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ON THE L^2-MOMENT CLOSURE OF TRANSPORT EQUATIONS: THE GENERAL CASE

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Abstract. Transport equations are intensively used in Mathematical Biology. In this article the moment closure for transport equations for an arbitrary finite number of moments is presented. With use of a variational principle the closure can be obtained by minimizing the $L^2(V)$-norm with constraints. An H-Theorem for the negative L^2-norm is shown and the existence of Lagrange multipliers is proven. The Cattaneo closure is a special case for two moments and was studied in Part I (Hillen 2003). Here the general theory is given and the three moment closure for two space dimensions is calculated explicitly. It turns out that the steady states of the two and three moment systems are determined by the steady states of a corresponding diffusion problem.

1. **Introduction.** In this article the moment closure for a class of transport equations is studied, which are used in mathematical biology. Based on a variational principle the moment system will be closed for general turning kernel, for general bounded spaces of velocities and for an arbitrary finite number of moments. The L^2-moment closure was introduced in an earlier paper (Part I) [6], where the 2-moment closure (Cattaneo approximation) for a specific transport equation was studied in detail. In Part I nonlinearities due to birth, death, cell interactions, and oriented movement were also studied. The general theory is developed further in this paper for linear transport equations. Extensions to the nonlinear case are briefly mentioned.

The moment closure procedure is based on an L^2-norm minimization method. Besides of a careful notation of tensor indices, the moment closure requires two main ingredients, which are proven in this paper. First an H-Theorem for the negative L^2-norm (Theorem 3.5), which ensures that the negative L^2-norm can be seen as a physical entropy for the transport equation. The closure then corresponds to entropy maximization. Secondly, the existence of Lagrange multipliers for the associated variational problem is proven in Theorem 3.6.

In Hadeler [4] and Hillen [5, 6, 7] the relevance of transport equations and moment closure to biological applications is discussed in detail. The relations to other moment closure methods as they are known for Boltzmann equations [3], for the

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semiconductor transport equation [15, 10] and in the theory of *Extended Thermodynamics* [11] are presented. A large number of references is given in Part I. For this paper, a short introduction is sufficient.

The paper proceeds as follows. In Section 2 a class of transport equation in a general form is introduced. Some basic notations for moments, moment tensors and velocity tensors are given. Section 3 presents the L^2-moment closure procedure. The H-Theorem (Theorem 3.5) and the existence of Lagrangian multipliers (Theorem 3.6) are proven. The latter is a key to obtain the moment closure. In Extended Thermodynamics the existence of Lagrangian multipliers was shown by Liu [9]. Liu's result is specific to the physical application and not applicable to the case studied here. Theorem 3.6 is proven with the use of basic variational principles. Also in Section 3 explicit formulas for the closed systems are given (equations (28)-(31)). In Section 4 two examples of the theory are given: the 2-moment Cattaneo closure, which was discussed in Part I [6] in detail, and the 3-moment closure in two spatial dimensions. "Three-moment closure" refers to closure for the fourth order moment. In 3-D the three-moment closure consists of 13 dependent functions $(M^0, M^1, M^2, M^3, M^{11}, \dots, M^{33})$. Finally, in Section 5 the steady states are calculated for the 2- and 3-moment closures and it is shown that they are steady states of the corresponding diffusion limit.

2. **Transport Equations.** As shown by Stroock [16] and Othmer *et al.* [12] the movement characteristics of flagellated bacteria and other organisms can be modeled by a linear transport equation for the population density $p(t, x, v)$ at time $t \geq 0$, space $x \in \mathbb{R}^n$ and velocity $v \in V \subset \mathbb{R}^n$. The set of velocities V is compact and in some cases, where indicated, symmetry is assumed. The linear transport equation reads

$$\frac{\partial}{\partial t} p(t, x, v) + v \cdot \nabla p(t, x, v) = -\mu p(t, x, v) + \mu \int_V T(v, v') p(t, x, v') dv', \qquad (1)$$

where μ denotes the turning rate and $T(v, v')$ the distribution of newly chosen velocities. Transport equations with nonlinearities and with terms for oriented movement are discussed in Part I [6] and also in [1]. In Hillen and Othmer [8, 13] the diffusion limit of transport equations was considered in great detail. General conditions were given such that a diffusion limit exists, which usually is non-isotropic. Moreover, applications to reaction-transport equations and to transport equations for chemosensitive movement were considered. See also the review [5].

