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Existence of Weak Solutions for a Hyperbolic Model of Chemosensitive Movement

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A hyperbolic model for chemotaxis and chemosensitive movement in one space dimension is considered. In contrast to parabolic models for chemotaxis the hyperbolic model allows us to take the dependence of the particle speed on external stimuli explicitly into account. This qualitatively covers recent experiments on chemotaxis in which it has been shown that particles adapt their speed to the surrounding environment. The model presented here consists of two hyperbolic differential equations of first order coupled with an elliptic equation. We assume that the speed depends on the external stimulus only (and not on its gradients). In that case solutions with steep gradients are expected which have the interpretation

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of moving swarms. A notion of weak solutions for this hyperbolic chemotaxis model is presented and the global existence of weak solutions is shown. The proof relies on the vanishing viscosity method; i.e., we obtain the weak solution as the limit of classical solutions of an associated parabolically regularized problem for vanishing viscosity parameter. Numerical simulations demonstrate phenomena like swarming behaviour and formation of steep gradients.

1. INTRODUCTION

In this paper we study existence of weak solutions of the following nonlinear hyperbolic model for chemosensitive movement on the real line,

$$u_t^+ + (\gamma(s)u^+)_x = -\mu^+(s, s_x)u^+ + \mu^-(s, s_x)u^-, \tag{1}$$

$$u_t^- - (\gamma(s)u^-)_x = \mu^+(s, s_x)u^+ - \mu^-(s, s_x)u^-, \tag{2}$$

$$Ds_{xx} - \beta s = -\alpha(u^+ + u^-), \tag{3}$$

where the total particle density $u = u^+ + u^-$ has been split into densities for right and left moving particles $u^\pm$, respectively. The turning rates $\mu^\pm$ and the particle speed γ depend on an external signal. The density of the external signal is denoted by s and its diffusion, reproduction, and degradation is assumed to be linear with D, α, $\beta > 0$ by Eq. (3). Equation (3) is the quasistationary approximation of the parabolic equation

$$\tau s_t = Ds_{xx} - \beta s + \alpha(u^+ + u^-). \tag{4}$$

Here we assume that the diffusion of s is fast compared to the movement of the particles, hence it suffices to consider (3).

To obtain a Cauchy problem the equations are supplemented with nonnegative compactly supported initial conditions

$$u^\pm(0, x) = u_0^\pm(x) \geq 0. \tag{5}$$

Taxis or *kinesis* describes the spatial movement of individuals which depends on the surrounding environment. Individual motion changes in response to external stimuli like temperature, light, concentration of food, or other substances. Taxis denotes response to an external vector field (e.g., gradient of an external chemical) and kinesis denotes response to some scalar field. If the external signal is a chemical nutrient which could be recognized by receptors of the individual particles, then the reaction to this stimulus is called *chemotaxis* or *chemokinesis*, respectively. In the mathematical literature the word chemotaxis has been used to summarize chemotaxis, chemokinesis, and other responses to chemicals. We prefer to follow the biological distinction of kinesis and taxis and we denote all this by *chemosensitive movement*.

For motion of slime molds, bacteria, or leukocytes chemosensitive movement has been observed experimentally (see, e.g., Soll [25] or Othmer and Schaap [19] for slime molds, Ford *et al*. [2] for bacteria, and Gallin and Quie [3] for leukocytes). Chemosensitive movement leads to different states of pattern formation and self organization. Local aggregations of slime molds are able to form multicellular organisms, swarms move as one organism, and street formation with periodical patterns can be observed. There is great evidence that chemotaxis is one process which leads to pigmentation pattern on animal skins and it is discussed whether or not this mechanism drives the early development of an embryo.

The first mathematical model to describe chemosensitive movement has been introduced by Keller and Segel [15]. Being parabolic this model allows for infinitely fast propagation speed which clearly is an unwanted effect. To avoid this difficulty Segel [23] introduced a hyperbolic model for chemotaxis in one space dimension based on the Goldstein–Kac model [4, 14] for one-dimensional correlated random walk. The Goldstein–Kac model reads

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= \mu(u^- - u^+), \\ u_t^- - \gamma u_x^- &= \mu(u^+ - u^-), \end{aligned} \tag{6}$$

where γ is the (constant) particle speed and μ the (constant) turning rate. This model combined with reactions and interactions of particles (which include birth, death, predator-prey, activator-inhibitor, epidemic spread, etc.) has been widely studied by Holmes [13], Hadeler [5, 6], Hillen [10, 11], Müller and Hillen [18], and Schneider and Müller [22]. The interplay of reaction and motion has been discussed in detail [6]. Moreover boundary conditions on a bounded interval have been introduced; existence and uniqueness results for weak and classical solutions have been given; energy methods have been used to describe the asymptotic behavior of solutions; invariance results and positivity results have been derived; and traveling wave solutions have been studied. These reaction random walk systems do not cover the phenomenon of chemosensitive movement; nevertheless basic properties carry over to the model studied here.

In experiments of Soll [25] it turned out that the speed and the turning rates in *Dictyostelium discoideum* depend on the external signal and on temporal and spatial gradients of the external signal. Hillen and Stevens [12] generalized Segel's model and introduced a hyperbolic model for chemosensitive movement such that these dependencies can be modeled. They showed local and global existence of bounded solutions for (1), (2), and (4) in case of constant speed $\gamma(s) = \gamma$ and with both $\tau = 0$ and $\tau > 0$.

In their investigation of the parabolic limit it turned out that basically two effects lead to aggregation. One is turning rates which depend on the signal and on its gradients. The other is a speed decreasing in the signal s. In numerical experiments shown at the end of this paper (Section 4) it turns out that in some cases decreasing speed is not enough to establish stable aggregations.

There are several ways to model the dependence of the external signal on the density. As pointed out, e.g., by Othmer and Stevens [20], one can consider nonlinear types of production of s. Here we restrict to the linear production to focus on the dependencies of $\gamma(s)$. If s is produced proportionally to $s(u^+ + u^-)$ then blow-up in finite time may result [12]. (Results on blow-up for the parabolic Keller–Segel model are well known from the literature; see, e.g., [9, 17, 20, 24]).

The following section gives detailed assumptions and states the main result of this paper (Theorem 2.1). Section 3 contains the proof using the vanishing viscosity method. Finally in Section 4 we show some interesting simulations. Phenomena like swarming behavior, formation of steep gradients, and pattern formation are illustrated.

2. ASSUMPTIONS AND MAIN RESULT

To obtain existence of weak solutions of (1)–(3) we introduce some reasonable basic assumptions.

(A1) The initial values $u_0^\pm \in L^\infty(\mathbb{R}) \cap C^3(\mathbb{R})$ satisfy $\operatorname{supp}(u_0^\pm) \subset [-R, R]$ for $R > 0$ and $u_0^\pm \geq 0$. The function $s_0 \in W^{1,2}(\mathbb{R})$ is the unique weak solution of

$$Ds_{0,xx} = \beta s_0 - \alpha(u_0^+ + u_0^-), \qquad s_0(\pm\infty) = 0. \tag{7}$$

(A2) The turning rates $\mu^\pm : \mathbb{R}^2 \to \mathbb{R}$ are nonnegative, i.e., $\mu^\pm \geq 0$.

(A3) Symmetry with respect to s_x, i.e., $\mu^+(s, s_x) = \mu^-(s, -s_x)$.

(A4) The turning rates satisfy $\mu^\pm \in C^1(\mathbb{R}^2)$ and are bounded

$$\|D_j \mu^\pm\|_\infty \leq C_\mu,\ j = 1,2 \quad \text{and} \quad 0 \leq \mu^\pm(s, s_x) \leq C_\mu(1 + \|s\|_{W^{1,\infty}})$$

with some constant $C_\mu > 0$.

(A5) The speed function $\gamma(s)$ satisfies $\gamma \in C^2(\mathbb{R})$ with

$$\|\gamma'\|_\infty \leq C_\gamma, \qquad \|\gamma''\|_\infty \leq C_\gamma,$$

for some constant $C_\gamma > 0$.

From (A5) it follows that

$$\|\gamma(s)\|_\infty \leq C_\gamma(1 + \|s\|_\infty).$$

The existence of a unique weak solution of (7) is obvious. Assumption (A1) and the maximum principle for elliptic equations ensure

$$s_0(x) \geq 0, \qquad x \in \mathbb{R}.$$

The assumptions (A1)–(A5) indeed cover biological relevant scenarios. Assumption (A3) ensures symmetry with respect to change of "left" and "right." To motivate assumption (A4) we refer to Hillen and Stevens [12]. There the parabolic limit of (1)–(3) for large speeds and large turning rates has been considered in detail. It turned out that the simplest choice, which actually leads to the parabolic Keller–Segel model, is γ = constant and

$$\mu^\pm(s, s_x) = \frac{1}{2}\left(c_1 \mp c_2(s) s_x\right)^+,$$

for constant $c_1 > 0$ and bounded $c_2(s)$. The upper case symbol $+$ is used to denote the positive part of the function. This example is covered by assumption (A4). In other situations the turning rate is bounded from above (e.g., for bacteria) and (A4) is satisfied. In most applications the particle speed is bounded which is covered by assumption (A5).

