Existence Theory for Correlated Random Walks on Bounded Domains

T. Hillen*

May 12, 2009

Abstract: In this paper we present a comprehensive existence theory for linear and nonlinear reaction random walk systems. The methods are based on semigroup theory for solutions of differential equations on Banach spaces. The solution properties on a bounded domain sensitively depend on the choice of boundary conditions. For Neumann or for periodic boundary conditions, singularities are transported along characteristics and the solutions form a group. Surprisingly, for Dirichlet boundary conditions, singularities are washed out, the problem regularizes in finite time, and the solution operator forms a semigroup.

Furthermore, we study the relation to damped wave equations and reaction-telegraph equations. The relation between random walk models and telegraph equations for Neumann and periodic boundary conditions requires a compatibility condition of the initial condition. For Dirichlet boundary conditions, however, there is no direct relation between the random walk model and the telegraph equation.

1 Introduction

The detailed investigation of partial differential equations (PDE) models for correlated random walks has flourished in the 1990th and many theoretical results and specific applications have been studied (see [16], [21]). The current paper is based on my thesis from 1995 [17], which is written in German. It contains results which are largely unknown and have not been published previously. In continued work on these systems, it turns out that the semigroup solution theory as well as the regularization property of the Dirichlet problem are needed for many follow up studies. Also, the

*University of Alberta, Centre for Mathematical Biology, Department of Mathematical and Statistical Sciences, thillen@ualberta.ca, supported by NSERC and MITACS

relation to telegraph equations is discussed by many scholars. The results presented here give a definitive answer.
The original model (2) was introduced by Goldstein and Kac [11, 26]. The Goldstein-Kac model (2) is a linear system of hyperbolic PDE and the existence theory on an unbounded domain $\mathbb{R}$ is clear (see for example Bressan [4]). Existence and uniqueness of solutions on bounded domains is less clear and we use semigroup theory to prove local and global existence.

The one-dimensional model for correlated random walk and generalizations have been used extensively for modelling of biological processes. A comprehensive review including applications to edpidemic spread, to chemotaxis, Turing pattern formation and travelling waves can be found in the CIME lecture notes of Hadeler [16], see also [18, 19, 23, 22]. Models for alignment were studied by Lutscher [31]. A comprehensive study of pattern formation under non-local aggregation and alignment terms was studied recently by Eftimie and co-workers in [6, 7].

The paper is organized as follows. Following this introduction we will introduce the main model in Section 2 and discuss local and global existence of weak and classical solutions in Section 3. It is well accepted that hyperbolic systems do not regularize (in contrast to parabolic systems) and solutions are as smooth as the initial conditions. We show in Section 4 that this is true for periodic and for Neumann boundary conditions, and that it is not true for Dirichlet boundary conditions. For Dirichlet boundary conditions, singularities are washed out and solutions regularize in finite time. This is a quite surprising result and it shows how sensitive these models are to boundary conditions. In Section 5 we introduce kinetic terms which describe birth and death events. These terms lead to so called nonlinear *reaction-random walk equations* (RRWE). We show in Section 6 that these are closely related to *reaction telegraph equations* (RTE). If we only look at the equations (and not the boundary conditions), then we see a close correlation between these models. For each solution of RRWE we find a solution of RTE, and for each solution of RTE there is a one-parameter family of solutions to RRWE. This relation changes if boundary conditions are included. For Neumann and for periodic boundary conditions an additional compatibility condition appears. Surprisingly, for the Dirichlet problem the above relation is untrue.
Besides the two main results mentioned above, this paper also provides an explicit solution based on Bessel functions (Section 3), and summarizes results on the existence of solutions for damped wave equations (Section 7).

2 One-dimensional Correlated Random Walk

For one space dimension Taylor [43] considers the following random walk process:

Suppose that a point starts moving with uniform velocity γ along a line, and that after a time τ it suddenly makes a fresh start and either continues moving forward with velocity γ or reverses its direction and moves back over the same path with the same velocity γ.

This form of random walk can be formulated on a spatial grid. A particle travels $\rho = \gamma\tau$ per unit of time τ and changes direction with probability $q = \mu\tau$. It keeps the direction with probability $p = 1 - \mu\tau$. We denote by $u_n^+(x)$ the density of right moving particles and with $u_n^-(x)$ the density of left moving particles at location x at time $t = n\tau$. This random walk can be described through the following Master equation

$$\begin{aligned} u_n^+(x) &= p\, u_{n-1}^+(x-\rho) + q\, u_{n-1}^-(x-\rho) \\ u_n^-(x) &= p\, u_{n-1}^-(x+\rho) + q\, u_{n-1}^+(x+\rho). \end{aligned} \tag{1}$$

We divide this equation by τ and consider the limit $\tau \to 0$. Then $\rho \to 0$ and $p/\tau \to \mu$. If we define

$$u(t,x) := \lim_{\tau\to 0, n\tau = t} u_n^+(x), \quad u^-(t,x) := \lim_{\tau\to 0, n\tau = t} u_n^-(x)$$

then $u^{\pm}(x,t)$ satisfy the following *hyperbolic system of correlated random walk.*

$$\begin{aligned} u_t^+(t,x) + \gamma u_x^+(t,x) &= \mu(u^-(t,x) - u^+(t,x)) \\ u_t^-(t,x) - \gamma u_x^-(t,x) &= \mu(u^+(t,x) - u^-(t,x)). \end{aligned} \tag{2}$$

Here we used index notation for partial derivatives.
The model can be understood in terms of death and birth events. A particle moving to the right dies with rate μ and is reborn as a left moving particle with the same rate.

2.1 Boundary Conditions

We study (2) on an interval $[0,1]$. We see that the classical Dirichlet, Neumann and periodic boundary conditions for this hyperbolic system need to be appropriately adapted to the hyperbolic model. For given $0 < T \leq \infty$ we denote the spatio-temporal domain $\Omega_T := [0,T) \times [0,l]$ and we define its "hyperbolic boundary".

Definition 2.1 The *hyperbolic boundary* of Ω_T is defined as

$$\begin{aligned} \partial^+\Omega_T &:= \{0\} \times [0,l] \,\cup\, [0,T) \times \{0\}, \\ \partial^-\Omega_T &:= \{0\} \times [0,l] \,\cup\, [0,T) \times \{l\}. \end{aligned}$$

The hyperbolic system (2) has two families of characteristics, $x+\gamma t$ and $x-\gamma t$. Hence we impose conditions for u^+ on $\partial^+\Omega_T$ and conditions for u^- on $\partial^-\Omega_T$ (see. Fig. 1)

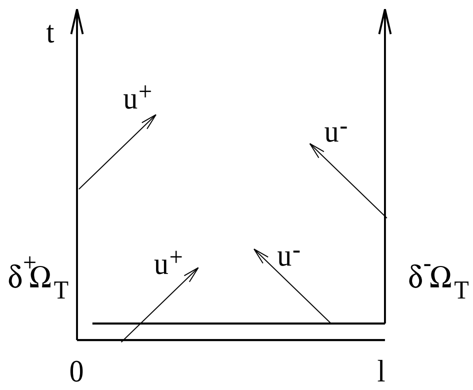

Figure 1: Hyperbolic boundary.

At $t = 0$ we impose initial conditions for both compartments

$$u^+(0, x) = u_0^+(x),\ u^-(0, x) = u_0^-(x), \quad \forall\, x \in [0, l] \tag{3}$$

The boundary conditions will be chosen to describe the physical situation of classical Dirichlet, Neumann and periodic boundary conditions.

- **Homogeneous Dirichlet boundary conditions** describe a domain which is open at the boundary. Particles can leave the domain but no particle can enter from the outside. Hence

$$u^+(t, 0) = 0,\ u^-(t, l) = 0. \tag{4}$$

- **Homogeneous Neumann boundary conditions** describe a closed domain, where no particle can leave. Hence particles are reflected

$$u^+(t, 0) = u^-(t, 0), \quad u^-(t, l) = u^+(t, l). \tag{5}$$

- **Periodic boundary conditions** are straightforward

$$u^+(t, 0) = u^+(t, l), \quad u^-(t, l) = u^-(t, 0). \tag{6}$$

We combine these three and more general boundary conditions as

$$u^+(t, 0) = \chi_0(u^+(t, l), u^-(t, 0)), \quad u^-(t, l) = \chi_l(u^+(t, l), u^-(t, 0)). \tag{7}$$

where $\chi_0(\alpha, \beta) = \chi_l(\alpha, \beta) = 0$ for Dirichlet, $\chi_0(\alpha, \beta) = \beta$, $\chi_l(\alpha, \beta) = \alpha$ for Neumann and $\chi_0(\alpha, \beta) = \alpha$, $\chi_l(\alpha, \beta) = \beta$ for periodic boundary conditions.

3 Existence and Uniqueness

To show existence and uniqueness for equation (2) with the above boundary conditions, we use semigroup theory and apply the Lumer-Phillips theorem (see Pazy [37]).

We define two operator matrices as

$$G := \gamma \begin{pmatrix} -D_x & 0 \\ 0 & D_x \end{pmatrix}, \qquad B := \begin{pmatrix} -\mu & \mu \\ \mu & -\mu \end{pmatrix}, \tag{8}$$

where D_x denotes the partial differential operator with respect to $x \in \mathbb{R}$. We define $y := (u^+, u^-)$ and (2) reads

$$y_t = Gy + By. \tag{9}$$

For $1 \le p \le \infty$ we denote

$$\mathcal{L}^p := (L^p([0,l]))^2$$

with norm

$$\|y\|_p = \|(y_1, y_2)\|_p := \begin{cases} \left(\|y_1\|^p_{L^p([0,l])} + \|y_2\|^p_{L^p([0,l])}\right)^{1/p} & \text{f'ur} \quad 1 \le p < \infty, \\ \max(\|y_1\|_\infty, \|y_2\|_\infty) & \text{f'ur} \quad p = \infty \end{cases}$$

The dual of $\mathcal{L}^p$ is $\mathcal{L}^q$ with $q = p/(p-1)$ for $1 < p < \infty$ and $q = \infty$ for $p = 1$. We will prove that G is generator of a semigroup for $1 \le p < \infty$. The space $\mathcal{L}^\infty$ is only used as dual of $\mathcal{L}^1$.
For $1 \le k \in \mathbb{N}$ we denote the Sobolev spaces as

$$\mathcal{W}^{k,p} := (W^{k,p}([0,l]))^2,$$

with norm

$$\|y\|_{k,p} = \sum_{j=0}^{k} \|D_x^j y\|_p.$$

For $1 \le p < \infty$ we denote the domain of definition of G as

$$\mathcal{D}(G) := \{y \in \mathcal{W}^{1,p} : \ y_1(0) = \chi_0(y_1(l), y_2(0)), \ y_2(l) = \chi_l(y_1(l), y_2(0))\} \tag{10}$$

with χ_0 and χ_l as in (7). The domain of definition $\mathcal{D}(G)$ is dense $\mathcal{L}^p$. The norm on $\mathcal{D}(G)$ is induced by $\mathcal{W}^{1,p}$ and corresponds to the graph norm of G, i.e. $\|y\|_{\mathcal{D}(G)} = \|y\|_{\mathcal{L}^p} + \|D_x y\|_{\mathcal{L}^p} \equiv \|y\|_{\mathcal{L}^p} + \frac{1}{\gamma}\|Gy\|_{\mathcal{L}^p}$. Hence G is a closed operator.
We first study the spectrum of G:

Lemma 3.1 *The spectrum of $G : \mathcal{D}(G) \to \mathcal{L}^p$ for $1 \le p < \infty$ is given as follows*

- *Dirichlet boundary conditions* (4)*:*

$$\sigma(G) = \emptyset$$

- *Neumann boundary conditions* (5)*:*

$$\sigma(G) = \{k\frac{\gamma\pi}{l}i : \ k \in \mathbb{Z}\}$$

- *Periodic boundary conditions* (6)*:*

$$\sigma(G) = \{2k\frac{\gamma\pi}{l}i : \ k \in \mathbb{Z}\}$$

Proof. Consider $\lambda \in \mathbf{C}$ and let $1 \le p < \infty$. If for each $z \in \mathcal{L}^p$ there exists a unique $y \in \mathcal{D}(G)$ with $(\lambda I - G)y = z$ and if the resolvent $R(\lambda, G) = (\lambda I - G)^{-1} : \mathcal{L}^p \to \mathcal{D}(G)$ is bounded, then $\lambda \in \rho(G)$.