As in Hillen *et al.* [7, 8], the following basic assumptions are made:

(T1) $T(v, v') \geq 0, \quad \int_V T(v, v') dv = 1, \quad$ and $\quad \int_V \int_V T^2(v, v') dv' dv < \infty.$

(T2) There exist some $u_0 \geq 0$ with $u_0 \not\equiv 0$, some integer N and a constant $\rho > 0$ such that for all $(v, v') \in V \times V$

$$u_0(v) \leq \; T^N(v', v) \; \leq \rho u_0(v),$$

where the N-th iterate of T is

$$T^N(v, v') := \int \dots \int T(v, w_1) T(w_1, w_2) \cdots T(w_{N-1}, v') dw_1 \dots dw_{N-1}.$$

(T3) We introduce an integral operator $\mathcal{T}$ by

$$\mathcal{T} p = \int_V T(v, v') p(v') dv$$

and we assume that $\|\mathcal{T}\|_{\langle 1\rangle^\perp} < 1$, where $\langle 1\rangle^\perp$ denotes the orthogonal complement of the subspace $\langle 1\rangle \subset L^2(V)$ of functions constant in v.

(T4) $\int_V T(v,v')dv' = 1$.

The *turning operator* is defined as $\mathcal{L} := -\mu(I-\mathcal{T})$ and Proposition 2.1 of [8] applies:

Proposition 2.1. *Assume* (T1)-(T4). *Then*

1. 0 *is a simple eigenvalue of* $\mathcal{L}$ *with eigenfunction* $\phi(v) \equiv 1$.
2. *There exists an orthogonal decomposition* $L^2(V) = \langle 1\rangle \oplus \langle 1\rangle^\perp$ *and for all* $\psi \in \langle 1\rangle^\perp$ *we have*
$$\int \psi\mathcal{L}\psi dv \le -\nu_2\|\psi\|^2_{L^2(V)}, \quad \textit{with} \quad \nu_2 \equiv \mu(1-\|\mathcal{T}\|_{\langle 1\rangle^\perp}).$$
3. *Each eigenvalue* $\lambda \neq 0$ *satisfies* $-2\mu < Re\,\lambda \le -\nu_2 < 0$*, and there is no other positive eigenfunction.*
4. $\|\mathcal{L}\|_{\mathcal{L}(L^2(V),L^2(V))} \le 2\mu$.
5. $\mathcal{L}$ *restricted to* $\langle 1\rangle^\perp \subset L^2(V)$ *has a linear inverse* $\mathcal{F}$ *with norm*
$$\|\mathcal{F}\|_{\mathcal{L}(\langle 1\rangle^\perp,\langle 1\rangle^\perp)} \le \frac{1}{\nu_2}.$$

In addition to (T1)-(T4) we assume that

(T5) For each $v' \in V$ there exists a moment-generating function for $T(.,v')$.

Assumption (T5) ensures that the v-moments of the kernel T are bounded and that the distribution $T(.,v')$ can be generated from its moments (see Billingsley [2]).

With use of Stone's theorem (see e.g. Pazy [14]) it is straightforward that under the above assumptions the transport equation (1) generates a strongly continuous solution group on $L^2(\mathbb{R}^n \times V)$. For initial data
$$\phi_0 \in \mathcal{D} := \{\phi \in L^2(\mathbb{R}^n \times V); \phi(.,v) \in H^1(\mathbb{R}^n)\}$$
a unique solution exists globally in
$$\mathcal{X} = C^1([0,\infty), L^2(\mathbb{R}^n \times V)) \cap C([0,\infty),\mathcal{D}). \tag{2}$$

2.1. **Notations.** A careful notation of tensor indices is absolutely necessary for the theory to be developed further. The following notations turns out to be very helpful.