System (1)–(3) is built up by two hyperbolic balance laws and constrained by an elliptic equation. It is well known that solutions of purely hyperbolic conservation laws can develop discontinuities even for arbitrarily smooth initial data. Here the situation is not so clear since the solution of the elliptic equation enters into the flux function of the conservation law. In the case of $D = 0$ (which is not considered here) it follows that $s = \frac{\alpha}{\beta}(u^+ + u^-)$ and (1)–(2) becomes a nonlinear conservation law. Consequently at least for $D \to 0$ we expect that the $u^\pm$ become discontinuous. Numerical simulations in Section 4 confirm the expected loss of regularity.

According to this regularity problem we introduce a notion of solutions that allows for discontinuities.

DEFINITION 2.1. For $T > 0$ let $\Omega_T = (0, T) \times \mathbb{R}$.

A function $(u^+, u^-, s) \in L^\infty(\overline{\Omega}_T) \cap L^\infty([0, T]; L^1(\mathbb{R})^2 \times W^{1,2}(\mathbb{R}))$ is called a weak solution of the Cauchy problem (1)–(3) and (5) iff the subsequent conditions (i), (ii), and (iii) are satisfied.

(i) For all $\phi \in C_0^\infty(\Omega_T)$ we have

$$-\int_{\overline{\Omega}_T} u^\pm \phi_t \pm \gamma(s) u^\pm \phi_x = \int_{\overline{\Omega}_T} [\mp \mu^+(s, s_x) u^+ \pm \mu^-(s, s_x) u^-] \phi. \quad (8)$$

(ii) For almost all $t \in [0, T]$ the function $s(t, \cdot) \in W^{1,2}(\mathbb{R})$ is the weak solution of (3) for $s(t, \pm\infty) = 0$; i.e., for all test functions $v \in W_0^{1,2}(\mathbb{R})$ we have

$$-\int_{\mathbb{R}} s_x(t,x) v_x \, dx = \int_{\mathbb{R}} [\beta s(t,x) - \alpha(u^+(t,x) + u^-(t,x))] v \, dx.$$

(iii) The functions u^+, u^-, s fulfill the initial condition (5) in the weak sense; i.e., there exists a set $\mathscr{L} \subset [0, T]$ of Lebesgue measure 0 such that $u^\pm(\tau, \cdot)$ and $s(\tau, \cdot)$ are defined almost everywhere in $\mathbb{R}$ for $\tau \in [0, T] \setminus \mathscr{L}$ and satisfy

$$\lim_{\tau \to 0, \tau \in [0,T] \setminus \mathscr{L}} \int_{-\infty}^{\infty} |u^\pm(\tau, x) - u_0^\pm(x)| \, dx = 0, \quad (9)$$

$$\lim_{\tau \to 0, \tau \in [0,T] \setminus \mathscr{L}} \int_{-\infty}^{\infty} |s(\tau, x) - s_0(x)|^2 \, dx = 0. \quad (10)$$

Based on Definition 2.1 we can present the main result of this paper.

THEOREM 2.1. *Let Assumptions* (A1)–(A5) *hold. Then there exists a weak solution* (u^+, u^-, s) *of the hyperbolic model of chemosensitive movement* (1), (2), (3), (5) *such that for all* $T < \infty$ *the weak solution satisfies for almost all* $(t, x) \in \overline{\Omega}_T$

$$\begin{aligned} 0 &\le u^\pm(t,x) \le C_\infty, \\ 0 &\le s(t,x) \le C_s. \end{aligned} \quad (11)$$

The constant $C_\infty \ge 0$ *grows exponentially in* T *and depends on* C_γ, C_μ, α, β, $N_0 = \int (u_0^+ + u_0^-) \, dx$, *and* $C_s = C_s(\alpha, \beta, N_0)$.

To prove Theorem 2.1 we will proceed along the lines of the vanishing viscosity method which has a long tradition in the existence theory of hyperbolic conservation laws. Here we rely on the methods introduced by Vol'pert and Kružkov for a scalar conservation law [16, 26]. Especially we use the results in [21] where a generalization of Kružkov's results to weakly coupled hyperbolic conservation laws is given.

Within the vanishing viscosity ansatz we analyze for $0 < \varepsilon \leq 1$ the subsequent parabolically regularized Cauchy problem

$$\begin{aligned} u_t^{\varepsilon+} - \varepsilon u_{xx}^{\varepsilon+} &= -(\gamma(s^\varepsilon)u^{\varepsilon+})_x - \mu^+(s^\varepsilon, s_x^\varepsilon)u^{\varepsilon+} + \mu^-(s^\varepsilon, s_x^\varepsilon)u^{\varepsilon-}, \\ u_t^{\varepsilon-} - \varepsilon u_{xx}^{\varepsilon-} &= (\gamma(s^\varepsilon)u^{\varepsilon-})_x + \mu^+(s^\varepsilon, s_x^\varepsilon)u^{\varepsilon+} - \mu^-(s^\varepsilon, s_x^\varepsilon)u^{\varepsilon-}, \\ s_{xx}^\varepsilon &= \beta s^\varepsilon - \alpha(u^{\varepsilon+} + u^{\varepsilon-}), \\ u^{\varepsilon\pm}(0,\cdot) &= u_0^\pm, \qquad s^\varepsilon(0,\cdot) = s_0, \\ s_{0,xx} &= \beta s_0 - \alpha(u_0^+ + u_0^-), \end{aligned} \tag{12}$$

where we used without loss of generality $D = 1$.

In terms of total particle density $u = u^+ + u^-$ and particle flow γv with $v = u^+ - u^-$ system (12) reads

$$\begin{aligned} u_t^\varepsilon - \varepsilon u_{xx}^\varepsilon &= -(\gamma(s^\varepsilon)v^\varepsilon)_x, \\ v_t^\varepsilon - \varepsilon v_{xx}^\varepsilon &= -(\gamma(s^\varepsilon)u^\varepsilon)_x - \xi(s^\varepsilon, s_x^\varepsilon)u^\varepsilon - \eta(s^\varepsilon, s_x^\varepsilon)v^\varepsilon, \\ s_{xx}^\varepsilon &= \beta s^\varepsilon - \alpha u^\varepsilon, \\ u^\varepsilon(0,\cdot) &= u_0 = u_0^+ + u_0^-, \\ v^\varepsilon(0,\cdot) &= v_0 = u_0^+ - u_0^-, \\ s_{0,xx} &= \beta s_0 - \alpha u_0, \qquad s^\varepsilon(0,\cdot) = s_0, \end{aligned} \tag{13}$$

with $\xi = \mu^+ - \mu^-$ and $\eta = \mu^+ + \mu^-$.

3. THE PARABOLICALLY REGULARIZED PROBLEM

We consider classical solutions of (12) which are bounded in L^p-norms. Hence we use the Sobolev space notation $L^2 = L^2(\mathbb{R})$, $W^{k,p} = W^{k,p}(\mathbb{R})$ with norms $\|\cdot\|_p$, $\|\cdot\|_{k,p}$, respectively. The spaces $W^{2,p}$ are denoted by H^p. We denote for $k \in \mathbb{N}$,

$$\mathscr{C}_0 := \left\{u \in C(\mathbb{R}) : \|u\|_\infty < \infty,\ \lim_{|x|\to\infty} u(x) = 0\right\},$$

$$\mathscr{C}_0^k := \left\{u \in C^k(\mathbb{R}) : \forall 0 \leq j \leq k,\ D_j u \in \mathscr{C}_0\right\}.$$

The space $C_{ub}(\mathbb{R})$ denotes bounded uniformly continuous functions on $\mathbb{R}$. Due to conventional notations we denote any constant by C, where we explicitly state the dependencies, when necessary. During this section we suppress the index ε, since here we consider the parabolic problem (12) for fixed $\varepsilon > 0$.

3.1. L^2-Theory

For given $u \in L^2$ we study the elliptic equation for s:

$$s_{xx} = \beta s - \alpha u. \tag{14}$$

Using Green's function on $\mathbb{R}$ we find a continuous solution operator $S: L^2 \to W^{2,2}$, $u \mapsto S(u)$ of this equation.

LEMMA 3.1. *There exists a constant $C = C(\alpha, \beta)$ such that for $u \in L^2$ we have for $s = S(u)$,*

$$\|s\|_{2,2} \leq C\|u\|_2 \quad \text{and} \quad \|s\|_{\mathscr{C}_0^1} \leq C\|u\|_2.$$

Proof. For the first estimate we apply the Fourier transformation to Eq. (14) and observe

$$\hat{s}(\kappa) = \frac{\alpha \hat{u}(\kappa)}{\kappa^2 + \beta}. \tag{15}$$

Since the Fourier transformation is isometric on L^2 we immediately get

$$\|s\|_2 \leq \frac{\alpha}{\beta}\|u\|_2.$$

Differentiation of Eq. (14) once respectively twice and Fourier transformation gives estimates for s_x and s_{xx}. The second estimate follows directly from the first with the embedding $W^{2,2} \to \mathscr{C}_0^1$. ∎

From the continuity of Green's operator S the following is also clear.

