Math 337, Summer 2010
Assignment 4

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Exercise 0.1. ✗

The neutron flux u in a sphere of uranium obeys the differential equation

$$\frac{\lambda}{3}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + (k-1)A\,u = 0$$

for $0 < r < a$, where λ is the effective distance traveled by a neutron between collisions, A is called the absorption cross section, and k is the number of neutrons produced by a collision during fission. In addition, the neutron flux at the boundary of the sphere is 0.

(a) Make the substitution

$$u = \frac{v}{r} \qquad \text{and} \qquad \mu^2 = \frac{3(k-1)A}{\lambda}$$

and show that $v(r)$ satisfies $\dfrac{d^2v}{dr^2} + \mu^2\, v = 0,\ 0 < r < a.$

(b) Find the general solution to the differential equation in part (a) and then find $u(r)$ that satisfies the boundary condition and boundedness condition:

$$u(a) = 0 \qquad \text{and} \qquad \lim_{r\to 0^+} |u(r)| \quad \text{bounded.}$$

(c) Find the critical radius, that is, the smallest radius a for which the solution is not identically 0.

Solution to Exercise 0.1:

(a) Letting $u = v/r$, then

$$\frac{du}{dr} = \frac{1}{r}\frac{dv}{dr} - \frac{1}{r^2}v \qquad \text{and} \qquad r^2\frac{du}{dr} = r\frac{dv}{dr} - v,$$

so that

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = r\frac{d^2v}{dr^2} + \frac{dv}{dr} - \frac{dv}{dr} = r\frac{d^2v}{dr^2},$$

and the differential equation for $v(r)$ is

$$\frac{1}{r}\frac{d^2v}{dr^2} + \frac{\mu^2}{r}\,v = 0, \qquad \text{that is,} \qquad \frac{d^2v}{dr^2} + \mu^2\,v = 0$$

for $0 < r < a$.

(b) The general solution to the differential equation in part (a) is

$$v(r) = c_1 \cos\mu r + c_2 \sin\mu r$$

for $0 < r < a$, and the solution to the neutron flux equation is

$$u(r) = \frac{v(r)}{r} = c_1\,\frac{\cos\mu r}{r} + c_2\,\frac{\sin\mu r}{r}$$

for $0 < r < a$. Applying the boundedness condition, since

$$\lim_{r\to 0^+} \frac{\sin\mu r}{r} = \mu \qquad \text{and} \qquad \lim_{r\to 0^+} \frac{\cos\mu r}{r} \quad \text{doesn't exist,}$$

then we need $c_1 = 0$, and the solution is

$$u(r) = c_2\,\frac{\sin\mu r}{r}$$

for $0 < r < a$.

(c) Applying the boundary condition

$$u(a) = \frac{c_2}{a}\,\sin\mu a = 0,$$

clearly, there is a nontrivial solution if and only if $\mu a = n\pi$ for some positive integer n. The critical radius is $a = \dfrac{\pi}{\mu}$.

Exercise 0.2. ✗✗

Solve Laplace's equation in the square $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$ with the boundary conditions given below

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leqslant x \leqslant \pi, \quad 0 \leqslant y \leqslant \pi$$

$$u(0,y) = 0, \quad 0 \leqslant y \leqslant \pi$$

$$u(\pi,y) = 0, \quad 0 \leqslant y \leqslant \pi$$

$$u(x,0) = 0, \quad 0 \leqslant x \leqslant \pi$$

$$u(x,\pi) = 1, \quad 0 \leqslant x \leqslant \pi.$$

Solution to Exercise 0.2: We use separation of variables and assume a solution to Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

of the form $u(x, y) = X(x) \cdot Y(y)$.
Separating variables we have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(t)}{Y(t)} = -\lambda,$$

and we obtain the two ordinary differential equations

$$\begin{aligned} X'' + \lambda X = 0 \quad 0 \leqslant x \leqslant \pi \qquad & Y'' - \lambda Y = 0, \quad 0 \leqslant y \leqslant \pi \\ X(0) = 0 \qquad & Y(0) = 0 \\ X(\pi) = 0. \qquad & \end{aligned}$$