Consider $z \in \mathcal{L}^p$. The resolvent equation reads

$$(\lambda I - G)y = z \iff \left\{ \begin{array}{rcl} y_1' &=& -\frac{\lambda}{\gamma}y_1 + \frac{z_1}{\gamma} \\ y_2' &=& \frac{\lambda}{\gamma}y_2 - \frac{z_2}{\gamma}, \end{array} \right. \tag{11}$$

The solution can be written explicitly using the boundary conditions (7) as

$$y_1(x) = e^{-\frac{\lambda}{\gamma}x}\chi_0(y_1(l), y_2(0)) + \frac{1}{\gamma}\int_0^x e^{\frac{\lambda}{\gamma}(\xi - x)} z_1(\xi) d\xi \tag{12}$$

$$y_2(x) = e^{\frac{\lambda}{\gamma}(x-l)}\chi_l(y_1(l), y_2(0)) + \frac{1}{\gamma}\int_x^l e^{-\frac{\lambda}{\gamma}(\xi - x)} z_2(\xi) d\xi. \tag{13}$$

We abbreviate

$$\varepsilon(\lambda) := e^{-\frac{\lambda}{\gamma}l}, \quad K_1 := \frac{1}{\gamma}\int_0^l e^{\frac{\lambda}{\gamma}(\xi - l)} z_1(\xi) d\xi, \quad K_2 := \frac{1}{\gamma}\int_0^l e^{-\frac{\lambda}{\gamma}\xi} z_2(\xi) d\xi. \tag{14}$$

We evaluate (12) at $x = l$ and (13) at $x = 0$ and find a linear system for the unknown $y_1(l)$ and $y_2(0)$.

$$\begin{array}{rcl} y_1(l) &=& \varepsilon(\lambda)\,\chi_0(y_1(l), y_2(0)) + K_1 \\ y_2(0) &=& \varepsilon(\lambda)\,\chi_l(y_1(l), y_2(0)) + K_2 \end{array} \tag{15}$$

- Dirichlet ($\chi_0(y_1(l), y_2(0)) = 0$, $\chi_l(y_1(l), y_2(0)) = 0$): Equation (15) reads in this case

  $$y_1(l) = K_1 \quad \text{and} \quad y_2(0) = K_2. \tag{16}$$

  Hence the resolvent equation (11) has a unique solution for alle $\lambda \in \mathbf{C}$. Hence the set $\Sigma_D := \emptyset$ is a candidate for the spectrum of G.

- Neumann ($\chi_0(y_1(l), y_2(0)) = y_2(0),\ \chi_l(y_1(l), y_2(0)) = y_1(l)$): In this case equation (15) becomes

$$y_1(l) = \varepsilon y_2(0) + K_1, \qquad y_2(0) = \varepsilon y_1(l) + K_2.$$

  This system has a unique solution if

$$\det \begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix} \neq 0 \iff \varepsilon^2 \neq 1 \iff \lambda \notin \Sigma_N := \left\{ k\frac{\gamma\pi}{l} i : \ k \in \mathbb{Z} \right\}.$$

  The corresponding solution reads

$$y_1(l) = \frac{K_1 + \varepsilon K_2}{1 - \varepsilon^2}, \qquad y_2(0) = \frac{K_2 + \varepsilon K_1}{1 - \varepsilon^2}. \tag{17}$$

- Periodic ($\chi_0(y_1(l), y_2(0)) = y_1(l),\ \chi_l(y_1(l), y_2(0)) = y_2(0)$): From (15) we get

$$(1 - \varepsilon) y_1(l) = K_1, \qquad (1 - \varepsilon) y_2(0) = K_2,$$

  which has a unique solution if

$$\varepsilon \neq 1 \iff \lambda \notin \Sigma_P := \left\{ 2k\frac{\gamma\pi}{l} i : \ k \in \mathbb{Z} \right\}.$$

  The corresponding solution is

$$y_1(l) = \frac{K_1}{1 - \varepsilon}, \qquad y_2(0) = \frac{K_2}{1 - \varepsilon}. \tag{18}$$

To ensure that the sets Σ_D, Σ_N and Σ_P are indeed the spectrum of G, we need to show that the resolvent is continuous for the complementary set, respectively.
Hence we need to show that each solution y of the resolvent equation (11) satisfies

$$\|y\|_{\mathcal{D}(G)} \leq c(\lambda, p) \|z\|_p$$

with a constant c, which might depend on λ und p. We first study the first component y_1. We get from (12) that

$$\|y_1\|_{L^p} \leq \underbrace{\left\| e^{-\frac{\lambda}{\gamma} x} \chi_0(y_1(l), y_2(0)) \right\|_{L^p}}_{T_1} + \underbrace{\frac{1}{\gamma} \left\| \int_0^x e^{\frac{\lambda}{\gamma}(\xi - x)} z_1(\xi) d\xi \right\|_{L^p}}_{T_2}. \tag{19}$$

The first term T_1 is studied for each boundary condition separately.

- Dirichlet: $T_1 = 0$.

- Neumann: Let $\lambda \notin \Sigma_N$. From (17) we obtain

$$T_1 = \left\| e^{-\frac{\lambda}{\gamma}x} \frac{K_2 + \varepsilon K_1}{1-\varepsilon^2} \right\|_{L^p} \leq \left| \frac{K_2 + \varepsilon K_1}{1-\varepsilon^2} \right|.$$

- Periodic: Let $\lambda \notin \Sigma_P$. From (18) we get

$$T_1 = \left\| e^{-\frac{\lambda}{\gamma}x} \frac{K_1}{1-\varepsilon} \right\|_{L^p} \leq \left| \frac{K_1}{1-\varepsilon} \right|.$$

The constants K_1 and K_2 from (14) are bounded by the norm of z as follows. For $1 < p < \infty$ we use Hölders inequality and get

$$|K_1| \leq c_1(\lambda, p)\|z_1\|_{L^p}, \quad \text{with} \quad c_1(\lambda, p) := \frac{1}{\gamma}\left\| e^{\frac{\lambda}{\gamma}(\xi - l)} \right\|_{L^q} \tag{20}$$

and

$$|K_2| \leq c_1(\lambda, p)\|z_2\|_{L^p}$$

with the same constant c_1.
For $p = 1$ we obtain

$$\begin{aligned} |K_1| &\leq \frac{1}{\gamma} \max_{\xi \in [0,l]} |e^{\frac{\lambda}{\gamma}(\xi - l)}| \, \|z_1\|_{L^1} = \frac{1}{\gamma}\|z_1\|_{L^1}, \\ |K_2| &\leq \frac{1}{\gamma} \max_{\xi \in [0,l]} |e^{-\frac{\lambda}{\gamma}\xi}| \, \|z_2\|_{L^1} = \frac{1}{\gamma}\|z_2\|_{L^1}. \end{aligned}$$

Hence for all $1 \leq p < \infty$ and the three boundary conditions we find $T_1 \leq c_3(\lambda, p)\|z\|_p$.

Now we study T_2. Again using Hölders inequality we find for each $x \in [0, l]$

$$\begin{aligned} \frac{1}{\gamma}\int_0^x e^{\frac{\lambda}{\gamma}(\xi - x)} z_1(\xi) d\xi &\leq \frac{1}{\gamma}\|e^{\frac{\lambda}{\gamma}(\xi - x)}\|_{L^q([0,x])} \, \|z_1\|_{L^p([0,x])} \\ &\leq \frac{1}{\gamma}\|e^{\frac{\lambda}{\gamma}(\xi - l)}\|_{L^q([0,l])} \, \|z_1\|_{L^p([0,l])} \\ &= c_1(\lambda, p)\|z_1\|_{L^p} \end{aligned}$$

with the same constant c_1 as in (20). Hence $T_2 \leq c_1(\lambda, p)\|z_1\|_{L^p}$.

Together we find $\|y_1\|_{L^p} \leq c_4(\lambda, p)\|z\|_p$ for all $1 \leq p < \infty$ and alle three boundary conditions, respectively. Similarly we obtain $\|y_2\|_{L^p} \leq c_5(\lambda, p)\|z\|_p$.
Since y satisfies the resolvent equation (11) we can estimate $D_x y$ through the norms of y and z.

$$\|D_x y\|_p \leq c_6(\lambda, p)\|y\|_p + c_7(\lambda, p)\|z\|_p \leq c_8(\lambda, p)\|z\|_p.$$

Finally we see

$$\|y\|_{\mathcal{D}(G)} = \|y\|_p + \gamma \|D_x y\|_p \leq c(\lambda, p) \|z\|_p.$$

Hence the resolvent $R(\lambda, G)$ is continuous for all $\lambda \notin \Sigma_i, i = D, N, P$, respectively. *qed.*

We recall a definition from Pazy [37].

Definition 3.1 Let E be a Banach space and $A : \mathcal{D}(A) \to E$ a linear operator. A is *dissipative*, if for each $y \in \mathcal{D}(A)$ there exists a continuous linear form $z \in E'$ with $\|z\|_{E'}^2 = \langle z, y \rangle = \|y\|_E^2$ and $\mathrm{Re}\,(\langle z, Ay \rangle) \leq 0$.

Lemma 3.2 *The operator $G : \mathcal{D}(G) \to \mathcal{L}^p$, equipped with Dirichlet, Neumann, or periodic boundary conditions, respectively, is dissipative.*

Proof. The action of a linear form z on y can be represented through integration

$$\langle z, y \rangle := \int_0^l (z_1 y_1 + z_2 y_2) dx. \tag{21}$$

We show for each $1 \leq p < \infty$ the following statement: For each $y \in \mathcal{D}(G)$ there exists a continuous linear form $z \in \mathcal{L}^q$ with $\|z\|_q^2 = \langle z, y \rangle = \|y\|_p^2$, such that $\langle z, Gy \rangle \leq 0$.

Consider $y \in \mathcal{D}(G)$.

Case 1: $1 < p < \infty$. We denote $\tilde{z} := (y_1 |y_1|^{p-2}, y_2 |y_2|^{p-2})$ and observe that

$$\begin{aligned} \|\tilde{z}\|_q^q &= \textstyle\int_0^l \left| y_1 |y_1|^{p-2} \right|^q dx + \int_0^l \left| y_2 |y_2|^{p-2} \right|^q dx \\ &= \textstyle\int_0^l |y_1|^{(p-1)q} dx + \int_0^l |y_2|^{(p-1)q} dx \\ &= \|y\|_p^p \end{aligned} \tag{22}$$

and

$$\langle \tilde{z}, y \rangle = \int_0^l y_1 |y_1|^{p-2} y_1 dx + \int_0^l y_2 |y_2|^{p-2} y_2 dx = \int_0^l |y_1|^p dx + \int_0^l |y_2|^p dx = \|y\|_p^p. \tag{23}$$

With $c := \|y\|_p$ and $z := c^{2-p}\, \tilde{z}$ we obtain with use of (22)

$$\|z\|_q^2 = c^{2(2-p)} \|\tilde{z}\|_q^2 = c^{2(2-p)} (\|y\|_p^p)^{2/q} = c^{2(2-p)+2p/q} = c^2 = \|y\|_p^2.$$

From (23) we get

$$\langle z, y \rangle = c^{2-p} \langle \tilde{z}, y \rangle = c^{2-p} \|y\|_p^p = \|y\|_p^2.$$

Now we apply z to Gy

$$\begin{aligned} c^{p-2}\, \langle z, Gy \rangle = \langle \tilde{z}, Gy \rangle &= \int_0^l y_1 |y_1|^{p-2} (-\gamma D_x y_1) dx + \int_0^l y_2 |y_2|^{p-2} (\gamma D_x y_2) dx \\ &= -\frac{\gamma}{p} \int_0^l D_x |y_1|^p dx + \frac{\gamma}{p} \int_0^l D_x |y_2|^p dx \\ &= \frac{\gamma}{p} \big(|y_1(0)|^p - |y_1(l)|^p + |y_2(l)|^p - |y_2(0)|^p \big). \end{aligned} \tag{24}$$

We will study this equality for the three boundary conditions separately later.