COROLLARY 3.1. *If $t \mapsto u(t) \in L^2$ is continuous then $t \mapsto S(u(t))$ is continuous as well.*

We proceed to study the whole system (12). If we formally solve the elliptic equation in (12) for $s = S(u^+ + u^-)$ then the system for (u^+, u^-) decouples into

$$\begin{aligned} y_t &= Ay + F(y) \\ y(0, \cdot) &= y_0, \end{aligned} \tag{16}$$

with

$$y = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}, \qquad A = \begin{pmatrix} \varepsilon\Delta & 0 \\ 0 & \varepsilon\Delta \end{pmatrix}$$

and

$$F(y) = \begin{pmatrix} -\gamma(S)u_x^+ - [\gamma'(S)S_x + \mu^+(S, S_x)]u^+ + \mu^-(S, S_x)u^- \\ \gamma(S)u_x^- + [\gamma'(S)S_x - \mu^-(S, S_x)]u^- + \mu^+(S, S_x)u^+ \end{pmatrix},$$

where $S = S(u)$ and $u = u^+ + u^-$. We aim to solve this system in the Banach space

$$\mathscr{X}_2 := L^2 \cap \mathscr{C}_0^2 \quad \text{with norm } \|\cdot\|_{\mathscr{X}_2} := \|\cdot\|_2 + \|\cdot\|_{C^2}.$$

The operator A defines an analytic semigroup on both spaces L^2 and $\mathscr{C}_0$. To prove local existence it suffices to show that $F : H^1 \cap \mathscr{C}_0^1 \to L^2 \cap \mathscr{C}_0$ locally Lipschitz continuous (see, e.g., Henry [8]). By the definition of the norm on $\mathscr{X}_2$ it suffices to show the Lipschitz condition of $F : H^1 \to L^2$ and of $F : \mathscr{C}_0^1 \to \mathscr{C}_0$ separately. For this we consider $u^{\pm}, w^{\pm} \in H^1$ and study the first component of $\|F(u^+, u^-) - F(w^+, w^-)\|$. Since in F the nonlocal solution operator S appears it is necessary to consider this Lipschitz condition in detail. We obtain four terms:

Term 1. The bound of γ' is a bound for the Lipschitz constant of γ. Then we use (A5) and Lemma 3.1 to see

$$\begin{aligned}
&\| \gamma(S(u))u_x^+ - \gamma(S(w))w_x^+ \|_2 \\
&\quad \le \| \gamma(S(u)) - \gamma(S(w)) \|_\infty \|u_x^+\|_2 + \| \gamma(S(w)) \|_\infty \|u_x^+ - w_x^+\|_2 \\
&\quad \le C\|\gamma'\|_\infty \|u_x^+\|_2 \|u - w\|_2 + C_\gamma(1 + \|S(w)\|_\infty)\|u - w\|_{1,2} \\
&\quad \le C\big(C_\gamma, \|w\|_2, \|u^+\|_{1,2}\big)\|u - w\|_{1,2},
\end{aligned}$$

where $u = u^+ + u^-$, $w = w^+ + w^-$.

Term 2. From the second term we get three parts

$$\begin{aligned}
&\| \gamma'(S(u))S_x(u)u^+ - \gamma'(S(w))S_x(w)w^+ \|_2 \\
&\quad \le \| \gamma'(S(u)) - \gamma'(S(w)) \|_\infty \| S_x(u) \|_\infty \|u^+\|_2 \\
&\qquad + \| \gamma'(S(w)) \|_\infty \| S_x(u) - S_x(w) \|_\infty \|u^+\|_2 \\
&\qquad + \| \gamma'(S(w))S_x(w) \|_\infty \|u^+ - w^+\|_2 .
\end{aligned}$$

The first is estimated as above with the bound of γ'' and Lemma 3.1 for a bound on $S_x(u)$. The second part uses the global bound on γ' and again Lemma 3.1 together with the fact that the equation for s is linear. The third part is easily estimated by the assumptions and the estimates on S.

Terms 3, 4. The terms

$$\| \mu^{\pm}(S(u), S_x(u))u^+ - \mu^{\pm}(S(w), S_x(w))w^+ \|_2$$

can be handled with assumption (A4) and Lemma 3.1 similar to the terms 1 and 2. All together we have

$$\| F(u^+, u^-) - F(w^+, w^-) \|_2 \le C\| (u^+, u^-) - (w^+, w^-) \|_{1,2},$$

where the constant C depends on

$$C_\gamma, C_\mu, \|(u^+, u^-)\|_{1,2}, \|(w^+, w^-)\|_{1,2}.$$

Hence $F : (H^1)^2 \to (L^2)^2$ is locally Lipschitz continuous.

To show that $F : \mathscr{C}_0^1 \to \mathscr{C}_0$ is locally Lipschitz continuous we only need estimates for $S(u)$ similar to Lemma 3.1 in the case of $u \in \mathscr{C}_0^1$. It is known that the full analog of Lemma 3.1 only holds in Hölder spaces but here the following weaker version suffices: for $u \in C^1$ with $\|u\|_{C^1} < \infty$ the solution $S(u)$ of Eq. (14) satisfies $S(u) \in C^2$ and

$$\|S(u)\|_{C^2} \leq C\|u\|_{C^1},$$

where the constant C does not depend on u. The rest is straightforward and we get the following result.

THEOREM 3.1. *For initial values $u_0^\pm \in \mathscr{X}_2$ there exists a unique solution of* (16) *with*

$$(u^+, u^-) \in C\big([0, T), \mathscr{X}_2^2\big)$$

for some time $T > 0$.

3.2. L^1-*Estimates*

Since the original problem (1), (2), (3) is in conservation form the L^1-norm, which corresponds to the total population size, is a natural measure for this system. Hence we aim to estimate s and its derivatives in terms of the population size.

LEMMA 3.2. *If $u \in L^1$ then there is a solution $s = S(u) \in W^{2,1} \cap C^1(\mathbb{R})$ of* (14) *which satisfies*

$$\|s\|_{2,1} \leq C(\alpha, \beta)\|u\|_1, \qquad \|s\|_{C^1} \leq C(\alpha, \beta)\|u\|_1$$

and

$$\lim_{|x|\to\infty} s(x) = 0, \qquad \lim_{|x|\to\infty} s_x(x) = 0$$

for some constant $C = C(\alpha, \beta) > 0$.

Proof. From the Riemann–Lebesgue Lemma it follows that with $u \in L^1$ we have $\hat{u} \in C_{ub}(\mathbb{R})$ with $\lim_{|\kappa|\to\infty} \hat{u}(\kappa) = 0$ and $\|\hat{u}\|_\infty \leq \|u\|_1$. Fourier transformation of (14) gives (15), which shows that $\hat{s} \in C_{ub}(\mathbb{R})$ with $\lim_{|\kappa|\to\infty} \hat{s}(\kappa) = 0$. Integration of the modulus of Eq. (15) gives

$$\int |\hat{s}|\, d\kappa \leq \|\hat{u}\|_\infty \frac{\alpha}{\sqrt{\beta}} \arctan \frac{\kappa}{\sqrt{\beta}} \Bigg|_{-\infty}^{+\infty} \leq \frac{\alpha\pi}{\beta}\|\hat{u}\|_\infty. \tag{17}$$

Using the inverse Fourier-transformation for $\hat{s}$ we get $s \in C_{ub}$ with $\lim_{|x| \to \infty} s(x) = 0$ and

$$\|s\|_\infty \le \|\hat{s}\|_1 \le C\|\hat{u}\|_\infty \le C\|u\|_1.$$

If we denote the inverse Fourier transformation by $\check{}$ we get a representation of the solution of (14) as $s = (\hat{s})\check{}$. Then it directly follows that $s \in L^1$ and $\|s\|_1 \le C(\alpha, \beta)\|u\|_1$. From (14) we see that also $s_{xx} \in L^1$ and $\|s_{xx}\|_1 \le C(\alpha, \beta)\|u\|_1$. For the derivative of s in Fourier space we get Eq. (15) multiplied by κ:

$$\kappa\hat{s} = \frac{\kappa\alpha\hat{u}(\kappa)}{\kappa^2 + \beta}.$$

The Fourier transform of a characteristic function of an interval is bounded by C/κ for large κ. Since u is in L^1 we can approximate it by a sequence of step functions and the Fourier transformation of the sequence approximates $\hat{u}$ in C_{ub}. Hence $\hat{u}$ is bounded by C/κ for large κ. Therefore $\kappa\hat{s}$ is integrable and inverse Fourier-transformation gives $s_x \in C_{ub}$ with $\lim_{|x| \to \infty} s_x(x) = 0$. Using the same arguments as above we find $s_x \in L^1$ and $\|s_x\|_1 \le C(\alpha, \beta)\|u\|_1$. ∎

LEMMA 3.3. *Assume $u^{\pm}(t, x) \ge 0$ for all $(t, x) \in \Omega_T$. Then $s(t, x) \ge 0$ for all $(t, x) \in \Omega_T$.*

Proof. This a consequence of the elliptic maximum principle. ∎

LEMMA 3.4. *If $u_0^{\pm} \ge 0$ then the solution (u^+, u^-, s) with $u^{\pm} \in \mathscr{C}_0^2$ of (12) satisfies $u^{\pm}(t, x) \ge 0$ as long as it exists.*

Proof. Since in (A2) we assumed nonnegative turning rates this follows from the concept of invariant regions for parabolic systems (see, e.g., Chueh *et al.* [1]). ∎

From the fact that the particle densities are nonnegative it follows that the total particle size is preserved. This can be seen by integrating the first equation of (13):

$$\frac{d}{dt}N(t) = 0, \quad \text{with } N(t) = \int u(t, x)\, dx.$$

Hence

$$N(t) = N_0 = \int (u_0^+ + u_0^-)\, dx. \tag{18}$$

We again consider Eq. (3) for s separately with nonnegative u to give explicit values of the constant from Lemma 3.2 and its dependency on the parameters.