The velocity moments of a distribution function $p(t,x,v)$ are defined as
$$\begin{aligned}
m^0(t,x) &= \int p(t,x,v)dv \\
m^i(t,x) &= \int v^i p(t,x,v)dv, \quad i \in \{1,\dots,n\} \\
&\vdots \\
m^{i_1\dots i_k}(t,x) &= \int v^{i_1}\cdots v^{i_k} p(t,x,v)dv, \quad k \in \mathbb{N}, \quad (i_1,\dots,i_k) \in \{1,\dots,n\}^k.
\end{aligned}$$

Here tensor notation is used, which means that $m^{i_1\dots i_k}$ denotes the $(i_1,\dots,i_k)$-component of a k-tensor. In Euclidean space $\mathbb{R}^n$, both sub and super indices are used and the summation convention is applied on repeated indices, e.g.

$$\Lambda_{i_1\dots i_k} m^{i_1\dots i_k} = \sum_{(i_1,\dots,i_k)\in\{1,\dots,n\}^k} \Lambda_{i_1\dots i_k} m^{i_1\dots i_k}.$$

It is, however, ensured that a specific function or parameter appears only covariant or contravariant, respectively.

For fixed $k \in \mathbb{N}$ the tuple of all tensor indices for tensors of lower than or equal to order k is denoted by

$$\begin{aligned} \alpha_k \quad := \quad & (0,1,2,\dots,n,(1,1),(1,2),\dots,(n,n),\dots, \\ & \dots,(\underbrace{1,\dots,1}_{k \text{ times}}),\dots,(\underbrace{n,\dots,n}_{k \text{ times}})). \end{aligned} \tag{3}$$

The index-vector α_k has the length

$$|\alpha_k| = \sum_{l=0}^{k} n^l =: N_k.$$

Then m^{α_k} denotes a vector of length N_k of all moments of order $\leq k$:

$$\begin{aligned} m^{\alpha_k} \quad := \quad & \big(m^0, m^1, m^2, \dots, m^n, m^{11}, m^{12}, \dots, m^{nn}, \dots, \\ & \dots, m^{1\dots 1}, \dots, m^{n\dots n}\big). \end{aligned} \tag{4}$$

This notation is used for products of velocity components as well; $v^{i_1\cdots i_l} = v^{i_1}\cdots v^{i_l}$ and it makes sense to write v^{α_k}.

To distinguish between different summations β_k is used equivalently with α_k.

2.2. **The Velocity Tensors.** The mean of the velocity tensors are defined as

$$\bar{v}^{i_1\dots i_k} := \int v^{i_1}\cdots v^{i_k} dv.$$

For the specific choice of $V = sS^{n-1}$ the $\bar{v}^{i_1\cdots i_k}$ can be calculated explicitly: It is clear that $\bar{v}^0 = \int dv = \omega = \omega_0 s^{n-1}$, with $\omega_0 = |S^{n-1}|$, and that $\bar{v}^i = \int v^i dv = 0$. Moreover, explicit formulas for the velocity tensors $\bar{v}^{i_1\cdots i_k}$ for odd and even orders are given.

Lemma 2.2. *Assume $V = sS^{n-1}$.*

1. *If $k \in \mathbb{N}$ is odd, then*

$$\bar{v}^{i_1\dots i_k} = 0, \quad \textit{for all} \quad i_1,\dots,i_k \in \{1,\dots,n\}.$$

2. *If $k \in \mathbb{N}$ is even, then there is a constant $c_k > 0$ such that*

$$\bar{v}^{i_1\dots i_k} = s^{k+n-1} c_k \left(\sum_{\mathcal{P}(i_1,\dots,i_k)} \delta^{i_{j_1} i_{j_2}} \dots \delta^{i_{j_{k-1}} i_{j_k}} \right), \tag{5}$$

where the set of all pairs of indices out of $(i_1,\dots,i_k)$ is defined as

$$\mathcal{P}(i_1,\dots,i_k) := \Big\{ \Big((i_{j_1}, i_{j_2}),\dots,(i_{j_{k-1}}, i_{j_k})\Big) : \{j_1,\dots,j_k\} = \{1,\dots,k\} \Big\}.$$

The constants c_k are given by

$$c_0 = \omega_0, \quad c_2 = \frac{\omega_0}{n}, \quad c_k = \frac{c_{k-2}}{k-2+n}, \quad \textit{for} \;\; k \geq 4.$$