Case 2: $p = 1$ Here we introduce $\tilde{z} := (\tilde{z}_1, \tilde{z}_2)$ with

$$\tilde{z}_i = \begin{cases} y_i \, |y_i|^{-1} & \text{for} \quad y_i \neq 0 \\ 1 & \text{for} \quad y_i = 0 \end{cases} \qquad i = 1, 2.$$

We have

$$\|\tilde{z}\|_\infty = \max\left(\left\| \frac{y_1}{|y_1|} \right\|_\infty, \left\| \frac{y_2}{|y_2|} \right\|_\infty \right) = 1 \tag{25}$$

and

$$\langle \tilde{z}, y \rangle = \int_0^l y_1 |y_1|^{-1} y_1 dx + \int_0^l y_2 |y_2|^{-1} y_2 dx = \int_0^l |y_1| dx + \int_0^l |y_2| dx = \|y\|_1. \tag{26}$$

With $c := \|y\|_1$ and $z := c\tilde{z}$ we use (25) and (26) and derive the following relations.

$$\|z\|_\infty^2 = c^2 = \|y\|_1^2 \quad \text{und} \quad \langle z, y \rangle = c\langle \tilde{z}, y \rangle = c^2 = \|y\|_1^2.$$

If we apply z to Gy for this case we get

$$\begin{aligned} \frac{1}{c}\langle z, Gy \rangle = \langle \tilde{z}, Gy \rangle &= \int_0^l y_1 |y_1|^{-1} (-\gamma D_x y_1) dx + \int_0^l y_2 |y_2|^{-1} (\gamma D_x y_2) dx \\ &= -\gamma \int_0^l D_x |y_1| dx + \frac{\gamma}{p} \int_0^l D_x |y_2| dx \\ &= \gamma\big(|y_1(0)| - |y_1(l)| + |y_2(l)| - |y_2(0)|\big). \end{aligned}$$

Hence in both cases, i.e. for $1 \leq p < \infty$, we obtain equation (24). Now we consider each of the three boundary conditions separately:

- Dirichlet (4): $\langle z, Gy \rangle = -\frac{\gamma}{pc^{p-2}} \left(|y_1(l)|^p + |y_2(0)|^p \right) \leq 0.$
- Neumann (5): $\langle z, Gy \rangle = 0.$
- periodic (6): $\langle z, Gy \rangle = 0.$

qed.

Lemma 3.3 *The linear operator $G : \mathcal{D}(G) \to \mathcal{L}^p$, with Dirichlet, Neumann, or periodic boundary conditions, respectively, is generator of a strongly continuous semigroup of contractions on $\mathcal{L}^p$.*

Proof. Theorem of Lumer–Phillips (see Pazy [37], Chap. 1, Theorem 4.3, p.14). *qed.*

Theorem 3.4 *The linear operator $G + B : \mathcal{D}(G) \to \mathcal{L}^p$, with Dirichlet, Neumann, or periodic boundary conditions, respectively, is generator of a strongly continuous semigroup on $\mathcal{L}^p$. For $\mu \geq 0$ this semigroup is positive.*

Proof. The perturbation operator B is bounded on $\mathcal{L}^p$. We use a theorem of Pazy ([37], Chap. 3, Theorem 1.1, p.76) on generators with bounded perturbations. The positivity follows form the fact if $u^+ = 0$ (or $u^- = 0$) at a certain point, then the right hand side of the first (second) equation of (2) is non-negative (see Smoller [42]).

For existence, we did not need an assumption on the sign of the turning rate μ. The notion of "turning rate" implicitly implies a positivity assumption. The arguments used so far, however also hold true for negative μ. If μ is negative, then the model for the correlated random walk describes alignment, where particles moving in positive direction enhance the positive direction. The two directions will be split and eventually only one direction remains. In some sense (made precise later) the case for negative μ is inverse to the random walk case for $\mu > 0$. We will use this fact to show that the hyperbolic system (2) generates a solution group for Neumann and for periodic boundary conditions. This fact is not true for the Dirichlet problem, which we will study separately. For Dirichlet boundary conditions we observe a regularity property, which prohibits backward well definedness.

Lemma 3.5 *The linear operator $G - B : \mathcal{D}(G) \to \mathcal{L}^p$, with Dirichlet, Neumann, or periodic boundary conditions is generator of a strongly continuous semigroup in $\mathcal{L}^p$.*

3.1 Group for Neumann and Periodic Boundary Conditions

Theorem 3.6 *The linear operator G+B, with Neumann, or periodic boundary conditions is generator of a strongly continuous group in $\mathcal{L}^p$.*

Proof. We use a remark from Goldstein ([10], Remark 2.16, p.22) which states that the property of generation of a group is equivalent to the fact that both $G + B$ and $-(G + B)$ are generators of a strongly continuous semigroup and that $\mathcal{D}(G + B) = \mathcal{D}(-(G + B))$.
In our case we have a bounded operator B, hence

$$\mathcal{D}(G + B) = \mathcal{D}(G), \qquad \mathcal{D}(-(G + B)) = \mathcal{D}(-G).$$

To match forward and backward solutions, we introduce a permutation matrix $\Pi := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $C := G - B$ satisfies

$$\Pi^{-1} G \Pi = -G \quad \text{and} \quad \Pi^{-1} C \Pi = -G - B = -(G + B).$$

The solutions of the inverse problem $y_t = -(G+B)y$ are after switch of coordinates y_1 and y_2, solutions of $z_t = Cz$ with $C = G - B$. Since C is generator of a strongly continuous semigroup, and since conjugation with Π is just a transformation of coordinates, we conclude that also $-(G+B)$ is generator of a strongly continuous semigroup.
Finally it remains to show that $\mathcal{D}(G) = \mathcal{D}(-G)$. By definition we have

$$\mathcal{D}(G) = \{(z_1, z_2) \in \mathcal{W}^{1,p} : \; z_1(0) = \chi_0(z_1(l), z_2(0)), \; z_2(l) = \chi_l(z_1(l), \; z_2(0))\}.$$

Using the coordinate transformation Π we find the domain of definition of $-G$ to be

$$\mathcal{D}(-G) = \{(z_1, z_2) \in \mathcal{W}^{1,p} : \; z_2(0) = \chi_0(z_2(l), z_1(0)), \; z_1(l) = \chi_l(z_2(l), \; z_1(0))\}.$$

Again, we study the boundary conditions separately.

- Neumann: $(\chi_0(\alpha, \beta) = \beta, \; \chi_l(\alpha, \beta) = \alpha)$

$$\begin{aligned} \mathcal{D}(-G) &= \{(z_1, z_2) \in \mathcal{W}^{1,p} : \; z_2(0) = z_1(0), \; z_1(l) = z_2(l)\} \\ &= \{(y_1, y_2) \in \mathcal{W}^{1,p} : \; y_1(0) = y_2(0), \; y_2(l) = y_1(l)\} = \mathcal{D}(G) \end{aligned}$$

- Periodic: $(\chi_0(\alpha, \beta) = \alpha, \; \chi_l(\alpha, \beta) = \beta)$

$$\begin{aligned} \mathcal{D}(-G) &= \{(z_1, z_2) \in \mathcal{W}^{1,p} : \; z_2(0) = z_2(l), \; z_1(l) = z_1(0)\} \\ &= \{(y_1, y_2) \in \mathcal{W}^{1,p} : \; y_1(0) = y_1(l), \; y_2(l) = y_2(0)\} = \mathcal{D}(G) \end{aligned}$$

qed.

Remarks 3.2 The above relation is not satisfied for Dirichlet boundary conditions $(\chi_0 = 0, \; \chi_l = 0)$ since in this case

$$\mathcal{D}(-G) = \{(z_1, z_2) \in \mathcal{W}^{1,p} : \; z_2(0) = 0, \; z_1(l) = 0\} \neq \mathcal{D}(G).$$

It is true that the Dirichlet problem for $G - B$ can be solved as well, but the solutions are not the backward solutions of $G + B$, since the boundary conditions do not match.

3.2 Weak and Classical Solutions

Now we use the semigroup properties to study weak and strong solutions as they are classically defined for PDEs. According to Pazy [37] and Ball [1], weak solutions are defined through integration with appropriate test functions. In this context, the space of test functions is the domain of definition of the adjoint operator of G. Using integration by parts we find that

$$G^* = -G : \mathcal{D}(G^*) \to \mathcal{L}^q,$$

with domain of definition

$$\mathcal{D}(G^*) = \{(\varphi_1, \varphi_2) \in \mathcal{W}^{1,q} : \ \varphi_2(0) = \chi_0(\varphi_2(l), \varphi_1(0)), \ \varphi_1(l) = \chi_l(\varphi_2(l), \varphi_1(0))\},$$

with χ_0 and χ_l from (7).

We study the following intial boundary value problem (IBWP)

$$\begin{array}{rcll} u_t^+ + \gamma u_x^+ & = & \mu(u^- - u^+) & \multirow{2}{*}{$\Big\} \quad (t,x) \in [0,\infty) \times (0,l)$} \\ u_t^- - \gamma u_x^- & = & \mu(u^+ - u^-) & \\ u^+(0,x) & = & u_0^+(x) & \Big\} \quad x \in [0,l] \\ u^-(0,x) & = & u_0^-(x) & \\ u^+(t,0) & = & \chi_0(u^+(t,l), u^-(t,0)) & \Big\} \quad t \in [0,\infty) \\ u^-(t,l) & = & \chi_l(u^+(t,l), u^-(t,0)) & \end{array} \qquad (27)$$

Definition 3.3 (Pazy [37], Chap. 4) Let $T > 0$.

1. The function $u(t,x) = (u^+(t,x), u^-(t,x))$ is called *classical solution* of (27), if

   $$u \in C^1([0,T), \mathcal{L}^p) \cap C([0,T), \mathcal{D}(G))$$

   and for all $t \in [0,T)$ we have

   $$\frac{d}{dt} u(t) = (G+B)u(t).$$

2. The function $u(t,x) = (u^+(t,x), u^-(t,x))$ is called *weak solution* of (27), if

   $$u \in C([0,T), \mathcal{L}^p)$$

   and for each $z \in \mathcal{D}(G^*)$ and all $t \in [0,T)$

   $$\frac{d}{dt}\langle z, u(t)\rangle = \langle G^* z, u(t)\rangle + \langle z, Bu(t)\rangle$$

   and the map $t \mapsto \langle z, u(t)\rangle$ is absolutely continuous.

Obviously, a classical solution is also a weak solution.

Theorem 3.7 *Let $(S(t))_{t\geq 0}$ denote the semigroup generated by $G+B$.*

1. *If $u_0 \in \mathcal{D}(G)$, then $u(t,x) := S(t)u_0(x)$ is a unique classical solution of the IBWP* (27)*.*

2. *If* $u_0 \in \mathcal{L}^p$, *then* $u(t,x) := S(t)u_0(x)$ *is a unique weak solution of the IBWP* (27).