LEMMA 3.5. *For $u \in L^1$, $u \geq 0$ with $\|u\|_1 = N_0$ the solution $s = S(u)$ of* (14) *satisfies*:

(i) $\|s\|_1 = \frac{\alpha}{\beta} N_0$,

(ii) $\|s_{xx}\|_1 \leq \alpha(1 + \frac{1}{\beta}) N_0$,

(iii) $\|s_x\|_\infty \leq 2\alpha N_0$,

(iv) *If moreover $u \in L^\infty(\mathbb{R})$ then $s_{xx} \in L^\infty$ and*

$$\|s_{xx}\|_\infty \leq \frac{\alpha\pi}{\beta} N_0 + \beta \|u\|_\infty.$$

Proof. From Lemma 3.2 we get that $\lim_{|x|\to\infty}(|s(x)| + |s_x(x)|) = 0$. Then:

(i) Integration of (14) along $\mathbb{R}$ gives

$$0 = \beta \int s\,dx - \alpha \int u\,dx.$$

(ii) Integration of the absolute value of (14) and use of (i) gives (ii).

(iii) Integration of (14) along $(-\infty, x]$ gives

$$s_x(x) = \beta \int_{-\infty}^{x} s\,dx - \alpha \int_{-\infty}^{x} u\,dx.$$

If $s_x(x) \geq 0$ then with (i) it follows that $|s_x(x)| \leq \beta \|s\|_1 \leq \alpha \|u\|_1$. If $s_x(x) < 0$ then directly $|s_x(x)| \leq \alpha \|u\|_1$.

(iv) This follows directly from (14) and (17). ∎

3.3. *Global Existence in L^2*

We derive an energy estimate which shows that the L^2-norm of solutions cannot grow faster than exponentially. This directly leads to global existence.

THEOREM 3.2. *Consider initial conditions $u_0^+, u_0^- \in L^1 \cap L^2$ with $\int(u_0^+ + u_0^-)\,dx = N_0$ and assume* (A1)–(A5). *Then the solution (u^+, u^-) of* (12) *exists globally in $C([0,\infty), L^1 \cap L^2)^2$. Moreover there is a constant $K = K(C_\gamma, C_\mu, \alpha, \beta, N_0) > 0$ which is independent of $\varepsilon > 0$ such that for all $t \geq 0$,*

$$\|(u^+(t,\cdot), u^-(t,\cdot))\|_2 \leq \|(u_0^+, u_0^-)\|_2 e^{Kt}. \tag{19}$$

Proof. We use the equivalent (u, v) notation in (13). Due to Theorem 3.1 a solution (u, v) of (13) exists at least up to some time $T > 0$. Then for each $t \leq T$ we have

$$\frac{d}{dt}\|(u,v)(t,\cdot)\|_2^2 = 2\int (uu_t + vv_t)\,dx$$

$$= -2\varepsilon\int u_x^2\,dx - 2\int (u(\gamma v)_x + v(\gamma u)_x)\,dx$$

$$- 2\varepsilon\int v_x^2\,dx - \int \xi uv\,dx - \int \eta v^2\,dx$$

$$\leq -\int (2\gamma_x + \xi)uv\,dx,$$

since $\eta \geq 0$. From Lemma 3.4 it follows that $u^+ + u^- \geq |u^+ - u^-|$, hence

$$\frac{d}{dt}\|(u,v)(t,\cdot)\|_2^2 \leq \big(2\|\gamma' s_x(t,\cdot)\|_\infty + \|\xi(t,\cdot)\|_\infty\big)\|u(t,\cdot)\|_2^2.$$

With use of Lemma 3.5(iv) and assumptions (A4) and (A5) we find a constant $K = K(C_\gamma, C_\mu, \alpha, \beta, N_0)$ such that

$$\frac{d}{dt}\|(u,v)(t,\cdot)\|_2^2 \leq \frac{K}{2}\|u(t,\cdot)\|_2^2.$$

Finally Gronwall's Lemma shows (19). Now any solution which exists up to a possibly maximal time T can be continued to some time $T + \delta$. Hence solutions exist globally. ▮

COROLLARY 3.2. *If for* $\varepsilon = 0$ *we have* L^2*-solutions of the hyperbolic model* (1)–(3) *then estimate* (19) *holds as well.*

3.4. L^p-*Estimates*

Again we consider $S(u)$ to be the formal solution of the elliptic equation (14). We define

$$\Gamma(t,x) := \gamma(S(u(t,\cdot)(x))),$$
$$M^\pm(t,x) := \mu^\pm(S(u(t,\cdot))(x), S_x(u(t,\cdot))(x)).$$

From assumption (A4), (A5), and Lemma 3.5 it follows that

$$\begin{aligned} \|\Gamma(t,\cdot)\|_\infty + \|\Gamma_x(t,\cdot)\|_\infty &\leq C_\Gamma, \\ \|M^+(t,\cdot)\|_\infty + \|M^-(t,\cdot)\|_\infty &\leq C_M, \end{aligned} \tag{20}$$

with constants $C_\Gamma = C_\Gamma(C_\gamma, \alpha, \beta, N_0) > 0$ and $C_M = C_M(C_\mu, \alpha, \beta, N_0) > 0$.

Then system (12) reads

$$\begin{aligned} u_t^+ + (\Gamma u^+)_x - \varepsilon u_{xx}^+ &= -M^+u^+ + M^-u^- \\ u_t^- - (\Gamma u^-)_x - \varepsilon u_{xx}^- &= M^+u^+ - M^-u^- \\ u_0^\pm \in \mathscr{C}_0^2 \quad &\text{with compact support.} \end{aligned} \tag{21}$$

LEMMA 3.6. *Assume* (A1)–(A5). *Then solutions of* (21) *given by Theorems* 3.1 *and* 3.2 *satisfy* $u^\pm(t,\cdot) \in L^p$ *for all* $t \geq 0$.

Proof. For solutions $(u^+, u^-) \in (L^2 \cap \mathscr{C}_0^2)^2$ given by Theorems 3.1 and 3.2 the parameter functions $\Gamma(t,x)$ and $M(t,x)$ as defined above are fixed and bounded in L^∞ (see (20)). Then (21) is a linear parabolic equation with bounded coefficients. Since the initial conditions are in L^p for each $1 \leq p < \infty$ there exists a solution (w^+, w^-) in $L^p \cap \mathscr{C}_0^2$. Since solutions are unique in $\mathscr{C}_0^2$ this solution coincides with the known solution (u^+, u^-) in $L^2 \cap \mathscr{C}_0^2$ of (12). ∎

THEOREM 3.3. *Assume* (A1)–(A5). *Then there are constants* $C_1 = C_1(u_0^\pm)$ *and* $C_2 = C_2(C_\gamma, C_\mu, \alpha, \beta, N_0)$ *such that*

$$\|u^+(t,\cdot)\|_p + \|u^-(t,\cdot)\|_p \leq C_1 e^{C_2 t}, \tag{22}$$

for all even integers $p \geq 2$.

Proof. We will use the identity

$$\frac{d}{dx}(\Gamma u^{+,p}) = (1-p)\Gamma_x u^{+,p} + p(\Gamma u^+)_x u^{+,p-1},$$

where $u^{+,p}$ denotes $(u^+)^p$ and $p \in \mathbb{N}$ is assumed to be even. Multiplication of the first equation of (21) with $pu^{+,p-1}$ and integration along $\mathbb{R}$ leads to

$$\begin{aligned} &\frac{d}{dt}\|u^+(t,\cdot)\|_p^p + \int \frac{d}{dx}(\Gamma u^{+,p})\,dx + \int (p-1)\Gamma_x u^{+,p} \\ &\quad = p\int(-\mu^+ u^{+,p} + \mu^- u^- u^{+,p-1})\,dx - \varepsilon p(p-1)\int u^{+,p-2}(u_x^+)^2\,dx, \end{aligned}$$

where we omit the argument (t, x) on the right hand side. Since p is even the last term is negative. Using Hölder's inequality leads to

$$\frac{d}{dt}\|u^+(t,\cdot)\|_p^p \le (p-1)C_\Gamma\|u^+\|_p^p + pC_M\left(\|u^+\|_p^p + \int |u^-|\,|u^{+,p-1}|\,dx\right)$$
$$\le pC\big(\|u^+\|_p + \|u^-\|_p\big)\|u^+\|_p^{p-1},$$

where $C = C(C_\Gamma, C_M)$. If we assume $\|u^+(t,\cdot)\|_p \neq 0$ we get

$$\frac{d}{dt}\|u^+(t,\cdot)\|_p \le C\big(\|u^+\|_p + \|u^-\|_p\big).$$

A similar estimate holds for $\|u^-\|_p$. Then Gronwall's Lemma implies that

$$\|u^+(t,\cdot)\|_p + \|u^-(t,\cdot)\|_p \le \big(\|u_0^+\|_p + \|u_0^-\|_p\big)\,e^{C_2 t},$$

where $C_2 = 2C$. Since the initial data are assumed to have compact support we have

$$\|u_0^+\|_p + \|u_0^-\|_p \le \text{meas}\{\text{supp}(u_0^+) \cup \text{supp}(u_0^-)\}(\|u_0^+\|_\infty + \|u_0^-\|_\infty) =: C_1$$

and (22) follows. ∎

3.5. *Uniform L^∞-Estimate*

THEOREM 3.4. *Assume* (A1)–(A5). *Then there exists a global unique solution* (u^+, u^-, s) *of* (12) *with*

$$(u^+, u^-, s) \in C\Big([0,\infty), (L^1 \cap L^2 \cap L^\infty)^2 \times (L^1 \cap W^{2,2} \cap W^{2,\infty})\Big).$$

