University of Alberta
Mathematical and Statistical Sciences
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## Math 667, Topics in Differential Equations

Winter 2005

### Assignment 4, due March. 21. 2005, 10 AM

**Exercise 15:** (20)

If we are just interested in local existence of weak solutions of reaction-diffusion equations we can relax the condition on $f$ which we used during the lecture. We assume now that $f$ is linearly bounded:
*There are two constants $C_1, C_2 \geq 0$ such that for each $u \in \mathbb{R}$*

$$|f(u)| \leq C(1+|u|), \qquad |f'(u)| \leq C_2. \tag{1}$$

In this exercise we will use the Galerkin method to prove the following result:

**Theorem 0.1** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary. For $T > 0$ we denote $\Omega_T = (0,T) \times \Omega$. Given $u_0 \in L^2(\Omega)$. If (1) holds then there exists a unique weak solution $u$ of the reaction-diffusion equation*

$$\frac{\partial u}{\partial t} - \Delta u = f(u).$$

$$u = 0 \quad on \quad \partial\Omega, \qquad u(0) = u_0,$$

*with*

$$u \in L^2(0,T;H_0^1(\Omega)).$$

**Proof.**

1. Use the method of truncated eigenfunction expansions to show the existence of approximate solutions $u_n$.
2. Show that these approximate solutions are uniformly bounded in the following spaces: in $L^\infty(0,T;L^2(\Omega))$, in $L^2(0,T;L^2(\Omega))$, in $L^\infty(0,T;H_0^1(\Omega))$, and in $L^2(0,T;H_0^1(\Omega))$.
3. Show that $\{f(u_n)\}$ is uniformly bounded in $L^2(\Omega_T)$.
4. Use compactness arguments to find weak convergent subsequences for $\{u_n\}$ and for $\{f(u_n)\}$.
5. Use a test-function $\phi \in L^2(\Omega)$ to show that also $P_n f(u_n)$ has a weakly convergent subsequence.
6. Show that $\frac{du_n}{dt}$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$, find a weak* convergent subsequence and show that
$$\frac{du_n}{dt} \rightharpoonup^* \frac{du}{dt}.$$
7. Use the dominated convergence theorem to show that
$$f(u_n) \rightharpoonup f(u).$$
8. Show that the limit function $u$ indeed is a weak solution.
9. Prove uniqueness.

□