Proof. Since the resolvent set of G is non-empty, also the resolvent set of $G+B$ is non-empty. Then a Theorem from Pazy [37] (Chapter 4, Theorem 1.3) applies and the generator property is equivalent with item 1. of the theorem. The second statement is based on a Theorem of Ball [1]

qed.

Remarks 3.4

1. A complete spectral analysis for a larger class of hyperbolic systems which also include the models studied here is given in Neves, Ribeiro, Lopes [36] and Neves, Lin [35]. They include dynamic boundary conditions of the form

$$\begin{aligned} u^+(t,0) &= E(t)u^-(t,0) \\ \frac{d}{dt}(u^-(t,l) - D(t)u^+(t,l)) &= F(t)u^+(t,l) + G(t)u^-(t,l) \end{aligned}$$

   For existence and uniqueness the authors refer to "standard literature on semi-groups". Hence we feel it is justified to carry out the above arguments in detail.

2. Beck [3] studies boundary conditions of the form

$$\begin{aligned} u^+(t,0) &= \chi_0(t) \in C^1([0,T]) \\ u^-(t,l) &= \chi_l(t) \in C^1([0,T]) \end{aligned}$$

   and he proves existence and uniqueness of classical solutions for (2) with initial conditions $u^+(0,x) = u_0^+(x), \quad u^-(0,x) = u_0^-(x)$ which satsfy the following continuity conditions

$$\lim_{t\to 0} \chi_0(t) = \lim_{x\to 0} u_0^+(x), \qquad \lim_{t\to 0} \chi_l(t) = \lim_{x\to l} u_0^-(x).$$

3. The hyperbolic system (2) on an unbounded domain $\mathbb{R}$ falls into the class of symmetric hyperbolic systems, as they are studied in many standard texts. For example Kato [27] showed that initial conditions $u_0 \in \mathcal{W}^{k,2}(\mathbb{R})$ with $k \geq 2$ lead to unique solutions in $C([0,T), \mathcal{W}^{k,2}(\mathbb{R})) \cap C^1([0,T), \mathcal{W}^{k-1,2}(\mathbb{R}))$. Renardy and Rogers [40] present a proof for $k = 1$ and also John [25] studies these kind of systems on $\mathbb{R}$.

4. Using the one-dimensional Sobolev embeddings (see e.g. [9], [44], [5]) we find that classical solutions of (2) are continuous on $\Omega_\infty = [0,\infty) \times [0,l]$.

5. It is well known that the hyperbolic model (2) can be transformed into a telegraph equation (damped wave equation) (see Hadeler [15] and section 5 below). Poincare [38] used Bessel functions to find explicit solutions for the telegraph equation. Hadeler [15] used these solutions to derive explicit solutions for our hyperbolic system (2) on $\mathbb{R}$.

   For $k \in \mathbb{N}$ let $I_k(x) := e^{k\pi i} J_k(ix)$ denote the Besselfunction with purely imaginary argument (see also Smirnow Vol. II [41]). For $k = 0$ and $k = 1$ we have the relations

$$I_0(x) = J_0(ix) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k} \quad \text{and} \quad I_1(x) = \frac{d}{dx} I_0(x).$$

   The functions $I_0(x)$, $I_1(x)$ and $I_1(x)/x$ are real analytic and positive for $x > 0$. For an initial condition $u_0 \in \mathcal{L}^p$ the solution of $u(t,x) = (u^+(t,x), u^-(t,x))$ of (2) on $\mathbb{R}$ can be written as

$$\begin{aligned} u^+(t,x) &= u_0^+(x-\gamma t)e^{-\mu t} + \int_{x-\gamma t}^{x+\gamma t} K(t,x,y) u_0^-(y) dy \\ &\quad + \int_{x-\gamma t}^{x+\gamma t} K_+(t,x,y) u_0^+(y) dy \qquad (28) \\ u^-(t,x) &= u_0^-(x+\gamma t)e^{-\mu t} + \int_{x-\gamma t}^{x+\gamma t} K(t,x,y) u_0^+(y) dy \\ &\quad + \int_{x-\gamma t}^{x+\gamma t} K_-(t,x,y) u_0^-(y) dy \qquad (29) \end{aligned}$$

   with integral kernels

$$\begin{aligned} K(t,x,y) &:= \tfrac{\mu e^{-\mu t}}{2\gamma} I_0\left(\tfrac{\mu}{\gamma}\sqrt{\gamma^2 t^2 - (y-x)^2}\right) \\ K_\pm(t,x,y) &:= \tfrac{\mu e^{-\mu t}}{2\gamma} \tfrac{I_1\left(\frac{\mu}{\gamma}\sqrt{\gamma^2t^2-(y-x)^2}\right)}{\sqrt{\gamma^2t^2-(y-x)^2}} (\gamma t \mp (y-x)). \end{aligned} \qquad (30)$$

   From this representation we see that solutions are in $L^\infty([0,\infty)\times\mathbb{R})$ for initial conditions in L^∞. The integrals are absolutely continuous such that possible discontinuities can only travel along the characteristics $x-\gamma t = c$ and $x+\gamma t = c$. This fact was observed by Reed [39] using different methods.

   Additionally, if u_0 is k–times differentiable then u is also. In general we conclude:

$$\begin{aligned} u_0 \in L^\infty(\mathbb{R}) &\implies u \in L^\infty([0,\infty)\times\mathbb{R}) \\ u_0 \in C^k(\mathbb{R}) &\implies u \in C^k([0,\infty)\times\mathbb{R}). \end{aligned}$$

6. A semigroup theory for matrix operators, such as $G = \begin{pmatrix} -\gamma D_x & 0 \\ 0 & \gamma D_x \end{pmatrix}$ has been developed in Nagel [33] and Engel [8]. The computation of the spectrum of G with periodic boundary conditions is an example in [34].

4 Regularity

For the diffusion equation it is known that solutions regularize. This means non-smooth initial conditions lead to smooth solutions for each $t > 0$. For hyperbolic systems this is not the case. We saw in the above remarks that for (2) on $\mathbb{R}$ solutions stay in the same smoothness class as the initial conditions and singularities are transported along characteristics and are damped exponentially.
Problem (2) on a bounded domain $[0, l]$ has similar properties. However, we need to consider an additional matching condition on the boundary. For example if for the Neumann problem we choose smooth initial conditions which do not satisfy the Neumann boundary condition, then a "kink" will develop and move into the domain along characteristics. As time evolves, this "kink" will be reflected at the boundaries and be damped exponentially. The Dirichlet problem is different. Here, singularities are washed out at the boundary and the solution becomes regular after time $T = l/\gamma$. Hence, as we show below, the Dirichlet problem does regularize in finite time.
We study the three boundary conditions separately. We begin with periodic boundary conditions, which can be extended to a problem on $\mathbb{R}$. Then we study Neumann boundary conditions on $[0, l]$ and use the fact that these can be extended to periodic boundary conditions on $[0, 2l]$. The Dirichlet problem is treated using the method of characteristics.

4.1 Periodic Boundary Conditions

First we show that periodic solutions on $\mathbb{R}$ stay periodic for all times.

Lemma 4.1 *Assume that the initial condition $u_0 = (u_0^+, u_0^-) \in (L^\infty(\mathbb{R}))^2$ is periodic with period l. Then for all $t > 0$ the solution of (2) on $\mathbb{R}$ is spatially periodic with period l.*

Proof. Given $u_0 \in (L^\infty(\mathbb{R}))^2$ with $u_0(x + l) = u_0(x)$ and corresponding solution $u(t, x)$. We define another solution $v(t, x) = (v^+(t, x), v^-(t, x)) := u(t, x + l)$. Then v satisfies

$$\begin{aligned}
(v_t^+ + \gamma v_x^+)|_{(t,x)} &= (u_t^+ + \gamma u_x^+)|_{(t,x+l)} = \mu(u^- - u^+)|_{(t,x+l)} = \mu(v^- - v^+)|_{(t,x)} \\
(v_t^- - \gamma v_x^-)|_{(t,x)} &= (u_t^- - \gamma u_x^-)|_{(t,x+l)} = \mu(u^+ - u^-)|_{(t,x+l)} = \mu(v^+ - v^-)|_{(t,x)}
\end{aligned}$$

with initial conditions $v(0, x) = u_0(x + l) = u_0(x)$. Hence u and v satisfy the same intial value problem on $\mathbb{R}$. Since the solution is unique we have $u(t, x) = v(t, x)$ for all (t, x). *qed.*

Now we study the initial boundary value problem (27) on $[0, l]$ with periodic boundary conditions (6). As shown earlier, an initial condition $u_0 \in \mathcal{D}(G)$ defines a unique classical solution $u \in C^1([0,\infty),\mathcal{L}^p) \cap C([0,\infty),\mathcal{D}(G))$. We further assume that the initial condition satisfies

$$u_0 \in (C^k([0,l]))^2 \cap \mathcal{D}(G). \tag{31}$$

The initial condition might have lower regularity at the boundary: Let $\kappa \in \mathbb{N}$ be such that $0 \le \kappa \le k$ and assume that

$$\forall j \le \kappa: \quad D_x^j u_0^+(0) = D_x^j u_0^+(l) \quad \text{and} \quad D_x^j u_0^-(0) = D_x^j u_0^-(l). \tag{32}$$

Theorem 4.2 *Assume* (31), (32) *then the unique solution of* (27) *with periodic boundary conditions satisfies* $u \in (C^\kappa(\Omega_\infty))^2$.

Proof. We periodically extend u_0 to $\mathbb{R}$

$$\hat{u}_0(x) := u_0(x \bmod(l)). \tag{33}$$

From assumptions (31), (32) we have $\hat{u}_0 \in (C^\kappa(\mathbb{R}))^2$ and based on Remark 3.2 item 5 we know that the unique solution satisfies $\hat{u} \in (C^\kappa(\mathbb{R}_+ \times \mathbb{R}))^2$. In Lemma 4.1 we showed that this solution keeps the period l. Hence the restriction of $u := \hat{u}|_{[0,l]}$ is our desired solution. *qed.*

If the initial condition satisfies $u_0 \notin \mathcal{D}(G)$, then the extension $\hat{u}_0$ on $\mathbb{R}$ is bounded but not necessarily continuous. Hence the corresponding solution will be in $(L^\infty(\Omega_\infty))^2$.

4.2 Neumann Boundary Conditions

To study Neumann boundary conditions, we use the following symmetry property.

Lemma 4.3 *Assume* $u = (u^+, u^-)$ *is a solution of* (2) *on* $\mathbb{R}$ *with initial conditions* $u_0 = (u_0^+, u_0^-)$. *Then for each* $a \in \mathbb{R}$ *the function* $v(t,x) := (u^-(t,a-x), u^+(t,a-x))$ *solves* (2) *with initial condition* $v_0(x) = (u_0^-(a-x), u_0^+(a-x))$.

Proof. Given a solution $u(t,x)$ of (2) on $\mathbb{R}$ and define v as above. The function v satisfies

$$\begin{aligned}
(v_t^+ + \gamma v_x^+)|_{(t,x)} &= (u_t^- - \gamma u_x^-)|_{(t,a-x)} = \mu(u^+ - u^-)|_{(t,a-x)} = \mu(v^- - v^+)|_{(t,x)} \\
(v_t^- - \gamma v_x^-)|_{(t,x)} &= (u_t^+ + \gamma u_x^+)|_{(t,a-x)} = \mu(u^- - u^+)|_{(t,a-x)} = \mu(v^+ - v^-)|_{(t,x)}
\end{aligned}$$

with initial conditions $v(0,x) = u_0(a-x)$. *qed.*

Corollary 4.4 *Let* $u = (u^+, u^-)$ *be a solution of* (27) *with periodic boundary conditions and initial condition* $u_0 \in \mathcal{D}(G)$. *Then* $v(t,x) := (u^-(t,l-x), u^+(t,l-x))$ *is a solution of* (27) *with periodic boundary conditions and initial condition* $v_0 = (u_0^-(l-x), u_0^+(l-x))$.