Moreover for each $T < \infty$ there is a constant $K(T)$ which depends on C_γ, C_μ, α, β, N_0, and T but not on $\varepsilon > 0$ such that for all $0 \le t \le T$

$$\|u^+(t,\cdot)\|_\infty + \|u^-(t,\cdot)\|_\infty \le K(T). \tag{23}$$

Proof. Since (22) holds for all even $p \in \mathbb{N}$ and the right hand side does not depend on p, estimate (23) is immediate. The boundedness of the solution $(u^+, u^-)(t,\cdot)$ in L^1 is the conservation of mass property. Existence in $L^2 \cap \mathscr{C}_0^2$ follows from Theorem 3.2. Boundedness of s in L^1 follows from Lemma 3.5, in $W^{2,2}$ from Lemma 3.1, and in $W^{2,\infty}$ from Lemma 3.5. All relevant constants are independent of $\varepsilon > 0$. ∎

3.6. *$W^{1,1}$-Estimates*

Theorem 3.4 ensures the existence of a family $\{(u^+, u^-, s)\}$ consisting of classical solutions of (12) for $\varepsilon > 0$. To establish ε-independent $W^{1,1}$-

estimates on the solutions $u^{\pm}$ let us first consider the subsequent linear Cauchy problem for $\tau \in (0, T]$ and $g_0 \in C_0^\infty(\mathbb{R})$

$$\begin{aligned} g_t^{\pm} \pm \gamma(s) g_x^{\pm} &= -\varepsilon g_{xx}^{\pm} \quad \text{in } (0,\tau) \times \mathbb{R}, \\ g^{\pm}(\tau, \cdot) &= g_0 \qquad\quad \text{on } \mathbb{R}. \end{aligned} \tag{24}$$

LEMMA 3.7. *For all* $g_0 \in C_0^\infty(\mathbb{R})$ *there exists a classical solution* $g^{\pm} \in C^2([0,\tau] \times \mathbb{R})$ *of* (24) *which satisfies*

$$|g^{\pm}(t,x)| \le \|g_0\|_{L^\infty}, \tag{25}$$

$$g^{\pm}(t,x), g_x^{\pm}(t,x) \xrightarrow{x \to \pm\infty} 0 \quad \textit{for all } t \in [0,\tau]. \tag{26}$$

Proof. $\gamma(s)$ is twice differentiable with respect to space and once with respect to time. These derivatives are bounded in $[0,\tau] \times \mathbb{R}$ due to Theorem 3.4 and (A5). Consequently a unique classical solution of (24) exists. The estimate follows from the maximum principle for parabolic equations. ∎

LEMMA 3.8. *There exists a constant* $C = C(C_\mu, C_\gamma, \|s\|_{W^{1,\infty}}) \ge 0$ *such that for all* $t \in [0, T]$

$$\|u^+(t,\cdot)\|_{W^{1,1}} + \|u^-(t,\cdot)\|_{W^{1,1}} \le C. \tag{27}$$

The estimate is independent of ε.

Proof. First we note that for solutions u^+, u^- given by Theorem 3.1 the L^1-norm is uniformly bounded with respect to $\varepsilon \in (0,1]$:

$$\|u^+(t,\cdot)\|_{L^1} + \|u^-(t,\cdot)\|_{L^1} \le C, \qquad 0 \le t \le T. \tag{28}$$

This is obvious in view of the positivity of u^-, u^+ and the fact that the total particle density $N(t) = \int_{\mathbb{R}} u^+(t,x) + u^-(t,x)\,dx$ is constant in time (and finite at $t = 0$).

To obtain $W^{1,1}$ estimates we differentiate the first equation of (12) once with respect to space. Setting $v^{\pm} = u_x^{\pm}$ we obtain

$$\begin{aligned} v_t^+ + (\gamma(s) v^+)_x = &-\big(\gamma''(s)(s_x)^2 + \gamma'(s) s_{xx}\big) u^+ - \gamma'(s) s_x v^+ \\ &- D_1 \mu^+(s, s_x) s_x u^+ - D_2 \mu^+(s, s_x) s_{xx} u^+ \\ &+ D_1 \mu^-(s, s_x) s_x u^- + D_2 \mu^-(s, s_x) s_{xx} u^- \\ &- \mu^+(s, s_x) v^+ + \mu^-(s, s_x) v^- + \varepsilon v_{xx}^+. \end{aligned} \tag{29}$$

Multiplication of (29) with g^+ and integration over $[0, \tau] \times \mathbb{R}$ leads to

$$\int_{\mathbb{R}} v^+(\tau, \cdot) g_0 = \int_{\mathbb{R}} u_0^{+\prime} g^+(0, \cdot) + \int_0^\tau \int_{\mathbb{R}} \big(\gamma''(s)(s_x)^2 + \gamma'(s) s_{xx}\big) u^+ g^+$$
$$+ \int_0^\tau \int_{\mathbb{R}} (-D_1 \mu^+(s, s_x) s_x - D_2 \mu^+(s, s_x) s_{xx}) u^+ g^+$$
$$+ \int_0^\tau \int_{\mathbb{R}} (D_1 \mu^-(s, s_x) s_x + D_2 \mu^-(s, s_x) s_{xx}) u^- g^+$$
$$+ \int_0^\tau \int_{\mathbb{R}} (\gamma'(s) s_x - \mu^+(s, s_x)) v^+ g^+ + \int_0^\tau \int_{\mathbb{R}} \mu^-(s, s_x) v^- g^+.$$

To obtain the last equation we used that g^+ satisfies Eq. (24) and vanishes together with its spatial derivative g_x^+ in $x = \pm\infty$ (cf. (26)). By Theorem 3.4 and the bound (25) we get

$$\left| \int_{\mathbb{R}} v^+(\tau, \cdot) g_0 \right|$$
$$\leq \Bigg(\|u_0^{+\prime}\|_{L^1} + C\big(C_\mu, C_\gamma, \|s\|_{W^{1,\infty}}\big) \|u^+\|_{L^1} + C\big(C_\mu, C_s\big) \|u^-\|_{L^1}$$
$$+ C\big(C_\mu, C_\gamma, \|s\|_{W^{1,\infty}}\big) \int_0^\tau \|v^+(t, \cdot)\|_{L^1}$$
$$+ C\big(C_\mu, \|s\|_{W^{1,\infty}}\big) \int_0^\tau \|v^-(t, \cdot)\|_{L^1} \Bigg) \|g^+\|_{L^\infty}.$$

Note that (30) holds for all $g_0 \in C_0^\infty(\mathbb{R})$. Since L^∞ is the dual space of L^1 we end up with

$$\|v^+(\tau, \cdot)\|_{L^1} \leq \|u_0^{+\prime}\|_{L^1}$$
$$+ C\big(C_\mu, C_\gamma, \|s\|_{W^{1,\infty}}\big) (\|u^+\|_{L^1} + \|u^-\|_{L^1})$$
$$+ C\big(C_\mu, C_\gamma, \|s\|_{W^{1,\infty}}\big) \int_0^\tau \|v^+(t, \cdot)\|_{L^1} + \|v^-(t, \cdot)\|_{L^1} \, dt.$$

Analogously we obtain for the second equation of (12) after differentiation with respect to space,

$$\|v^-(\tau, \cdot)\|_{L^1} \leq \|u_0^{-\prime}\|_{L^1} + C\big(C_\mu, C_\gamma, \|s\|_{W^{1,\infty}}\big) (\|u^+\|_{L^1} + \|u^-\|_{L^1})$$
$$+ C\big(C_\mu, C_\gamma, \|s\|_{W^{1,\infty}}\big) \int_0^\tau \|v^+(t, \cdot)\|_{L^1} + \|v^-(t, \cdot)\|_{L^1} \, dt.$$

The L^1 norm $\|u^+\|_{L^1} + \|u^-\|_{L^1}$ is known to be bounded uniformly in $0 < \varepsilon \le 1$ by (3.8). Since now the solution of the parabolic problem (12) defines a semigroup on $W^{1,1}$ for $u^\pm$ the norm $\|v^\pm(t,\cdot)\|_{L^1}$ is finite for each $0 \le t \le \tau$. Hence again Gronwall's inequality applies and we obtain a uniform bound of $\|v^+(\tau,\cdot)\|_{L^1} + \|v^-(\tau,\cdot)\|_{L^1}$ for all $\tau \in [0,T]$. The estimate (27) is proven. ∎

Finally we establish estimates of $u^\pm$ and s with respect to time t.

