Theorem this means that \( X \) is on the circle \( \xi \).

(2) Let \( X \) be a point on the circle \( \xi \). Draw \( BP \parallel DX \) and \( BQ \parallel CX \) as shown on the right. Since \( X \) is on the circle, then \( \angle CXD = 90^\circ \). It follows that \( PBQ = 90^\circ \). Since \( \triangle APB \sim \triangle AXD \) and \( \triangle AQB \sim \triangle AXC \) we also have the following:

\[
\frac{AX}{XP} = \frac{AD}{DB} \quad \text{and} \quad \frac{AX}{XQ} = \frac{AC}{CB}.
\]

Since \( \frac{AD}{DB} = \frac{AC}{CB} = k \), it follows that \( \frac{AX}{XP} = \frac{AX}{XQ} \), from which we get \( XP = XQ \).

Since \( PBQ \) is a right angle, the point \( B \) is on the circle centred at \( X \) with radius \( XP \) (Thales’ Theorem). Thus, \( XB = XP \), so,

\[
\frac{AX}{XB} = \frac{AX}{XP} = \frac{AD}{DB} = k,
\]

which shows that statement (2) holds.
Theorem 5.4. Let O be the centre and r the radius of the Circle of Apollonius for A, B, and k. Then:

(i) O is on the line AB.
(ii) The points A and B are to the same side of O.
(iii) A and B are inverses with respect to the circle.
(iv) If the circle meets AB at C and D, then C and D divide AB harmonically in the ratio k.

Proof. Statements (i) and (iv) follow directly from Theorem 5.3.

(ii) We may assume that the line AB is horizontal and that A is to the left of B, that C is between A and B, and that D is not. Thus, D is located either to the left of A (as shown in figure (a)), or else to the right of B (figure (b)). We will show that statement (ii) is true for case (a)—the proof for case (b) is similar.

For case (a), we have $CB < DB$. Since C and D are on the circle of Apollonius, we also have $\frac{AC}{CB} = \frac{AD}{DB}$, and so

$$\frac{AC}{CB} \cdot CB < \frac{AD}{DB} \cdot DB,$$

$$\implies AC < AD,$$

which shows that the midpoint O of CD is to the left of A, and hence to the left of both A and B.

(iii) Assuming that O is to the left of A we have the following relationships (see the figure below):

$$AC = r - OA, \quad AD = r + OA, \quad CB = OB - r, \quad DB = OB + r.$$

Since C and D are on the circle,

$$\frac{AC}{CB} = \frac{AD}{DB},$$

$$\implies \frac{r - OA}{OB - r} = \frac{r + OA}{OB + r},$$

and carrying out the Algebra will show that $OA \cdot OB = r^2$.\qed
Harmonic conjugates and inverses

**Theorem 5.5.** A and B are harmonic conjugates with respect to C and D iff A and B are inverses with respect to the circle with diameter CD.

Proof. Suppose that A, B are harmonic conjugates for CD. Then C and D are harmonic conjugates for AB, that is

\[
\frac{AC}{CB} = \frac{AD}{DB}.
\]

Letting r be the radius of the circle with diameter CD, we want to show that \(OA \cdot OB = r^2\). The proof proceeds as in the proof of statement (iii) of Theorem 5.4.

Conversely, suppose that A, B are inverses with respect to the circle \(\omega\) with diameter CD. Assuming that A is outside \(\omega\) as shown, to prove that A and B are harmonic conjugates for CD, it suffices to show that

\[
\frac{CA}{AD} = 1.
\]

Referring to the figure, we have

\[
\frac{CA}{AD} = \frac{CA \cdot BD}{AD \cdot CB} = \frac{(OA - r) \cdot (OB + r)}{(OA + r) \cdot (r - OB)} = \frac{OA \cdot OB - r \cdot OB + r \cdot OA - r^2}{r \cdot OA + r^2 - OA \cdot OB - r \cdot OB}
\]

and since \(OA \cdot OB = r^2\), we get

\[
\frac{CA}{AD} = \frac{r \cdot OA - r \cdot OB}{r \cdot OA - r \cdot OB} = 1,
\]

which finishes the proof.

The relationship between harmonic conjugates and inverses enables us to show how a straightedge alone can be used to find the inverse of a point \(P\) that is outside the circle of inversion.
**Problem 5.6.** Given a point $P$ outside a circle $\omega$ with centre $O$, construct the inverse of $P$ using only a straightedge.

**Solution.**

![Diagram](image)

(1) Draw the line $OP$ meeting $\omega$ at $C$ and $D$.

(2) Draw a second line through $P$ meeting $\omega$ at $A$ and $B$ as shown.

(3) Draw $AC$ and $BD$ meeting at $X$. Draw $AD$ and $BC$ meeting at $Y$.

(4) Draw $X$ and $Y$ meeting $OP$ at $Q$. Then $Q$ is the inverse of $P$.

**Proof.** Apply Ceva’s Theorem to $\triangle XCD$ and cevians $XQ$, $CB$, and $DA$. The cevians are concurrent at $Y$, so

$$\frac{XA}{AC} \cdot \frac{CQ}{QD} \cdot \frac{DB}{BX} = 1 \quad (1)$$

Apply Menelaus’ Theorem to $\triangle XCD$ with Menelaus points $P$, $A$, $B$. The points $P$, $A$, and $B$ are collinear so

$$\frac{XA}{AC} \cdot \frac{CP}{PD} \cdot \frac{DB}{BX} = 1 \quad (2)$$

From (1) and (2) we get

$$\frac{CQ}{QD} = \frac{CP}{PD},$$

which implies that $P$ and $Q$ are harmonic conjugates with respect to $CD$. By the previous theorem, this means that $P$ and $Q$ are inverses with respect to $\omega$. $\square$

**Inversion and the circle of Apollonius**

We state here several theorems that are easy consequences of the previous sections.

**Theorem 5.7.** If $\omega$ is the circle of Apollonius for $A$, $B$, and $k$, then $A$ and $B$ are inverses with respect to $\omega$.

**Theorem 5.8.** The Apollonian circle for $A$, $B$, and $k$ is the same as the Apollonian circle for $B$, $A$, and $\frac{1}{k}$.

**Remark:** Note the change in order of the points $A$ and $B$ in the previous theorem.

**Theorem 5.9.** If $A$ and $B$ are inverse points for a circle $\omega$, then $\omega$ is the circle of Apollonius for $A$, $B$, and some positive number $k$.  

28
**Theorem 5.10.** If $\alpha$ and $\beta$ are orthogonal circles, then whenever either circle intersects a diameter of the other, it divides that diameter harmonically.

The following is the converse of the previous theorem.

**Theorem 5.11.** If $\alpha$ and $\beta$ are two circles, and $\beta$ divides a diameter of $\alpha$ harmonically, then the two circles are orthogonal.