Proof. **1.:** Let $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$. In case of k odd we split V into V^+ and V^- defined by

$$V^+ := \{v \in V : v^{i_1} > 0\}, \qquad V^- := \{v \in V : v^{i_1} < 0\}.$$

Then for each $v \in V^+$ we have $-v \in V^-$. Since the set of $\{v^{i_1} = 0\} \subset V$ is a set of measure zero we get

$$\begin{aligned} \bar{v}^{i_1 \dots i_k} &= \int_{V^+} v^{i_1} \cdots v^{i_k} dv + \int_{V^-} v^{i_1} \cdots v^{i_k} dv \\ &= \int_{V^-} (-1)^k v^{i_1} \cdots v^{i_k} dv + \int_{V^-} v^{i_1} \cdots v^{i_k} dv \ = \ 0 \end{aligned}$$

since k is assumed to be odd.

2.: In the case of k even we use an induction argument and the divergence theorem on the ball $B_s(0)$ in $\mathbb{R}^n$.
<u>$\boldsymbol{k = 0}$</u>: $\bar{v}^0 = \omega_0 s^{n-1}$.
<u>$\boldsymbol{k = 2}$</u>: For any two vectors $a^1, a^2 \in \mathbb{R}^n$ we obtain

$$\begin{aligned} a^1_{i_1} a^2_{i_2} \bar{v}^{i_1 i_2} &= \int_V (a^1_{i_1} v^{i_1} a^2_{i_2} v^{i_2}) dv \\ &= s \int_V \frac{v_{i_1}}{|v|} (a^{1,i_1} a^2_{i_2} v^{i_2}) dv \\ &= s \int_{B_s(0)} \partial_{v_{i_1}} (a^{1,i_1} a^2_{i_2} v^{i_2}) dv \\ &= s \int_{B_s(0)} dv \ a^{1,i_1} a^2_{i_2} \delta^{i_2}_{i_1} \end{aligned}$$

Now we have

$$|B_s(0)| = s^n |B_1(0)| = \frac{s^n}{n} \int_{B_1(0)} \partial_{v_i} v^i dv = \frac{s^n}{n} \int_{S^{n-1}} \sigma_i \sigma^i d\sigma = \frac{s^n}{n} \omega_0.$$

Then we get

$$a^1_{i_1} a^2_{i_2} \bar{v}^{i_1 i_2} = s^{n+1} \frac{\omega_0}{n} a^1_{i_1} a^2_{i_2} \delta^{i_1 i_2},$$

which shows that

$$\bar{v}^{i_1 i_2} = s^{n+1} \frac{\omega_0}{n} \delta^{i_1 i_2}. \tag{6}$$

Since in the case $k = 2$ the set of pairs $\mathcal{P}(i_1, i_2)$ for $i_1, i_2 \in \{1, \dots, n\}$ reduces to

$$\begin{aligned} \mathcal{P}(i_1, i_2) &= \big\{(i_{j_1}, i_{j_2}) : \ \{j_1, j_2\} \in \{1, 2\}^2 \text{ with } \{j_1, j_2\} = \{1, 2\}\big\} \\ &= \big\{(i_1, i_2)\big\}, \end{aligned}$$

we obtain

$$\sum_{\mathcal{P}(i_1, i_2)} \delta^{i_{j_1} i_{j_2}} = \delta^{i_1, i_2}.$$

and (6) is (5) for $k = 2$.

$k-2 \to k$: Assume (5) holds for $k-2$. For any vectors $a^1, \dots, a^k \in \mathbb{R}^n$ we have

$$
\begin{aligned}
a^1_{i_1} \dots a^k_{i_k} \bar{v}^{i_1 \dots i_k} &= \int_V (a^1_{i_1} v^{i_1} \dots a^k_{i_k} v^{i_k}) dv \\
&= s \int_V \frac{v_{i_1}}{|v|} a^{1,i_1} \left(a^2_{i_2} v^{i_2} \dots a^k_{i_k} v^{i_k} \right) dv \\
&= s \int_{B_s(0)} \partial_{v_{i_1}} a^{1,i_1} \left(\prod_{l=2}^k a^l_{i_l} v^{i_l} \right) dv \\
&= s \int_{B_s(0)} dv\, a^{1,i_1} \sum_{r=2}^k a^r_{i_r} \delta^{i_r}_{i_1} \left(\prod_{l=2, l\neq r}^k a^l_{i_l} v^{i_l} \right) dv \\
&= s a^{1,i_1} \sum_{r=2}^k a^r_{i_r} \delta^{i_r}_{i_1} \int_{B_s(0)} \prod_{l=2, l\neq r}^k a^l_{i_l} v^{i_l}\ dv. \qquad (7)
\end{aligned}
$$