LEMMA 3.9. *There exists for each $\rho > 0$ a nondecreasing function $\omega_\rho^u \in C^0([0,\infty))$ with $\omega_\rho^u(0) = 0$ such that for each $\varepsilon \in (0,1]$ and for each t, $\Delta t \ge 0$ with $t, t + \Delta t \in [0,T]$ we have*

$$\int_{-\rho}^{\rho} \left| u^{\varepsilon\pm}(t+\Delta t, x) - u^{\varepsilon\pm}(t,x) \right| dx \le \omega_\rho^u(\Delta t). \tag{31}$$

Proof. We consider only $u^{\varepsilon+}$; the proof of $u^{\varepsilon-}$ will follow along the same lines. Let $g \in C_0^2(\mathbb{R})$ with $\operatorname{supp}(g) \subset [-\rho,\rho]$. Then we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} (u^{\varepsilon+}(t+\Delta t, x) - u^{\varepsilon+}(t,x)) g(x)\, dx \right| \\
&\quad = \left| \int_{\mathbb{R}} g(x) \int_t^{t+\Delta t} u_t^{\varepsilon+}(r,x)\, dr\, dx \right| \\
&\quad \le \left| \int_t^{t+\Delta t} \int_{\mathbb{R}} \gamma(s^\varepsilon) u^{\varepsilon+}(r,x) g_x(x)\, dx\, dr \right| \\
&\qquad + \left| \int_t^{t+\Delta t} \int_{\mathbb{R}} \mu^+(s^\varepsilon, s_x^\varepsilon) u^{\varepsilon+}(r,x) g(x)\, dx\, dr \right| \\
&\qquad + \left| \int_t^{t+\Delta t} \int_{\mathbb{R}} \mu^-(s^\varepsilon, s_x^\varepsilon) u^{\varepsilon-}(r,x) g(x)\, dr\, dx \right| \\
&\qquad + \left| \int_t^{t+\Delta t} \int_{\mathbb{R}} \varepsilon u^{\varepsilon+}(r,x) g_{xx}(x)\, dr\, dx \right| \\
&\quad \le \Delta t \rho C \|g\|_{C^2}.
\end{aligned}$$

The last line is a consequence of Theorem 3.4. Now, using convolution techniques as in the work of Kružkov [16, Lemma 5] the statement (31) follows from Lemma 3.8. ∎

The analogous statement for s is as follows.

LEMMA 3.10. *For each $\rho > 0$ there exists a nondecreasing function $\omega_\rho^s \in C^0([0,\infty))$ with $\omega_\rho^s(0) = 0$ such that for each $\varepsilon \in (0,1]$ and for each t,*

$\Delta t \geq 0$ *with* $t, t + \Delta t \in [0, T]$ *we have*

$$\int_{-\rho}^{\rho} |s^{\varepsilon}(t + \Delta t, x) - s^{\varepsilon}(t, x)| \, dx \leq \omega_{\rho}^{s}(\Delta t). \tag{32}$$

Proof. The proof is similar to the proof above. We show the analogous estimate for s and use Lemma 3.2. We choose $g \in C_0^2(\mathbb{R})$ with compact support in $[-\rho, \rho]$ and write G for the Green's function of the elliptic equation (14) for s,

$$\begin{aligned}
&\left| \int_{\mathbb{R}} (s^{\varepsilon}(t + \Delta t, x) - s^{\varepsilon}(t, x)) g(x) \, dx \right| \\
&\quad = \alpha \left| \int_t^{t+\Delta t} \int_{\mathbb{R}} \int_{\mathbb{R}} G(y) [\varepsilon u_{xx}^{\varepsilon} - (\gamma(s^{\varepsilon}) v^{\varepsilon})_x](r, x - y) g(x) \, dx \, dy \, dr \right| \\
&\quad \leq \alpha \left| \int_t^{t+\Delta t} \int_{\mathbb{R}} \int_{\mathbb{R}} G(y) \varepsilon u^{\varepsilon}(r, x - y) g''(x) \, dx \, dy \, dr \right| \\
&\qquad + \alpha \left| \int_t^{t+\Delta t} \int_{\mathbb{R}} \int_{\mathbb{R}} G(y) \gamma(s^{\varepsilon}) v^{\varepsilon}(r, x - y) g'(x) \, dx \, dy \, dr \right| \\
&\quad \leq \Delta t \rho C \|g\|_{C^2}.
\end{aligned}$$

▮

3.7. *Limit Process* $\varepsilon \to 0$/*Proof of Theorem* 2.1

LEMMA 3.11. *For all* $t \in [0, T]$ *there exists a function* $s(t, \cdot) \in C^1(\mathbb{R})$ *such that*

$$s^{\varepsilon_m}(t, \cdot) \to s(t, \cdot) \quad \text{in } C^1(\mathbb{R})$$

for some sequence $\varepsilon_m \to 0$ *with* $s(t, \cdot) \in W^{1,2}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ *for all* $t \in [0, T]$.

Proof. By Theorem 3.4 the set $\{s^{\varepsilon}(t, \cdot)\}$ is bounded in $W^{2,2}$ for each t and $0 < \varepsilon \leq 1$. The embedding $W^{2,2} \to C^1$ is compact and hence there is a convergent subsequence. The ε-independent estimates from Lemmas 3.1 and 3.2 ensure that s belongs to the spaces indicated. ▮

Now we are ready to present the proof of Theorem 2.1:

Proof. From Theorem 3.4 we know that for all $0 < \varepsilon \leq 1$ and all $T > 0$ there exists a classical solution $(u^{\varepsilon+}, u^{\varepsilon-}, s^{\varepsilon})$ of the parabolic Cauchy problem (12) which is uniformly bounded in $L^\infty(\overline{\Omega}_T)$. Consider now for $m \in \mathbb{N}$ a sequence ε_m with $\varepsilon_m \to 0$ for $m \to \infty$. Lemmata 3.8 and 3.9 imply that the sequences $\{u^{\varepsilon_m \pm}\}$ are precompact in $L^1_{loc}(\overline{\Omega}_T)$ by the Fréchet–Kolmogorov theorem. Similarly $\{s^{\varepsilon_m}\}$ is precompact by Lemmata 3.10 and 3.2. Using a standard diagonal extraction argument for an exhaustion of $\mathbb{R}$ we obtain subsequences—also labeled $\{u^{\varepsilon_m \pm}\}, \{s^{\varepsilon_m}\}$—and functions $u^\pm, s \in L^1_{loc}(\overline{\Omega}_T)$ with

$$\left.\begin{array}{c} u^{\varepsilon_m \pm} \to u^\pm \\ s^{\varepsilon_m} \to s \end{array}\right\} \quad \text{in } L^1_{loc}\big(\overline{\Omega}_T\big). \tag{33}$$

By standard theory the convergence is even pointwise a.e. for a suitable subsequence. By Lemma 3.11 we may assume that for almost all $t \in [0, T]$ we have $s(t, \cdot) \in W^{1,2}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ and

$$s^\varepsilon(t, \cdot) \to s(t, \cdot) \text{ a.e.} \qquad s^\varepsilon_x(t, \cdot) \to s_x(t, \cdot) \text{ a.e.} \tag{34}$$

The uniform L^1-bounds of $\{u^{\varepsilon_m \pm}(t, \cdot)\}$ imply that also $u^\pm(t, \cdot) \in L^1(\mathbb{R})$. Together with the uniform L^∞-bound we get $u^\pm \in L^\infty([0, T];\ L^1(\mathbb{R}))$. Next we show that the limit functions satisfy the weak formulation of the equations. Therefore we multiply the first two equations of (12) with a test function $\phi \in C_0^\infty(\Omega_T)$ and obtain after partial integration

$$\begin{aligned} -\int_{\overline{\Omega}_T} u^{\varepsilon_m \pm}\phi_t \pm \gamma(s^{\varepsilon_m})u^{\varepsilon_m \pm}\phi_x &= \int_{\overline{\Omega}_T} \big[\mp\mu^+(s^{\varepsilon_m}, s_x^{\varepsilon_m})u^{\varepsilon_m +} \pm \mu^-(s^{\varepsilon_m}, s_x^{\varepsilon_m})u^{\varepsilon_m -}\big]\phi \\ &\quad + \varepsilon_m \int_{\overline{\Omega}_T} u^{\varepsilon_m \pm}\phi_{xx}. \end{aligned} \tag{35}$$

The pointwise convergence properties as stated in (33), (34) and the smoothness of γ, $\mu^\pm$ now assure by Lebesgue's theorem that the limit (u^+, u^-, s) satisfies (8). Note that the last term in (35) vanishes in the limit due to the uniform L^∞-bound on $\{u^{\varepsilon_m \pm}\}$. The pointwise convergence of $\{u^{\varepsilon \pm}\}$ implies $u^\pm \in L^\infty(\overline{\Omega}_T)$ and especially the bounds (11) follow directly from Theorem 3.4.

The fact that $s(t, \cdot)$ is the weak solution of (3) for almost all $t \in [0, T]$ follows in a similar manner as (8). Consider for $t \in [0, T]$

$$\int_{\mathbb{R}} s^{\varepsilon_m}(t, \cdot)\psi_{xx} = \int_{\mathbb{R}} \big[\beta s^{\varepsilon_m}(t, \cdot) - \alpha\big(u^{\varepsilon_m +}(t, \cdot) + u^{\varepsilon_m -}(t, \cdot)\big)\big]\psi, \tag{36}$$

which trivially holds for all $\psi \in C_0^\infty(\mathbb{R})$ since s^{ε_m} is a classical solution of the associated equation in (12). With (33) and (34) the Lebesgue theorem implies that $s(t,\cdot)$ is a distributional solution of the elliptic equation (3). From Lemma 3.11 we also obtain $s(t,\cdot) \in W^{1,2}(\mathbb{R})$ which implies that $s(t,\cdot)$ is a weak solution of (3) since $C_0^\infty(\mathbb{R})$ is dense in $W^{1,2}(\mathbb{R})$.