Solving the complete boundary value problem for X, the eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2 \qquad \text{and} \qquad X_n(x) = \sin nx$$

for $n \geqslant 1$.
The corresponding problem for Y is

$$\begin{aligned} Y'' - n^2 Y &= 0 \\ Y(0) &= 0 \end{aligned}$$

with solutions

$$Y_n(y) = \sinh ny$$

for $n \geqslant 1$.
From the superposition principle, we write

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh ny \, \sin nx,$$

and setting $y = \pi$, we need

$$1 = u(x, \pi) = \sum_{n=1}^{\infty} b_n \sinh n\pi \, \sin nx,$$

and from the orthogonality of the eigenfunctions,

$$b_n \sinh n\pi = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = -\frac{2}{n\pi} \cos nx \Big|_0^{\pi} = -\frac{2}{n\pi} \left[(-1)^n - 1 \right],$$

so that

$$u(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n\sinh n\pi}\,\sin nx\,\sinh ny$$

for $0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \pi$.

Exercise 0.3. ✗✗

Consider the regular Sturm-Liouville problem

$$\varphi''(x) + \lambda\,\varphi(x) = 0, \quad 0 \leqslant x \leqslant 1$$
$$\varphi(0) = 0$$
$$\varphi(1) - h\,\varphi'(1) = 0$$

where $h > 0$.

Show that there is a single negative eigenvalue λ_0 if and only if $h < 1$. Find λ_0 and the corresponding eigenfunction $\varphi_0(x)$.

Hint: Assume $\lambda = -\mu^2$ for some real number $\mu \neq 0$.

Solution to Exercise 0.3: Following the hint, the differential equation becomes

$$\varphi''(x) - \mu^2\,\varphi(x) = 0,$$

with general solution

$$\varphi(x) = A\cosh\mu x + B\sinh\mu x$$

for $0 \leqslant x \leqslant 1$.

Applying the first boundary condition, we have

$$\varphi(0) = A = 0,$$

so that

$$\varphi(x) = B\sinh\mu x \qquad \text{with} \qquad \varphi'(x) = \mu B\cosh\mu x.$$

Applying the second boundary condition, we have

$$\varphi(1) - h\,\varphi'(1) = B\,[\sinh\mu - h\,\mu\cosh\mu] = 0.$$

so that

$$B\cosh\mu\,[\tanh\mu - h\,\mu] = 0,$$

and we have a nontrivial solution if and only if

$$\tanh\mu = h\,\mu$$

for some $\mu \neq 0$.
However, the graphs of $y = \tanh \mu$ and $y = h\mu$ intersect only at $\mu = 0$ if $h \geqslant 1$, and they intersect at exactly one positive value μ_0 if $0 < h < 1$.
Therefore, there is exactly one negative eigenvalue for this Sturm-Liouville problem if and only if $0 < h < 1$, and the eigenvalue is

$$\lambda_0 = -\mu_0^2$$

where μ_0 is the positive root of the equation $\tanh \mu = h\mu$.
The corresponding eigenfunction is

$$\varphi_0(x) = \sinh \mu_0 x$$

for $0 \leqslant x \leqslant 1$.

Exercise 0.4. ✗✗
Legendre's differential equation reads

$$(1 - x^2)y'' - 2xy' + \lambda y = 0, \qquad -1 < x < 1$$

(a) Write the differential equation in Sturm-Liouville form. Decide if the resulting Sturm-Liouville problem is regular or singular.

(b) Show that the first four Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1), \quad P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

are eigenfunctions of the Sturm-Liouville problem and find the corresponding eigenvalues.

(c) Use an appropriate weight function and show that P_1 and P_2 are orthogonal on the interval $(-1, 1)$ with respect to this weight function.