Now we study (27) on $[0, l]$ with Neumann boundary conditions (5). Based on Theorem 3.7 for each $u_0 \in \mathcal{D}(G)$ there exists a unique solution u. We assume higher regularity for $k \in \mathbb{N}$

$$u_0 \in (C^k([0, l]))^2 \cap \mathcal{D}(G). \tag{34}$$

For the boundary of the initial condition we assume

$$\forall j \leq \kappa \quad D_x^j u_0^+(0) = (-1)^j D_x^j u_0^-(0) \quad \text{and} \quad D_x^j u_0^+(l) = (-1)^j D_x^j u_0^-(l). \tag{35}$$

for $0 \leq \kappa \leq k$.

Theorem 4.5 *Assume* (34), (35) *then the unique solution u of* (27) *with Neumann boundary conditions satisfies $u \in (C^\kappa(\Omega_\infty))^2$.*

Proof. We define periodic intial conditions on $[0, 2l]$

$$w_0(x) = (w_0^+(x), w_0^-(x)) = \begin{cases} (u_0^+(x), u_0^-(x)) & 0 \leq x \leq l, \\ (u_0^-(2l - x), u_0^+(2l - x)) & l < x \leq 2l. \end{cases} \tag{36}$$

Based on assumptions (34), (35) we have $w_0 \in (C^\kappa([0, 2l]))^2$. Additionally, w_0 satisfies the periodic boundary conditions (32) on $[0, 2l]$ with the same κ. By Theorem 4.2 the corresponding solution satisfies $w \in (C^\kappa([0, \infty) \times [0, 2l]))^2$. Now we define for $x \in [0, 2l]$ the functions $(v^+(t, x), v^-(t, x)) = (w^-(t, 2l-x), w^+(t, 2l-x))$. By Corollary 4.4 v is solution of (27) on $[0, 2l]$ with periodic boundary conditions and initial condition

$$\begin{aligned} v_0^+(t, x), v_0^-(t, x)) &= (w_0^-(2l - x), w_0^+(2l - x)) \\ &= \begin{cases} (u_0^+(x), u_0^-(x)) & 0 \leq x \leq l, \\ (u_0^-(2l - x), u_0^+(2l - x)) & l < x \leq 2l. \end{cases} \end{aligned} \tag{37}$$

Hence v and w satisfy the same initial boundary value problem and we obtain

$$(w^+(t, x), w^-(t, x)) = (v^+(t, x), v^-(t, x)) = (w^-(t, 2l - x), w^+(t, 2l - x)). \tag{38}$$

The restriction $u := w|_{[0,l]}$ satisfies

$$\begin{aligned} u^+(t, 0) = w^+(t, 0) = w^-(t, 2l) = w^-(t, 0) &= u^-(t, 0) \\ u^-(t, l) = w^-(t, l) = w^+(t, l) &= u^+(t, l). \end{aligned}$$

Hence $u \in (C^\kappa(\Omega_\infty))^2$ solves the Neumann problem (27) on $[0, l]$. *qed.*

If the initial condition satisfies $u_0 \notin \mathcal{D}(G)$, then the extension w_0 is bounded but not necessarily continuous. Hence the corresponding solution will be in $(L^\infty(\Omega_\infty))^2$.

4.3 Dirichlet Boundary Conditions

Theorem 4.6 *Let $u = (u^+, u^-)$ be a solution of (27) with Dirichlet boundary conditions and initial condition $u_0 \in \mathcal{D}(G)$. Then $u \in (C^1((l/\gamma, \infty) \times [0, l]))^2$.*

Proof. The idea of the proof is as follows. For a given point (t, x) we follow the characteristics backwards until we either hit the domain boundary or the initial condition. If we hit the boundary, then the boundary terms in (28) and (29) vanish and the remaining integral terms are differentiable.
A solution given by (28), (29) only depends on the values of the solution in the characteristic cone $\Theta := \{(s, \xi) \in \Omega_\infty : \ 0 \le s \le t,\ x - \gamma(t-s) \le \xi \le x + \gamma(t-s)\}$.
For each $0 \le \tau \le t$ with $x - \gamma(t-\tau) \ge 0$ and $x + \gamma(t-\tau) \le l$ we can write

$$\begin{aligned} u^+(t,x) &= u^+(\tau, x-\gamma(t-\tau))e^{-\mu(t-\tau)} + \int_{x-\gamma(t-\tau)}^{x+\gamma(t-\tau)} K(t,x,y)u^-(\tau,y)dy \\ &\quad + \int_{x-\gamma(t-\tau)}^{x+\gamma(t-\tau)} K_+(t,x,y)u^+(\tau,y)dy \qquad (39) \\ u^-(t,x) &= u^-(\tau, x+\gamma(t-\tau))e^{-\mu(t-\tau)} + \int_{x-\gamma(t-\tau)}^{x+\gamma(t-\tau)} K(t,x,y)u^+(\tau,y)dy \\ &\quad + \int_{x-\gamma(t-\tau)}^{x+\gamma(t-\tau)} K_-(t,x,y)u^-(\tau,y)dy \qquad (40) \end{aligned}$$

with the integral kernels K, $K_\pm$ from (30).
For each $(t, x) \in \Omega_\infty$ we define two characteristic time values τ^+ and τ^- at which the characteristics leave the domain $[0, l]$ (see Fig. 2).

$$\begin{aligned} x - \gamma(t-\tau^+) = 0 \quad &\Longrightarrow \quad \tau^+(t,x) = t - \frac{x}{\gamma} \\ x + \gamma(t-\tau^-) = l \quad &\Longrightarrow \quad \tau^-(t,x) = t + \frac{x-l}{\gamma} \end{aligned}$$

The values $\tau^\pm$ are smooth functions of t and x. Next we show the differentiability of u^+. The arguments for u^- are similar.
Let $t > \frac{l}{\gamma}$, then τ^+ and τ^- are positive for each $x \in [0, l]$.

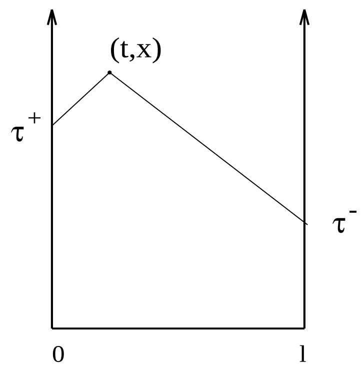

Fig. 2: The exit times τ^+ and τ^- of the backward characteristics.

Case 1: Assume $(t,x) \in Q_1 := (l/\gamma, \infty) \times [0, l/2)$. Then (39) reads

$$\begin{aligned} u^+(t,x) &= \underbrace{u^+(\tau^+(t,x),0)}_{=0} e^{-\mu(t-\tau^+(t,x))} + \int_0^{2x} K(t,x,y) u^-(\tau^+(t,x),y) dy \\ &\quad + \int_0^{2x} K_+(t,x,y) u^+(\tau^+,y) dy. \end{aligned} \tag{41}$$

As show in Theorem 3.7 we have $u \in C^1([0,\infty), \mathcal{L}^p)$. Additionally, $\tau^+ \in C^1(Q_1, \mathbb{R}_+)$ and the kernels K and K_+ are analytic in their arguments. Hence $u(t,x)$ is continuously differentiable on Q_1.

Case 2: Assume $x = l/2$:
Here we use (39) with $\tau = t - l/(4\gamma)$. Again, the integral terms are smooth. The first term from (39) reads now $u^+(\tau, x - \gamma(t-\tau)) = u^+(\tau, l/4)$ and we apply Case 1.

Case 3: Assume $(t,x) \in Q_2 := (l/\gamma, \infty) \times (l/2, l)$:
In this case the equation (41) does not hold, since $2x > l$. Hence now we use (39) with $\tau = \tau^-(t,x)$. Then we obtain

$$\begin{aligned} u^+(t,x) &= u^+(\tau^-(t,x), 2x-l) e^{-\mu(t-\tau^-(t,x))} + \int_{2x-l}^{l} K(t,x,y) u^-(\tau^-(t,x),y) dy \\ &\quad + \int_{2x-l}^{l} K_+(t,x,y) u^+(\tau^-(t,x),y) dy. \end{aligned} \tag{42}$$

The integral terms are smooth, and we only need to study the first term $u^+(\tau^-(t,x), 2x-l)$. We introduce $t_1(t,x) := \tau^-(t,x)$ and $x_1(t,x) := 2x - l$. If $x_1 < l/2$, then we apply Case 1 and $u^+(t_1(t,x), x_1(t,x))$ is continuously differentiable in (t,x). If $x_1 = l/2$ we apply Case 2. If $x_1 > l/2$ we repeat the construction above (42) with $u^+(t_1, x_1)$ and $t_2 = \tau^-(t_1, x_1)$ (see Fig. 3). We obtain a finite sequence $\{(t_k, x_k)\}_{k\in\{1,\ldots,n\}}$ with the property that

$$\begin{aligned} u^+(t_k, x_k) &= u^+(t_{k+1}, x_{k+1})e^{-\mu(t_k - t_{k+1})} + \int_{2x_k - l}^{l} K(t_k, x_k, y)u^-(t_{k+1}, y)dy \\ &\quad + \int_{2x_k - l}^{l} K_+(t_k, x_k, y)u^+(t_{k+1}, y)dy. \end{aligned} \tag{43}$$

The sequences $\{t_k\}$ and $\{x_k\}$ are both monotonically decreasing and we have $x_k - x_{k+1} = l - x_k \geq l - x =: r > 0$. Hence $x_{k+1} \leq x - kr$ and there exists $n \in \mathbb{N}$ such that $x_n \leq l/2$. Now we can again apply either Case 1 or Case 2. Notice that the sequence t_k stays non-negative, since for $x_k > l/2$ we have $t_{k+1} - \tau^+ = \frac{1}{\gamma}(x_k + x - l) > 0$, hence $t_{k+1} > \tau^+ > 0$.

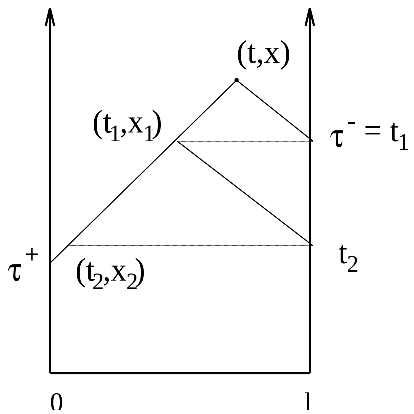

Fig. 3: Construction of the finite sequence $\{(t_k, x_k)\}_{k\in\{1,\ldots,n\}}$

Case 4: Assume $x = l$: In this case we use an implicit representation of the solution if we use the method of characteristics for (2). The characteristics of (2) are

$$x^+(s) = x - \gamma(t - s), \qquad x^-(s) = x + \gamma(t - s).$$

Let u^+ and u^- denote the components of the solution u of (2). Then $v^+(s) := u^+(s, x^+(s))$ and $v^-(s) := u^-(s, x^-(s))$ satisfy the ODE's (ordinary differential equatioins)

$$\begin{aligned} \dot{v}^+(s) &= -\mu v^+(s) + \mu u^-(s, x^+(s)) \\ \dot{v}^-(s) &= -\mu v^-(s) + \mu u^+(s, x^-(s)). \end{aligned}$$

Using the variation of constant formula we get the following implicit representation

$$u^+(t,x) = e^{-\mu(t-\tau)}u^+(\tau,x^+(\tau)) + \mu\int_\tau^t e^{\mu(s-t)}u^-(s,x^+(s))ds \tag{44}$$

$$u^-(t,x) = e^{-\mu(t-\tau)}u^-(\tau,x^-(\tau)) + \mu\int_\tau^t e^{\mu(s-t)}u^+(s,x^-(s))ds, \tag{45}$$

where $0<\tau<t$, such that $0<x-\gamma(t-\tau)$. In particular for $x=l$ we obtain

$$u^+(t,l) = e^{-\mu(t-\tau)}u^+(\tau,l-\gamma(t-\tau)) + \mu\int_\tau^t e^{\mu(s-t)}u^-(s,l-\gamma(t-s))ds. \tag{46}$$

Now we choose τ such that $0<l-\gamma(t-\tau)<l/2$. Then we see that the integral term in (46) is continuously differentiable and we apply the Case 1 to the first term.