To exclude one entry from a tuple we will now use the notation for $l \leq r \leq k, l < k$

$$
(i_l, \dots, i_k)_{\setminus\{r\}} := \begin{cases} (i_{l+1}, \dots, i_k), & \text{if } r = l, \\ (i_l, \dots, i_{r-1}, i_{r+1}, \dots i_k) & \text{if } l < r < k, \\ (i_l, \dots, i_{k-1}), & \text{if } r = k. \end{cases}
$$

With use of this notation we study the integral term in (7) separately. We will use the assumption that (5) holds for $k-2$.

$$
\begin{aligned}
& \int_{B_s(0)} \prod_{l=2, l\neq r}^k a^l_{i_l} v^{i_l}\ dv \\
= & \int_0^s \int_{\sigma S^{n-1}} \left(\prod_{l=2, l\neq r}^k a^l_{i_l} v^{i_l} \right) dv d\sigma \\
= & \int_0^s \sigma^{k-2+n-1} c_{k-2} \left(a^2_{i_2} \dots a^k_{i_k} \right)_{\setminus\{r\}} \sum_{\mathcal{P}((i_1,\dots,i_k)_{\setminus\{r\}})} \delta^{i_{j_1} i_{j_2}} \dots \delta^{i_{j_{k-3}} i_{j_{k-2}}}\ d\sigma \\
= & \frac{s^{k-2+n}}{k-2+n} c_{k-2} \left(a^2_{i_2} \dots a^k_{i_k} \right)_{\setminus\{r\}} \sum_{\mathcal{P}((i_1,\dots,i_k)_{\setminus\{r\}})} \delta^{i_{j_1} i_{j_2}} \dots \delta^{i_{j_{k-3}} i_{j_{k-2}}}.
\end{aligned}
$$

Using this equality in (7) we finally get

$$
\begin{aligned}
& a^1_{i_1} \dots a^k_{i_k} \bar{v}^{i_1 \dots i_k} \\
= & s^{k+n-1} \frac{c_{k-2}}{k-2+n} a^{1,i_1} \sum_{r=2}^k a^r_{i_r} \delta^{i_r}_{i_1} \\
& \left(\left(a^2_{i_2} \dots a^k_{i_k} \right)_{\setminus\{r\}} \sum_{\mathcal{P}((i_1,\dots,i_k)_{\setminus\{r\}})} \delta^{i_{j_1} i_{j_2}} \dots \delta^{i_{j_{k-3}} i_{j_{k-2}}} \right) \\
= & s^{k+n-1} c_k \left(\sum_{\mathcal{P}(i_1,\dots,i_k)} \delta^{i_{j_1} i_{j_2}} \dots \delta^{i_{j_{k-1}} i_{j_k}} \right).
\end{aligned}
$$

□

Example for $\mathcal{P}(i_1,\ldots,i_4)$:

$$\begin{aligned}\mathcal{P}(i_1,\ldots,i_4) &= \Big\{\Big((i_{j_1},i_{j_2}),(i_{j_3},i_{j_4})\Big):\\ &\qquad \{j_1,j_2\},\{j_3,j_4\}\in\{1,2,3,4\}^2,\ \text{with}\ \{j_1,j_2,j_3,j_4\}=\{1,2,3,4\}\Big\}\\ &= \Big\{\Big((i_1,i_2),(i_3,i_4)\Big),\Big((i_1,i_3),(i_2,i_4)\Big),\Big((i_1,i_4),(i_2,i_3)\Big)\Big\}. \qquad (8)\end{aligned}$$