Solution to 0.4:

(a) Since

$$\frac{d}{dx}\left((1 - x^2)\,\frac{dy}{dx}\right) = (1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx},$$

Legendre's equation can be written as

$$\big((1 - x^2)\,y'\big)' + \lambda y = 0, \quad -1 < x < 1, \qquad (*)$$

which is the classical Sturm-Liouville form

$$[p(x)\,y']' + [q(x) + \lambda r(x)]\,y = 0, \quad a < x < b$$

with

$$p(x) = 1 - x^2, \qquad q(x) = 0, \qquad \text{and} \qquad r(x) = 1,$$

for $a < x < b$, where $a = -1$ and $b = 1$.

For a regular Sturm-Liouville problem we require the regularity conditions:

$$p(x), \qquad p'(x), \qquad q(x), \qquad \text{and} \qquad r(x)$$

are continuous on the closed interval $a \leqslant x \leqslant b$, and

$$p(x) > 0 \qquad \text{and} \qquad r(x) > 0$$

for $a \leqslant x \leqslant b$.

We also require the boundary conditions

$$c_1 y(a) + c_2 y'(a) = 0 \qquad \text{and} \qquad d_1 y(b) + d_2 y'(b) = 0$$

where at least one of c_1 and c_2 is nonzero and at least one of d_1 and d_2 is nonzero.

Thus, it is clear that $(*)$ is a singular Sturm-Liouville problem (no matter what the boundary conditions are) since one of the regularity conditions is violated, namely, $p(-1) = p(1) = 0$.

(b) For $P_0(x) = 1$, we have

$$P_0'(x) = 0 \qquad \text{and} \qquad P_0''(x) = 0$$

for $-1 < x < 1$, so that

$$(1 - x^2)P_0'' - 2xP_0' + \lambda P_0 = 0, \quad -1 < x < 1$$

is satisfied for $\lambda = 0$, and the eigenvalue corresponding to the eigenfunction $P_0(x) = 1$ is $\lambda_0 = 0$.

For $P_1(x) = x$, we have

$$P_1'(x) = 1 \qquad \text{and} \qquad P_1''(x) = 0$$

for $-1 < x < 1$, so that

$$(1 - x^2)P_1'' - 2xP_1' + \lambda P_1 = 0, \quad -1 < x < 1$$

becomes

$$-2x \cdot 1 + \lambda x = 0, \quad -1 < x < 1$$

which is satisfied for $\lambda = 2$, and the eigenvalue corresponding to the eigenfunction $P_1(x) = x$ is $\lambda_1 = 2$.

For $P_2(x) = \dfrac{1}{2}(3x^2 - 1)$, we have

$$P_2'(x) = 3x \qquad \text{and} \qquad P_2''(x) = 3$$

for $-1 < x < 1$, so that

$$(1 - x^2)P_2'' - 2xP_2' + \lambda P_2 = 0, \quad -1 < x < 1$$

becomes

$$3(1 - x^2) - 6x^2 + \frac{\lambda}{2}(3x^2 - 1) = 0, \quad -1 < x < 1$$

that is,

$$-3(3x^2 - 1) + \frac{\lambda}{2}(3x^2 - 1) = 0, \quad -1 < x < 1$$

which is satisfied for $\lambda = 6$, and the eigenvalue corresponding to the eigenfunction $P_2(x) = \dfrac{1}{2}(3x^2 - 1)$ is $\lambda_2 = 6$.

For $P_3(x) = \dfrac{1}{2}(5x^3 - 3x)$, we have

$$P_3'(x) = \frac{1}{2}(15x^2 - 3) \qquad \text{and} \qquad P_3''(x) = 15x$$

for $-1 < x < 1$, so that

$$(1 - x^2)P_3'' - 2xP_3' + \lambda P_3 = 0, \quad -1 < x < 1$$

becomes

$$15x(1 - x^2) - (15x^3 - 3x) + \frac{\lambda}{2}(5x^3 - 3x) = 0, \quad -1 < x < 1$$

that is,

$$-6(5x^3 - 3x) + \frac{\lambda}{2}(5x^3 - 3x) = 0, \quad -1 < x < 1$$

which is satisfied for $\lambda = 12$, and the eigenvalue corresponding to the eigenfunction $P_3(x) = \dfrac{1}{2}(5x^3 - 3x)$ is $\lambda_3 = 12$.