The proof of smoothness for $u^-(t,x)$ is similar, where Case 1 corresponds to $l/2 < x \le l$, Case 2 corresponds to $x=l/2$, Case 3 corresponds to $0<x<l/2$ and Case 4 corresponds to $x=0$. *qed.*

5 Correlated Random Walk and Kinetics

We like to include population dynamics described by $u_t = f(u)$ into the random walk equations. For diffusion models it is typically assumed that diffusion and reaction are independent processes and they are modeled through addition of the corresponding terms, leading to reaction-diffusion equations (see [42]). In case of correlated random walk, we split the population into right and left moving compartments. This allows for a finer inclusion of reaction kinetics. In particular, we need to make sure that right moving particles die as right moving particles, i.e. we need appropriate death terms.

We follow Hadeler ([13], [14], [15]) and discuss a hierarchy of models:

(i) Holmes [24] introduced reaction as to be symmetric between the two classes. She assumed that (i,a) reaction is independent from the movement direction and (i,b) that newborn particles choose either direction with the same probability. The corresponding model reads

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= \mu(u^- - u^+) + \tfrac{1}{2}f(u^+ + u^-) \\ u_t^- - \gamma u_x^- &= \mu(u^+ - u^-) + \tfrac{1}{2}f(u^+ + u^-). \end{aligned} \tag{47}$$

(ii) The reaction can be split into gain and loss (birth and death) terms as $f(u) = u\,m(u) - u\,g(u)$, where $m(u)$ denotes a birth rate and $g(u)$ a death rate. Now we assume (ii,a) that the death rate $g(u)$ is independent of the movement direction, but right moving particles can only die as right moving particles and vice versa. Hence a death term for right moving particles appears only in the equation for right moving particles and vice versa. (ii,b) We assume that the birth rate $m(u)$ is independent of direction and newborn particles choose either direction with the same probability. The corresponding reaction-random walk model reads

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= \mu(u^- - u^+) + \tfrac{1}{2}(u^+ + u^-)\, m(u^+ + u^-) - u^+\, g(u^+ + u^-) \\ u_t^- - \gamma u_x^- &= \mu(u^+ - u^-) + \tfrac{1}{2}(u^+ + u^-)\, m(u^+ + u^-) - u^-\, g(u^+ + u^-). \end{aligned} \tag{48}$$

(iii) Here we consider the same reaction terms as in (ii) but we additionally assume (iii,a) that the movement direction of newborn particles correlates to the direction of their mother by a parameter $\tau \in [0,1]$. The corresponding model equations are

$$\begin{aligned} u_t^+ + \gamma u_x^+ &= \mu(u^- - u^+) + (\tau u^+ + (1-\tau)u^-)\, m(u^+ + u^-) - u^+\, g(u^+ + u^-) \\ u_t^- - \gamma u_x^- &= \mu(u^+ - u^-) + ((1-\tau)u^+ + \tau u^-)\, m(u^+ + u^-) - u^-\, g(u^+ + u^-). \end{aligned} \tag{49}$$

If $\tau = 1/2$ we have case (ii). For $\tau > 1/2$ the daughter particles tend to prefer the same direction as the mother and for $\tau < 1/2$ they prefer the opposite.

Holmes [24] and Hadeler [13], [14] studied travelling front solutions for these systems and they compare the wave speed with the corresponding reaction-diffusion equations. Hillen studied the corresponding hyperbolic Turing model [18] and he found a Lyapunov function for a class of nonlinearities [20].

Using operator notation we can write the above systems (47), (48) and (49) as

$$y_t = (G+B)\, y + F(y), \tag{50}$$

with $F(y) = (f_1(y_1,y_2), f_2(y_1,y_2))$ and

$$f_1(y_1,y_2) = \begin{cases} \frac{1}{2} f(y_1+y_2) & \text{in case (i)}, \\ \frac{1}{2}(y_1+y_2)m(y_1+y_2) - y_1 g(y_1+y_2) & \text{in case (ii)}, \\ (\tau y_1 + (1-\tau)y_2)m(y_1+y_2) - y_1 g(y_1+y_2) & \text{in case (iii)}, \end{cases}$$

$$f_2(y_1,y_2) = \begin{cases} \frac{1}{2} f(y_1+y_2) & \text{in case (i)}, \\ \frac{1}{2}(y_1+y_2)m(y_1+y_2) - y_2 g(y_1+y_2) & \text{in case (ii)}, \\ ((1-\tau) y_1 + \tau y_2)m(y_1+y_2) - y_2 g(y_1+y_2) & \text{in case (iii)}. \end{cases}$$

We use a perturbation argument for Lipschitz continuous perturbations to show existence for the above models.

Theorem 5.1 *Let $y_0 \in \mathcal{L}^p$.*

1. *If $F : \mathcal{L}^p \to \mathcal{L}^p$ is locally Lipschitz continuous then there exists a unique (local) solution $y \in C([0, t_{max}), \mathcal{L}^p)$ of (50). If the maximal time of existence is $t_{max} < \infty$, then $\lim_{t \to t_{max}} \|y(t)\| = \infty$. Furthermore, if F is differentiable with Lipschitz continuous derivative and $y_0 \in \mathcal{D}(G)$, then $y(t)$ is a classical solution.*

2. *If $F : \mathcal{L}^p \to \mathcal{L}^p$ is globally Lipschitz continuous then $t_{max} = \infty$.*

Proof.

1. See Pazy [37] Kap. 6, Theorem 1.4 and 1.5, as well as Grabosch und Heijmans [12] Theorem 3.2.

2. Grabosch und Heijmans [12] Theorem 4.6.

qed.

6 Reaction–Telegraph Equations

It is well known that correlated random walk systems can be transformed into a telegraph equation [16] (second order, damped wave equation). These systems are nearly equivalent. In this section we will study the reaction-telegraph equations which correspond to the reaction-random walk equations studied above. If we ignore boundary conditions, then we find that to each solution of the reaction-telegraph equation corresponds a one-parameter family of solutions to the reaction-random walk system (Section 6.1). For Neumann or for periodic boundary conditions, we find equivalence only if the initial conditions for the reaction-telegraph equation satisfy a certain compatibility condition (Sections 6.2, 6.3). Surprisingly, the corresponding Dirichlet problems cannot be related in a straightforward way (Section 6.4).
To use Kac's trick [26] we write the systems (47), (48) and (49) in terms of the total particle density $u = u^+ + u^-$ and the particle flux $v = u^+ - u^-$.

$$\begin{aligned} u_t + \gamma v_x &= f(u) \\ v_t + \gamma u_x &= -h(u)\, v, \end{aligned} \tag{51}$$

with

$$\begin{aligned} f(u) &= \begin{cases} f(u) & \text{in case (i)} \\ u\, m(u) - u\, g(u) & \text{in case (ii) and (iii),} \end{cases} \\ h(u) &= \begin{cases} 2\mu & \text{in case (i)} \\ 2\mu + g(u) & \text{in case (ii)} \\ 2\mu + (1-2\tau) m(u) + g(u) & \text{in case (iii).} \end{cases} \end{aligned}$$

The boundary conditions transform as

- Dirichlet (4):

$$u(t,0) = -v(t,0), \quad u(t,l) = v(t,l), \qquad t \in [0, t_{\max}). \tag{52}$$

- Neumann (5):

$$v(t,0) = 0, \quad v(t,l) = 0, \qquad t \in [0, t_{\max}). \tag{53}$$

- Periodic (6):

$$u(t,0) = u(t,l), \quad v(t,0) = v(t,l) \qquad t \in [0, t_{\max}). \tag{54}$$

To investigate the relation to a telegraph equation, we focus on case (i), i.e. $F(u) := \frac{1}{2}(f(u), f(u))$. System (51) reads in this case

$$\begin{aligned} u_t + \gamma v_x &= f(u) \\ v_t + \gamma u_x &= -2\mu\, v. \end{aligned} \tag{55}$$

We assume that solutions are twice differentiable and we differentiate the first equation with respect to t and the second with respect to x

$$u_{tt} + \gamma v_{xt} = f'(u) u_t. \tag{56}$$

$$v_{tx} + \gamma u_{xx} = -2\mu\, v_x. \tag{57}$$

Then we multiply (57) with γ and substitue into (56).

$$u_{tt} + (-\gamma^2 u_{xx} - 2\mu v_x) = f'(u) u_t.$$

The term $-2\mu v_x$ can be substituted from the first equation of (55).

$$u_{tt} + (2\mu - f'(u)) u_t = \gamma^2 u_{xx} + 2\mu\, f(u). \tag{58}$$

We call this equation a *reaction-telegraph equation.*
The above construction defines a map which maps solutions (u, v) of (55) onto solutions of (58). We see immediately, that a term of the form $v^* e^{-2\mu t}$ with $v^* \in \mathbb{R}$ can be added to v without changing the equations. Hence the systems are not equivalent. Initial conditions for u and v transform as

$$u_t(0,x) = -\gamma v_x(0,x) + f(u(0,x)) = -\gamma v_0'(x) + f(u_0(x)). \tag{59}$$

The above transformation applied to cases (ii) and (iii) would not lead to a single telegraph equation, unless $\tau = 1/2$ and $g = 0$, which is case (i) (see also [16, 15]).

Definition 6.1 Let $1 \le p < \infty$. The function $u \in C^1([0,T), L^p([0,l])) \cap C([0,T), W^{1,p}([0,l]))$ is a weak solution of (58), if for each test function $\varphi \in C^2([0,T)\times[0,l])$ with $\varphi|_{\partial\Omega_T} = 0$ we have

$$\int_0^T \int_0^l \Big(-u_t\varphi_t + (2\mu - f'(u))u_t\varphi + \gamma^2 u_x\varphi_x - 2\mu f(u)\varphi \Big) dx dt = 0.$$

We abbreviate the relevant function spaces as

$$\begin{aligned} E &:= C^1([0,T), L^p(I)) \cap C([0,T), W^{1,p}(I)) \\ E^2 &:= C^1([0,T), \mathcal{L}^p) \cap C([0,T), \mathcal{W}^{1,p}), \end{aligned}$$

with $I = [0,l]$ and $0 < T \le t_{\max}$.

We formulate the corresponding initial value problems

$$\begin{aligned} u_t + \gamma v_x &= f(u) \\ v_t + \gamma u_x &= -2\mu\, v \end{aligned} \tag{60}$$

with initial conditions

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \qquad x \in I \tag{61}$$

and

$$u_{tt} + (2\mu - f'(u))u_t = \gamma^2 u_{xx} + 2\mu\, f(u) \tag{62}$$

with initial conditions

$$u(0,x) = a_0(x), \quad u_t(0,x) = a_1(x) \qquad x \in I. \tag{63}$$

We study the boundary conditions separately.