To ensure the initial conditions (9), (10), first we define the set $\mathscr{L} \subset [0,T]$ such that for all $\tau \in [0,T] \setminus \mathscr{L}$ we have for almost all $x \in \mathbb{R}$ that (τ, x) is Lebesgue point of $u^\pm$ and s. This is a set of measure zero by the regularity properties of $u^\pm$ and s. We note that $u^\pm$ as solutions of a hyperbolic Cauchy problem with compactly supported initial data have compact support. Now let $\tau \in [0,T] \setminus \mathscr{L}$, $\rho > 0$. Then

$$\begin{aligned}
\int_{-\rho}^{\rho} \left| u^\pm(\tau,x) - u_0^\pm(x) \right| dx
&\le \int_{-\rho}^{\rho} \left| u^\pm(\tau,x) - u^{\varepsilon_m \pm}(\tau,x) \right| dx + \int_{-\rho}^{\rho} \left| u^{\varepsilon_m \pm}(\tau,x) - u_0^\pm(x) \right| dx \\
&\le \int_{-\rho}^{\rho} \left| u^\pm(\tau,x) - u^{\varepsilon_m \pm}(\tau,x) \right| dx + \omega_\rho^u(\tau).
\end{aligned}$$

The last estimate follows from Lemma 3.9. The pointwise convergence of $\{u^{\varepsilon_m \pm}\}$ shows

$$\int_{-\rho}^{\rho} \left| u^\pm(\tau,x) - u_0^\pm(x) \right| dx \le \omega_\rho^u(\tau). \tag{37}$$

The properties of ω_ρ^u now give (9) since $u^\pm$ has compact support.

It remains to prove the initial condition for s. Equation (3) for s is linear and so Lemma 3.1 shows

$$\begin{aligned}
\int_{-\infty}^{\infty} |s(\tau,x) - s_0(x)|^2\, dx \le C\bigg(&\int_{-\infty}^{\infty} |u^+(\tau,x) - u_0^+(x)|^2\, dx \\
&+ \int_{-\infty}^{\infty} |u^-(\tau,x) - u_0^-(x)|^2\, dx \bigg).
\end{aligned}$$

Due to the compact support of $u^\pm(t,\cdot)$ for $t \in [0,T]$ we can restrict the integration to a compact domain. Then, from the Cauchy–Schwarz inequality, the boundedness of $u^\pm(t,\cdot)$, $u^\pm$ in L^∞, and (37) we obtain (10). ∎

Note 3.5. We proved the existence of weak solutions by a compactness argument that does not allow us to obtain uniqueness of the solution. Anyhow, weak solutons of nonlinear hyperbolic balance laws are not unique in general. But, enforcing an additional entropy constraint on the weak solution, wellposedness within the associated class of (constrained) weak solutions can be achieved for scalar problems [16]. In our case, i.e., problem (1), (2), (3), (5), the corresponding entropy condition reads

$$\int_{\overline{\Omega}_T} |u^{\pm} - k|\phi_t \pm \operatorname{sgn}(u^{\pm} - k)\gamma(s)(u^{\pm} - k)\phi_x$$
$$\geq \int_{\overline{\Omega}_T} \operatorname{sgn}(u^{\pm} - k)\left[\gamma'(s)s_x \pm \mu^{+}(s, s_x)u^{+} \mp \mu^{-}(s, s_x)u^{-}\right]\phi$$
$$\forall \phi \in C_0^{\infty}(\Omega_T),\ \phi \geq 0, \forall k \in \mathbb{R}. \quad (38)$$

Using this condition and following the technique developed by Kružkov, we believe that uniqueness can be obtained for the weak solution that we obtained by the viscosity method. However, a detailed proof is out of the scope of this paper.

4. NUMERICAL EXAMPLES

In this section we illustrate the behavior of the solutions of the hyperbolic model for chemosensitive movement by a number of numerical experiments. To solve the hyperbolic equations (1), (2) we use an ENO-scheme of formally second order using the Engquist–Osher flux function [7]. The elliptic equation (3) is solved by a simple finite difference scheme.

EXAMPLE 1. We consider the time evolution of a swarm moving to the right, where the speed increases with increasing signal concentration s. The simulations demonstrate the fact that the function $\|u^{\pm}(t, \cdot)\|_{L^\infty}$ in general does not decrease with time and that $u^{\pm}$ can develop extremely steep gradients or possibly shocks for arbitrarily smooth initial data. This fact—similar to the behavior of purely nonlinear hyperbolic conservation laws—motivated us to consider weak solutions for (1)–(3).

From a biological perspective this behavior indicates that no individual leaves the swarm to the right. For smooth fronts there are always individuals, which might leave the swarm at its leading edge and eventually be captured by the swarm again. Here the right boundary of the swarm is sharp. This can be observed in nature in slug-swarms which are formed by *Dictyostelium discoideum* amoeba (see, e.g., Othmer and Schaap [19]).

For simplicity we take $\mu^{\pm} = 0$, $\beta = \alpha = 1.0$, and $u_0^- \equiv 0$, which leads to $u^- \equiv 0$ for $t \geq 0$. As initial datum for u_0^+ we choose a Gaussian distribution

$$u_0^+(x) = \exp\left(-(x - 4.0)^2\right), \qquad x \in [0, 10].$$

D is given by $D = 0.01$ and $\gamma(s) = s + 2$. The elliptic equation for s is solved for Dirichlet boundary conditions with $s(0) = s(10) = 0$. In Fig. 1 the results obtained for u^+ respectively s with 800 mesh points are shown at different time levels $t = 0, 0.5, 1.0, 1.5$. Steepening of the gradient for the right-moving population can clearly be observed. Furthermore the steep front leads also to a steep gradient of s which in turn produces the peak for u^+ behind the front becoming dramatic for $t > 0.5$.

EXAMPLE 2. In this simulation we illustrate the behavior of solutions of (1)–(3) for vanishing diffusion parameter D. In the limit case of $D = 0$ we

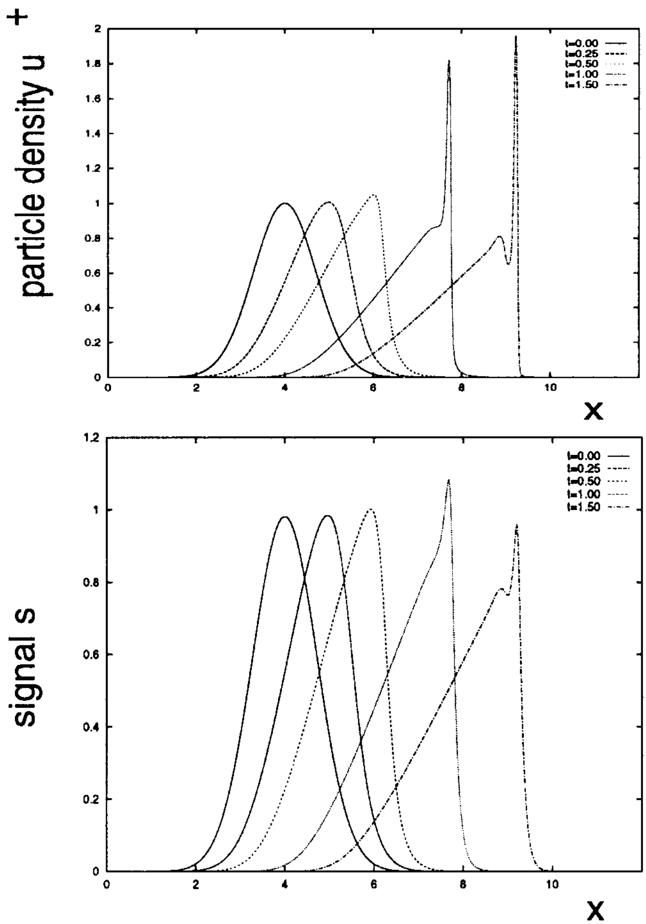

FIG. 1. The components u^+ (left) and s (right) for different time values.

obtain

$$s = \frac{\alpha}{\beta}(u^+ + u^-)$$

and a purely hyperbolic (2×2)-system for $u^\pm$ follows. Let $\mu^\pm = 0$, $\beta = \alpha = 1.0$, $u_0^- \equiv 0$, and

$$\gamma(s) = \frac{10s}{s+1} + 1.$$

Note that the quantity u^+ is then described for $D = 0$ by a scalar conservation law with convex flux for the range of the initial datum.

As initial datum for u_0^+ we choose

$$u_0^+(x) = \begin{cases} \exp\left(-400(x-0.2)^2\right), & -2 \leq x < 0.2, \\ 1, & 0.2 \leq x < 0.4, \\ \exp\left(-400(x-0.4)^2\right), & 0.4 \leq x < 2. \end{cases}$$

Since $u_{0,x}^+$ can become negative there exists a critical time such that the exact entropy solution for the hyperbolic conservation law exhibits a right-propagating shock beyond this critical time and is smooth otherwise. The elliptic equation for s is solved with Dirichlet boundary conditions $s(-2) = s(2) = 0$.

The calculations in Fig. 2 showing results for $t = 0.075$ on the interval $[0, 1]$ were performed on a grid with 1600 cells for $D = 0.0, 0.1, 0.01, 0.001,$

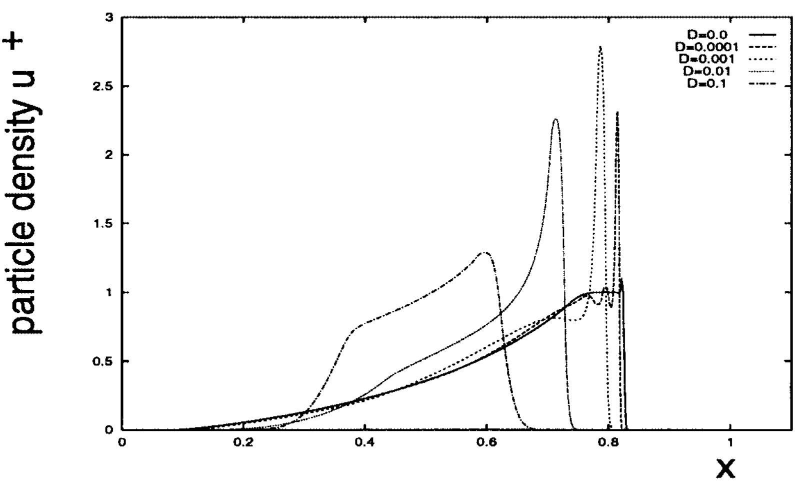

FIG. 2. The component u^+ for $t = 0.075$ and different values of D.