(c) Using the weight function $w(x) = 1$, for $-1 < x < 1$, we have

$$\langle P_1, P_2 \rangle = \int_{-1}^{1} P_1(x) \cdot P_2(x)\, dx = 0$$

since the product $P_1(x)P_2(x)$ is an odd function integrated between symmetric limits, thus $P_1(x)$ and $P_2(x)$ are orthogonal on the interval $-1 < x < 1$ with respect to the weight function $w(x) = 1$.

Exercise 0.5. ✗✗

Find the solution to Laplace's equation on the rectangle:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad 0 < x < a, \quad 0 < y < b \\ u(0,y) &= 1, \quad 0 < y < b \\ u(a,y) &= 1, \quad 0 < y < b \\ \frac{\partial u}{\partial y}(x,0) &= 0, \quad 0 < x < a \\ \frac{\partial u}{\partial y}(x,b) &= 0, \quad 0 < x < a \end{aligned}$$

using the method of separation of variables. Is your solution what you expected?

Solution to Exercise 0.5: Writing $u(x,y) = X(x)\,Y(y)$ we obtain

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2 \qquad \text{(constant)}$$

and hence the two ordinary differential equations

$$X'' - \lambda^2 X = 0 \qquad \text{and} \qquad \begin{aligned} Y'' + \lambda^2 Y &= 0 \qquad 0 < y < b \\ Y'(0) &= 0 \\ Y'(b) &= 0 \end{aligned}$$

Solving the regular Sturm-Liouville problem for Y, for the eigenvalue $\lambda_0^2 = 0$ the corresponding eigenfunction is

$$Y_0(y) = 1,$$

and the corresponding solution to the first equation is

$$X_0(x) = b_0\,x + a_0.$$

For the eigenvalues $\lambda_n^2 = \left(\frac{n\pi}{b}\right)^2$, the corresponding eigenfunctions are

$$Y_n(y) = \cos \lambda_n y,$$

and the corresponding solutions to the first equation are

$$X_n(x) = a_n \cosh \lambda_n x + b_n \sinh \lambda_n x,$$

for $n = 1, 2, 3, \ldots$

Using the superposition principle, we write

$$u(x,y) = b_0\,x + a_0 + \sum_{n=1}^{\infty} \big(a_n \cosh \lambda_n x + b_n \sinh \lambda_n x\big) \cos \lambda_n y.$$

From the boundary condition $u(0,y) = 1$, we have

$$1 = a_0 + \sum_{n=1}^{\infty} a_n \cos \lambda_n y$$

so that

$$a_0 = \frac{1}{b}\int_0^b 1\,dy = 1$$

while

$$a_n = \frac{2}{b}\int_0^b \cos \lambda_n y\,dy = \frac{2}{n\pi} \sin \lambda_n y \bigg|_0^b = 0$$

for $n = 1, 2, 3, \ldots$

From the boundary condition $u(a,y) = 1$, we have

$$1 = b_0\,a + 1 + \sum_{n=1}^{\infty} b_n \sinh \lambda_n a\, \cos \lambda_n y$$

and integrating this equation from 0 to b we get $b_0\,a\,b = 0$, and therefore $b_0 = 0$, so that

$$0 = \sum_{n=1}^{\infty} b_n \sinh \lambda_n a\, \cos \lambda_n y.$$

In order to evaluate the b_n's, we multiply this equation by $\cos \frac{m\pi}{b} y$ and integrate from 0 to b, to obtain $b_m \sinh \frac{m\pi}{b} a = 0$, that is, $b_m = 0$ for $m = 1, 2, 3, \ldots$.

Therefore the solution is $u(x,y) = 1$, which is not totally unexpected, since the solution is unique and it is clear from the statement of the problem that $u(x,y) = 1$ satisfies Laplace's equation on the rectangle and satisfies all of the boundary conditions.