6.1 No Boundary Conditions

Theorem 6.1

(a) *Assume* $(u_0, v_0) \in \mathcal{W}^{1,p}(I)$ *with corresponding classical solution* $(u,v) \in E^2$ *of* (60), (61), *then* u *is a weak solution of* (62) *with initial condition*

$$a_0 = u_0 \in W^{1,p}(I) \quad \text{and} \quad a_1 = -\gamma v_0' + f(u_0) \in L^p(I). \tag{64}$$

(b) *If* $a_0 \in W^{1,p}(I)$, $a_1 \in L^p(I)$ *and* $u \in E$ *the corresponding weak solution of* (62), (63), *then the pair* $(u,v) \in E^2$ *defined as*

$$\begin{aligned} v(t,x) &= \tilde{v}(x)e^{-2\mu t} - \int_0^t e^{-2\mu(t-\tau)}\gamma u_x(\tau,x)d\tau \\ \tilde{v}(x) &= v^* + \frac{1}{\gamma}\int_0^x (f(a_0(s)) - a_1(s))ds, \end{aligned} \tag{65}$$

defines a solution of (60) *for each* $v^* \in \mathbb{R}$. *The initial conditions are*

$$u_0 = a_0 \in W^{1,p}(I) \quad \text{and} \quad v_0 = v^* + \frac{1}{\gamma}\int_0^{\cdot} (f(a_0(s)) - a_1(s))ds \in W^{1,p}(I). \quad (66)$$

Proof.

(a) Let $(u,v) \in E^2$ solve (60) with initial conditions $(u_0, v_0) \in \mathcal{W}^{1,p}(I)$. Then for each $\varphi \in C^2([0,T) \times [0,l])$ with $\varphi|_{\partial\Omega_T} = 0$ we have

$$\begin{aligned}
\int_0^T \int_0^l -u_t\varphi_t dxdt &= \int_0^T \int_0^l (\gamma v_x \varphi_t - f(u)\varphi_t)dxdt \\
&= \int_0^T \int_0^l (-\gamma v \varphi_{tx} - f(u)\varphi_t)dxdt \\
&= \int_0^T \int_0^l (\gamma v_t \varphi_x + f'(u)u_t\varphi)dxdt \\
&= \int_0^T \int_0^l (-\gamma^2 u_x\varphi_x - 2\mu\gamma v\varphi_x + f'(u)u_t\varphi)dxdt \\
&= \int_0^T \int_0^l (-\gamma^2 u_x\varphi_x + 2\mu\gamma v_x\varphi + f'(u)u_t\varphi)dxdt \\
&= \int_0^T \int_0^l (-\gamma^2 u_x\varphi_x - 2\mu u_t\varphi + 2\mu f(u)\varphi + f'(u)u_t\varphi)dxdt.
\end{aligned}$$

Hence

$$\int_0^T \int_0^l \Big(- u_t\varphi_t + (2\mu - f'(u))u_t\varphi + \gamma^2 u_x\varphi_x - 2\mu f(u)\varphi \Big) dxdt = 0$$

and u is a weak solution of (62). The initial conditions are $a_0 = u_0$ and $a_1 = -\gamma v_0' + f(u_0)$.

(b) Let $u \in E$ solve (62), (63) with $a_0 \in W^{1,p}(I)$ and $a_1 \in L^p(I)$. Define v by (65). According to the assumptions we have $\tilde{v} \in W^{1,p}(I)$. The first term in (65) satisfies $\tilde{v}e^{-2\mu t} \in E$. The gradient of u satisfies $u_x \in C([0,T), L^p(I))$, and the integral term $\int_0^t e^{-2\mu(t-\tau)}\gamma u_x(\tau,x)d\tau \in C^1([0,T), L^p(I))$. Hence indeed $v \in C^1([0,T), L^p(I))$. We still need to confirm that $v \in C([0,T), W^{1,p}(I))$, which we achieve by using (62)

$$\begin{aligned}
\frac{\partial}{\partial x} v(t,x) &= \tilde{v}'(x)e^{-2\mu t} - \int_0^t e^{-2\mu(t-\tau)}\gamma \frac{\partial}{\partial x} u_x(\tau,x)d\tau \\
&= \frac{1}{\gamma}(f(a_0(x)) - a_1(x))e^{-2\mu t} \\
&\quad -\frac{1}{\gamma}\int_0^t e^{-2\mu(t-\tau)} \{u_{tt} + (2\mu - f'(u))u_t - 2\mu f(u)\} d\tau.
\end{aligned}$$

With the abbreviation $w := f(u) - u_t$ we have $w(0,x) = f(a_0(x)) - a_1(x)$ and after multiplication with γ

$$\begin{aligned}
\gamma v_x &= w(0,x)e^{-2\mu t} - \int_0^t e^{-2\mu(t-\tau)}(-2\mu w - w_t)d\tau \\
&= w(0,x)e^{-2\mu t} + \int_0^t \frac{\partial}{\partial \tau}\left(e^{-2\mu(t-\tau)}w(\tau,x)\right)d\tau \\
&= w(t,x) \\
&= f(u) - u_t. \qquad (67)
\end{aligned}$$

Hence $v_x \in C([0,T), L^p(I))$ and consequently $v \in E$. Furthermore, (67) is identical to the first equation of (60). To obtain the second equation of (60) we compute

$$\begin{aligned}
\frac{\partial}{\partial t}v(t,x) &= -2\mu\tilde{v}(x)e^{-2\mu t} - \gamma u_x(t,x) + \int_0^t 2\mu e^{-2\mu(t-\tau)}\gamma u_x(\tau,x)d\tau \\
&= -\gamma u(t,x) - 2\mu v(t,x).
\end{aligned}$$

For the corresponding initial conditions (61) of (u,v) we find $u_0(x) = a_0(x) \in W^{1,p}(I)$ and

$$v_0(x) = v^* + \frac{1}{\gamma}\int_0^x (f(a_0(s)) - a_1(s))ds \in W^{1,p}(I).$$

Here we clearly see where the free parameter v^* enters the equations. There is a one-parameter family of solutions v, which all correspond to the same u.

qed.

6.2 Neumann Boundary Conditions

The homogeneous Neumann boundary conditions read in the (u,v) notation

$$v(t,0) = 0, \quad v(t,l) = 0, \qquad t \in [0,T), \tag{68}$$

and for the reaction-telegraph equation (62), (63)

$$u_x(t,0) = 0, \quad u_x(t,l) = 0, \qquad t \in [0,T). \tag{69}$$

Theorem 6.2

(a) *Let $(u,v) \in E^2$ solve* (60) *and* (61) *with homogeneous Neumann boundary conditions* (68)*, then $u \in E$ solves* (62) *with initial conditions* (64) *and Neumann boundary conditions* (69)*.*

(b) *Let* $u \in E$ *solve* (62), (63) *with homogeneous Neumann boundary conditions* (69), *then the pair* $(u, v) \in E^2$ *defined by* (65) *with initial conditions* (66) *is a solution of* (60) *with homogeneous Neumann boundary conditions* (68) *if and only if*

$$v^* = 0 \quad \textit{and} \quad \int_0^l (f(a_0(s)) - a_1(s))ds = 0.$$

Proof.

(a) Since (68) is true for all $t \in [0, T)$, we can deduce that $v_t(t, 0) = v_t(t, l) = 0$ for all $t \in [0, T)$. The second equation of (60) evaluated at $x = 0$, and at $x = l$ reads

$$\begin{aligned} u_x(t, 0) &= \frac{1}{\gamma}v(t, 0) - \frac{2\mu}{\gamma}v_t(t, 0) = 0 \\ u_x(t, l) &= \frac{1}{\gamma}v(t, l) - \frac{2\mu}{\gamma}v_t(t, l) = 0. \end{aligned}$$

(b) We evaluate (65) at $x = 0$ and obtain

$$\begin{aligned} v(t, 0) &= \tilde{v}(0)e^{-2\mu t} + \int_0^t e^{-2\mu(t-\tau)}\gamma u_x(\tau, 0)dx \\ &= \tilde{v}(0)e^{-2\mu t} = v^* e^{-2\mu t}. \end{aligned}$$

Hence $v(t, 0) = 0$ if and only if $v^* = 0$. If we evaluate v at $x = l$ and use $v^* = 0$ then we see that

$$v(t, l) = 0 \quad \Longleftrightarrow \quad \int_0^l (f(a_0(s)) - a_1(s))ds = 0.$$

qed.

6.3 Periodic Boundary Conditions

The periodic boundary conditions for (60), (61) are

$$u(t, 0) = u(t, l), \quad v(t, 0) = v(t, l), \qquad t \in [0, T), \tag{70}$$

and for (62), (63)

$$u(t, 0) = u(t, l), \quad u_x(t, 0) = u_x(t, l), \qquad t \in [0, T). \tag{71}$$

Theorem 6.3

(a) *Let* $(u,v) \in E^2$ *solve* (60) *and* (61) *with periodic boundary conditions* (70)*, then* $u \in E$ *solves* (62) *with initial conditions* (64) *and periodic boundary conditions* (71)*.*

(b) *Let* $u \in E$ *solve* (62), (63) *with periodic boundary conditions* (71)*, then the pair* $(u,v) \in E^2$*, defined by* (65) *with initial conditions* (66) *solves* (60) *with periodic boundry conditions* (68) *if and only if*

$$\int_0^l (f(a_0(s)) - a_1(s))ds = 0.$$

Proof.

(a) Since (70) holds for all $t \in [0,T)$ we find $v_t(t,0) = v_t(t,l)$ for all $t \in [0,T)$. The second equation of (60) evaluated at $x = 0$ reads

$$\begin{aligned} u_x(t,0) &= \frac{1}{\gamma}v(t,0) - \frac{2\mu}{\gamma}v_t(t,0) \\ &= \frac{1}{\gamma}v(t,l) - \frac{2\mu}{\gamma}v_t(t,l) = u_x(t,l). \end{aligned}$$

(b) By definition (65) we have

$$\begin{aligned} v(t,0) - v(t,l) &= (\tilde{v}(0) - \tilde{v}(l))e^{-2\mu t} + \int_0^t e^{-2\mu(t-\tau)}\gamma(u_x(\tau,0) - u_x(\tau,l))dx \\ &= \int_0^l (f(a_0(s)) - a_1(s))ds\, e^{-2\mu t}. \end{aligned}$$

Hence $v(t,0) = v(t,l)$ if and only if $\int_0^l (f(a_0(s)) - a_1(s))ds = 0$.

qed.

6.4 Dirichlet Boundary Conditions

The homogeneous Dirichlet boundary conditions of (60), (61) are

$$u(t,0) = -v(t,0), \quad u(t,l) = v(t,l), \qquad t \in [0,T), \tag{72}$$

and for (62), (63)

$$u(t,0) = 0, \quad u(t,l) = 0, \qquad t \in [0,T). \tag{73}$$

These boundary conditions only match if $v(t,0) = v(t,l) = 0$. These are, in fact, Neumann boundary conditions (60) for the hyperbolic system. From Section 6.2 we know that then the solution of the reaction-telegraph equation satisfies $u_x(t,0) = u_x(t,l) = 0$, which is not generally true for the Dirichlet problem. It is quite surprising that there is no direct equivalence between the two models for homogeneous Dirichlet conditions.

7 Global Existence for Reaction–Telegraph Equations

In this section we review results on existence and uniqueness for reaction telegraph equations. The equivalences from the previous section make some of these results available to the reaction random walk systems as well.
Matsumura [32] studies a general class of damped wave equations of the form

$$u_{tt} - A_0(t, x, u, u_t, \nabla u)u + \alpha u_t + f(u, u_t, \nabla u) = 0. \tag{74}$$

He proves global existence and uniqueness of solutions $u(t, x) \in \mathbb{R}^m$ in $\mathbb{R}^n$. He assumes that A_0 is smooth, symmetric with respect to the spatial coordinates, coercive with respect to u, and some further regularity properties of A_0. For f he assumes that there is a $0 < s \in \mathbb{N}$ such that for all multi indices $|\alpha| \leq s + 1$

$$f(y) \in C^{s+1}(\mathbb{R}^{n+2}), \quad Df(0) = 0, \quad |D^\alpha f(y)| \leq h_1(|y|)$$

and that

$$f(u, u_t, \nabla u) = f_1(u) + f_2(u, u_t, \nabla u)$$

with

$$f_1(u)\, u \geq 0, \quad f_2(y) \leq |y|^2 h_1(|y|),\ \forall\, |y| \leq 1.$$

Where h_1 denotes a continuous non negative and non decreasing function. Notice that the nonlinearity is used with the opposite sign as in (62).