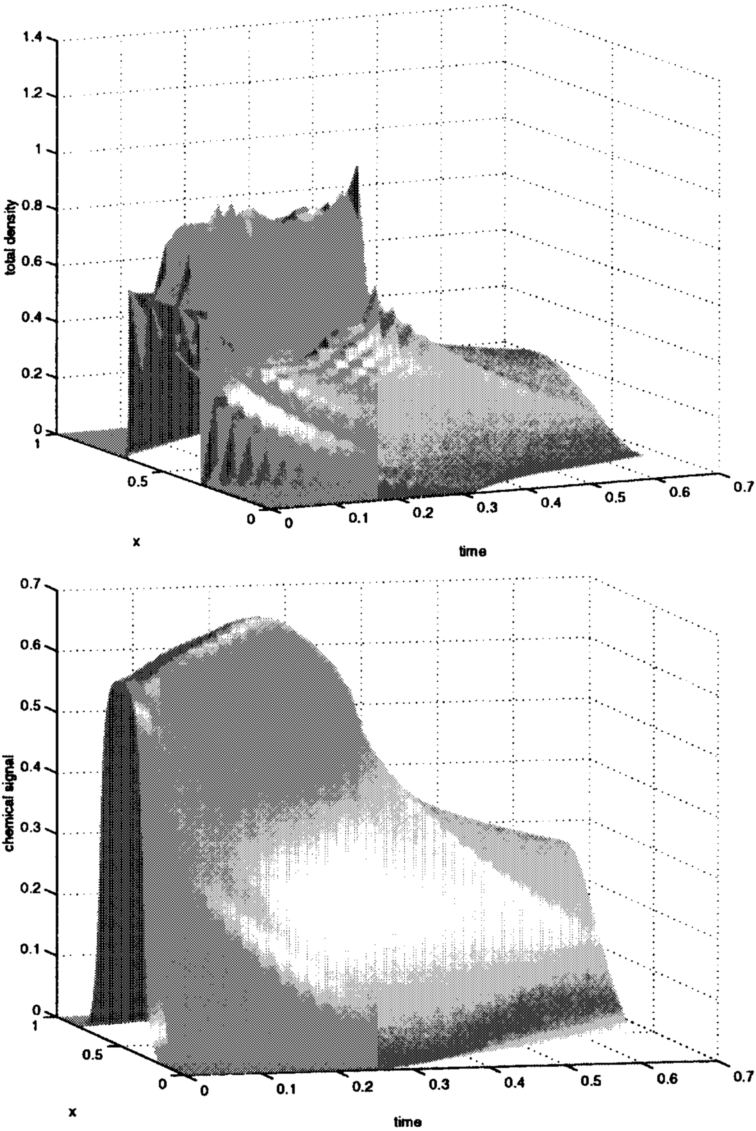

FIG. 3. Time evolution of the total density $u^+ + u^-$ (left) and of the signal distribution (right).

respectively 3200 cells for $D = 0.0001$. We carefully checked that further refinement of the grid does not change the results virtually. Note that the end time $t = 0.075$ is bigger than the critical time which can be easily derived from an analysis of the characteristic speeds.

We clearly obtain convergence to the entropy solution in the smooth parts for vanishing D. In the neighborhood of the shock we detect severe oscillations of the approximating sequence. Note that the estimates used to establish the existence of a weak solution of (1)–(3) are not uniform in the parameter D. The question about convergence/divergence of the sequence $\{u^{\pm, D}\}_{D>0}$ remains an interesting open problem.

EXAMPLE 3. In Hillen and Stevens [12] it has been predicted that a decreasing speed $\gamma = \gamma(s)$ can lead to aggregation. This can be observed in the following simulation of Fig. 3 only up to time $t = 0.32$. Suddenly the aggregation breaks down and the particle distribution adapts to the distribution of the external signal. This is a surprising effect but it can be explained as follows. The initial swarm starts to aggregate in the center of the interval and is able to establish a relatively high maximum for the signal distribution. Once an aggregation is established (at $t = 0.32$) the maximum of s decreases and finally the swarm spreads out.

The parameter values are

$$N = 200,\ T = 0.6,\ D = 2,\ \beta = 1000,\ \alpha = 1000,\ \mu^{+} = 10,\ \mu^{-} = 10,$$

$$\gamma(s) = \begin{cases} 1, & 0 \le s \le 0.2 \\ \dfrac{-0.5s + 0.49}{0.39}, & 0.2 < s < 0.59 \\ 0.5, & 0.59 \le s. \end{cases}$$

We checked carefully that this behavior occurs on finer grids and with similar shapes of $\gamma(s)$.

REFERENCES

1. K. N. Chueh, C. C. Conley, and J. A. Smoller, Positively invariant regions for systems of nonlinear diffusion equations, *Indiana Univ. Math. J.* **26** (1977), 373–392.
2. R. M. Ford, B. R. Phillips, J. A. Quinn, and D. A. Lauffenburger, Measurement of bacterial random motility and chemotaxis coefficients. I. Stopped-flow diffusion chamber assay, *Biotechnol. Bioengrg.* **37** (1991), 647–660.
3. J. I. Gallin and P. G. Quie, "Leukocyte Chemotaxis: Methods, Physiology, and Clinical Implications," Raven Press, New York, 1978.
4. S. Goldstein, On diffusion by discontinuous movements and the telegraph equation, *Quart. J. Mech. Appl. Math.* **4** (1951), 129–156.

5. K. P. Hadeler, Travelling fronts in random walk systems, *FORMA Japan* **10**, No. 3 (1995), 223–233.
6. K. P. Hadeler, Random walk systems and reaction telegraph equations, *in* "Dynamical Systems and Their Applications in Science" (S. v. Strien and S. V. Lunel, Eds.), Royal Academy of the Netherlands, 1996.
7. A. Harten, B. Engquist, S. Osher, and S. Chakravarthy, Uniformly high order accurate essentially non-oscillatory schemes, III, *J. Comput. Phys.* **271** (1987), 231–303.
8. D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Mathematics, Vol. 840, Springer-Verlag, Berlin, 1981.
9. M. Herrero and J. Velázquez, Singularity patterns in a chemotaxis model, *Math. Ann.* **306** (1996), 583–623.
10. T. Hillen, A Turing model with correlated random walk, *J. Math. Biol.* **35** (1996), 49–72.
11. T. Hillen, Invariance principles for hyperbolic random walk systems, *J. Math. Anal. Appl.* **210** (1997), 360–374.
12. T. Hillen and A. Stevens, Hyperbolic models for chemotaxis in 1-D, *Nonlinear Anal. Real World Appl's.* **1** (2000), 409–433.
13. E. E. Holmes, Are diffusion models too simple? A comparison with telegraph models of invasion, *Amer. Natur.* **142** (1993), 779–795.
14. M. Kac, A stochastic model related to the telegrapher's equation, *Rocky Mountain J. Math.* **4** (1956), 497–509.
15. E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.* **26** (1970), 399–415.
16. S. Kruẑkov, First order quasilinear equations in several independent variables, *Math. USSR Sb.* **10** (1970), 217–243.
17. H. A. Levine and B. D. Sleeman, A system of reaction diffusion equations arising in the theory of reinforced random walks, *SIAM J. Appl. Math.* **57** (1997), 683–730.
18. J. Müller and T. Hillen, Modulation equations and parabolic limits of reaction random walk systems, *Math. Methods Appl. Sci.* **21** (1998), 1207–1226.
19. H. G. Othmer and P. Schaap, Oscillatory cAMP signaling in the development of Dictyostelium discoideum, *Comments Theoret. Biol.* **5** (1998), 175–282.
20. H. G. Othmer and A. Stevens, Aggregation, blowup and collapse: The ABC's of taxis in reinforced random walks, *SIAM J. Appl. Math.* **57** (1997), 1044–1081.
21. C. Rohde, Entropy solutions for weakly coupled hyperbolic systems in several space dimensions, *Z. Angew. Math. Phys.* **49** (1998).
22. G. Schneider and J. Müller, Bifurcations analysis for a spatially extended random walk system in mathematical biology, *Comm. Appl. Anal.* **5** (2001), 247–264.
23. L. A. Segel, A theoretical study of receptor mechanisms in bacterial chemotaxis, *SIAM Appl. Math.* **32** (1977), 653–665.
24. T. Senba, Blow-up of radially symmetric solutions to some systems of partial differential equations modelling chemotaxis, *Adv. Math. Sci. Appl.* **7** (1997), 79–92.
25. D. R. Soll, Behavioral studies into the mechanism of eukaryotic chemotaxis, *J. Chem. Ecol.* **16** (1990), 133–150.
26. A. Vol'pert, The spaces *BV* and quasilinear equations, *Math. USSR Sb.* **2** (1967), 225–267.