Reed [39] shows global existence of solutions for equations of the form

$$u_{tt} - u_{xx} = f(x, t, u, u_x, u_t) \tag{75}$$

on $\mathbb{R}$. Reed uses energy estimates and assumes $f \in C^\infty$ and that $\partial/(\partial u_x)f$ and $\partial/(\partial u_t)f$ are uniformly bounded on compact sets for (x, u) and there exists a potential function $F \in C^\infty$ with the properties that

$$\frac{d}{dt}F(x, u, u_x, u_t) = f(x, u, u_x, u_t)\, u_t \quad \text{and} \quad F(x, 0, 0, 0) = 0.$$

It appears that we also need $F \geq 0$ (see [39], p.165 middle), which is not spelled out in [39].

Ball [2] studies the telegraph equation

$$\begin{aligned} u_{tt} + a(u, t)u_t - \Delta u + f(u) &= 0 \qquad \text{in } \Omega \subset \mathbb{R} \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{76}$$

To prove global existence it is assumed that $a(u,t)$ and $f(u,t)$ are smooth and there exists a continuous function $\theta_1 : \mathbb{R} \to \mathbb{R}$ and a locally integrable function $m(t)$ with $\lim_{s\to 0} \sup_{t\geq 0} \int_t^{t+s} m(\tau)d\tau = 0$ such that

$$|a(u,t)| + |f(u,t)| \leq m(t)\theta_1(u).$$

Webb [45] studies an eigenvalue problem for the telegraph equation

$$u_{tt} + 2\alpha u_t - u_{xx} = \lambda f(u), \quad \text{auf} \quad (0,\pi), \tag{77}$$

with periodic boundary conditions. The assumptions on f are

$$\begin{array}{ll} (i) & f \in C^2, \quad f(0) = 0, \quad f'(0) > 0, \\ (ii) & \limsup\limits_{\|x\|\to\infty} \dfrac{f(x)}{x} \leq 0, \quad \mathrm{sign} f''(x) = -\mathrm{sign} x. \end{array}$$

To prove existence of eigenfunctions, Webb uses an explicit representation of solutions as given in Weinberger [46] and the following Lyapunov function. Let $w = u_t$ and define

$$V(u,w) = \frac{1}{2}(\|u\|_B^2 + \|w\|^2) - \lambda \int_0^\pi F(u(x))dx, \quad \text{with} \quad F(u) = \int_0^u f(s)ds, \tag{78}$$

where the B–Norm denotes a norm of an appropriate dense subspace of $L^2([0,\pi])$.

Lopes [29] [30] studies telegraph equations

$$u_{tt} - \Delta u + cu_t + f(u) = h(t,x) \tag{79}$$

on $[0,2\pi]^3$ with periodic boundary conditions. He assumes that $f(u)$ is twice continuously differentiable and that there are constants $k_1 > 0$, k_2, k_3, $0 \leq \beta < 4$ with

$$uf(u) \geq k_1 u^2 + k_2, \qquad |f'(u)| \leq k_3(1 + |u|^\beta).$$

Lopes shows a–priori estimates for the norms of the solution in appropriate Sobolev spaces.

Li [28] shows existence of global classical solutions for

$$u_{tt} - \Delta u + g(u_t) + f(u) = 0 \tag{80}$$

in a bounded domain in $\mathbb{R}^n$, $n > 2$ with Dirichlet boundary conditions. He assumes that g and f are smooth enough and that

$$\begin{array}{ll} (i) & g(0) = 0 \quad \text{and} \quad g \text{ is monotonically increasing}, \\ (ii) & \forall\, y \in \mathbb{R} : \quad \displaystyle\int_0^y f(s)ds \geq 0, \\ (iii) & |f'(y)| \leq k(1 + |y|^{p-1}), \quad \forall\, y \in \mathbb{R}, \text{ with } k > 0,\ 1 \leq p \leq \dfrac{n}{n-2}. \end{array}$$

As can be seen here, the existence of solutions to nonlinear telegraph equations has been studied by many people. The same questions for the nonlinear random walk models of this paper have not been studied in such detail. The relation between telegraph equations and random walk systems as described in the previous section allow us to apply many of the result of this section to reaction-random walk equations.

References

[1] *J.M. Ball.* Strongly continuous semigroups, weak solutions and the variation of constants formula. Proc. Amer. Math. Soc., 63, 370–373, (1977).

[2] *J.M. Ball.* On the asymptotic behavior of generalized processes with applications to nonlinear evolution equations. J. of Diff. Eq., 27, 224–265, (1978).

[3] *Th. Beck.* Korrelierte Bewegung und nichtlineare Telegraphengleichung. Diplomarbeit, Tübingen, (1993).

[4] *A. Bressan.* Hyperbolic Systems of Conservation Laws. Oxford University Press, (2000).

[5] *D.E. Edmunds and W.D. Evans.* Spectral Theory and Differential Operators. Clarendon Press, Oxford, (1987).

[6] *R. Eftimie, G. de Vries, and M.A. Lewis.* Complex spatial group patterns result from different animal communication mechanisms. Proc. National Acad. Sci., 104, 6974–6979, (2007).

[7] *R. Eftimie, G. de Vries, and M.A. Lewis.* Weakly nonlinear analysis of a hyperbolic model for animal group formation. J. Math. Biol., (2008). online: DOI 10.1007/s00285-008-0209-8.

[8] *K. Engel.* Polynomial Operator Matrices. Dissertation, Uni. Tübingen, (1988).

[9] *D. Gilbarg and N.S. Trudinger.* Elliptic Partial Differential Equations of Second Order. Springer, Berlin, (1977).

[10] *J.A. Goldstein.* Semigroups of Linear Operators and Applications. Oxford Univ. Press, New York, (1985).

[11] *S. Goldstein.* On diffusion by discontinuous movements and the telegraph equation. Quart. J. Mech. Appl. Math., 4, 129–156, (1951).

[12] *A. Grabosch and H.J.A.M. Heijmans.* Cauchy problems with state–dependent time evolution. Japan J. of Appl. Math., 7, 433–457, (1990).

[13] *K.P. Hadeler*. Travelling fronts for correlated random walks. Canadian Applied Math. Quart., 2, 27–43, (1994).

[14] *K.P. Hadeler*. Travelling fronts in random walk systems. FORMA, Japan, 10, 223–233, (1995).

[15] *K.P. Hadeler*. Reaction telegraph equations and random walk systems. In: S.J. van Strien and S.M. Verduyn Lunel, (eds), *Stochastic and Spatial Structures of Dynamical Systems*, pages 133–161. Royal Academy of the Netherlands, (1996).

[16] *K.P. Hadeler*. Reaction transport systems. In: V. Capasso and O. Diekmann, (eds), *Mathematics inspired by biology, Lecture Notes in Mathematics 1714*, pages 95–150. CIME Letures 1997, Florence, Springer, (1999).

[17] *T. Hillen*. Nichtlineare hyperbolische Systeme zur Modellierung von Ausbreitungsvorgängen und Anwendung auf das Turing Modell. PhD thesis, University of Tübingen, (1995). in German.

[18] *T. Hillen*. A Turing model with correlated random walk. J. Math. Biol., 35, 49–72, (1996).

[19] *T. Hillen*. Invariance principles for hyperbolic random walk systems. J. Math. Anal. Appl., 210, 360–374, (1997).

[20] *T. Hillen*. Qualitative analysis of semilinear Cattaneo systems. Math. Models Meth. Appl. Sci., 8(3), 507–519, (1998).

[21] *T. Hillen*. Hyperbolic models for chemosensitive movement. Math. Models Methods Appl. Sci., 12(7), 1007–1034, (2002).

[22] *T. Hillen, C. Rohde, and F. Lutscher*. Existence of weak solutions for a hyperbolic model for chemosensitive movement. J. Math. Anal. Appl., 260, 173–199, (2001).

[23] *T. Hillen and A. Stevens*. Hyperbolic models for chemotaxis in 1-d. Nonlinear Analysis: Real World Applications, 1(1), 409–433, (2001).

[24] *E.E. Holmes*. Are diffusion models too simple ? a comparison with telegraph models of invasion. American Naturalist, 142, 779–795, (1993).

[25] *F. John*. Partial Differential Equations. Springer, (1971).

[26] *M. Kac*. A stochastic model related to the telegrapher‘s equation. Rocky Mountain J. Math., 4, 497–509, (1956).

[27] *T. Kato*. Quasilinear equations of evolution with applications to partial differential equations. In: W.N. Everitt, (ed), *Spectral Theory and Differential Equations*, pages 25–70. Lect. Notes in Math. 448, Springer, Berlin, (1975).

[28] *M.-R. Li*. Nichtlineare Wellengleichungen 2. Ordnung auf beschränkten Gebieten. Dissertation Tübingen, (1994).

[29] *O. Lopes*. Asymptotic behavior of solutions of a nonlinear wave equation. Bol. Soc. Bras. Mat., 19(1), 141–150, (1988).

[30] *O. Lopes*. Compact attractor for a nonlinear wave equation with critical exponent. Proc. Royal Soc. Edinburgh, 115A, 61–64, (1990).

[31] *F. Lutscher*. Modeling alignment and movement of animals and cells. J. Math. Biol., 45, 234–260, (2002).

[32] *A. Matsumura*. Global existence and asymptotics of the solutions of the second–order quasilinear hyperbolic equations with first–order dissipation. Publ. Research Institute for Math. Sci., 13, 349–379, (1977).

[33] *R. Nagel*. Towards a "matrix theory" for unbounded operator matrices. Math. Zeitschr., 201, 57–68, (1989).

[34] *R. Nagel*. The spectrum of unbounded operator matrices with nondiagonal domain. J. of Functional Analysis, **89**, 291–302, (1990).

[35] *A.F. Neves and X.B. Lin*. A multiplicity theorem for hyperbolic systems. J. Diff. Eq., 76, 339–352, (1988).

[36] *A.F. Neves, H. Ribeiro, and O. Lopes*. On the spectrum of evolution operators generated hyperbolic systems. J. of Functional Analysis, 67, 320–344, (1986).

[37] *A. Pazy*. Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York, (1983).

[38] *H. Poincaré*. Sur la propagation de l'électricité. Compt. Rend. Ac. Sci., 117, 1027–1032, (1893).

[39] *M.C. Reed*. Propagation of singularities for non–linear wave equations in one dimension. Comm. Part. Diff. Eq., 3(2), 153–199, (1978).

[40] *M. Renardy and R.C. Rogers*. An Introduction to Partial Differential Equations. Springer, New York, (1993).

[41] *W.I. Smirnow*. Lehrgang der Höheren Mathematik. VEB Deutscher Vlg. der Wiss., 9th edition, (1970).

[42] *J. Smoller*. Shock Waves and Reaction–Diffusion Equations. Springer, (1983).

[43] *G.I. Taylor*. Diffusion by discontinuous movements. Proc. London Math. Soc., pages 25–70, (1920).

[44] *R. Temam*. Infinite–Dimensional Dynamical systems in Mechanics and Physics. Springer, (1988).

[45] *G.F. Webb*. A bifurcation problem for a nonlinear hyperbolic partial differential equation. SIAM J. Math. Anal., 10(10), 922–932, (1979).

[46] *H.F. Weinberger*. Partial Differential Equations. Ginn–Blaisdell, Waltham, (1965).