In case of $n=2$ with polar representation $v=s(\cos\theta,\sin\theta)$ we explicitly calculate, e.g.

$$\bar v^{1111}=\int_0^{2\pi}\cos^4\theta d\theta=3\frac{\pi}{4},\quad \bar v^{1122}=\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta=\frac{\pi}{4},$$
$$\bar v^{1222}=\int_0^{2\pi}\cos\theta\sin^3\theta d\theta=0.$$

2.3. Symmetry of the Moments and the Velocity Tensors.

Lemma 2.3. *The tensors $m^{i_1\cdots i_k}$, and $\bar v^{i_1\cdots i_k}$ are invariant with respect to exchange of two indices.*

This follows directly from the definitions of $m^{i_1\cdots i_k}$, and $\bar v^{i_1\cdots i_k}$. For later use we will introduce an operator for change of two indices. For $1\le r\le l\le k$, $1<k$ we define

$$\eta_{r,l}(i_1,\ldots,i_r,\ldots,i_l,\ldots,i_k):=(i_1,\ldots,i_l,\ldots,i_r,\ldots,i_k).$$

And we allow $\eta_{r,l}$ to act on tensors and vectors as well, i.e.

$$\eta_{r,l}a^{i_1\ldots i_k}:=a^{\eta_{r,l}(i_1\ldots i_k)},\qquad \text{etc..}$$

3. Moment Closure.

We derive the system of moment equations by multiplying with combinations of $v^{i_1}\cdots v^{i_k}$ and integrating along V: Integration of (1) leads, with $\int T(v,v')dv=1$, to a conservation law for the particle number:

$$m_t^0+\partial_j m^j=0. \qquad (9)$$

For higher-order moment equations we use the following abbreviation. Let the *T-modulated moments of* $p(t,x,v)$ be denoted by

$$w^{i_1\ldots i_k}:=\int_V\int_V v^{i_1}\cdots v^{i_k}T(v,v')p(t,x,v')dv'dv. \qquad (10)$$

Using this definition, multiplication of equation (1) by v^i and integration leads to

$$\begin{aligned}m_t^i+\partial_j m^{ij} &= -\mu m^i+\mu\int\int v^iT(v,v')dv\,p(t,x,v')dv'\\ &= \mu(w^i-m^i). \qquad (11)\end{aligned}$$

and analogously we get for the l-moment, $l\le k$:

$$m^{i_1\ldots i_l}+\partial_j m^{i_1\ldots i_l j}=\mu(w^{i_1\ldots i_l}-m^{i_1\ldots i_l}). \qquad (12)$$

Finally, for all $k\in\mathbb{N}$ we have the system of moments which consists of equations (9), (11) and (12) for all $l\le k$. In the highest-order equation for $m^{i_1\cdots i_k}$ the divergence of the next higher moment $m^{i_1\cdots i_k j}$ appears, hence the system is not closed. If, moreover, the T-modulated moments depend on moments of p of order $>k$, then these higher moments appear as well. We will show that in some important cases the T-modulated moments of order k are linear functions of p-moments of order less than or equal to k (Lemma 3.3). We give two examples first:

Example 3.1. 1. *Assume $T(v,v') = 1/\omega$ describes uniform choice of any direction. Then*

$$w^{i_1\dots i_k} = \frac{\bar{v}^{i_1\dots i_k}}{\omega}\, m^0.$$

2. *Assume, for example, that $T(v,v') = \delta(v-v')$ (which is not included in our general hypotheses, but illustrates possible dependencies). Then*

$$w^{i_1\dots i_k} = m^{i_1\dots i_k}.$$

Since we aim to close the moment system (9), (11) and (12) with respect to the k-th order moment we distinguish two cases:

Definition 3.2. *The system of moments (9), (11) and (12) is called* <u>*k-quasi closed*</u> *if all T-modulated moments of order less than or equal to k depend on p only via the moments of p of order less than or equal to k, but not higher, i.e.*

$$w^{\alpha_k} = w^{\alpha_k}(m^{\alpha_k}).$$

The moment systems in both examples in Example 3.1 are k-quasi closed for each $k \in \mathbb{N}, k \geq 1$.

If the moment system is not k-quasi closed then we have to use the minimization procedure below to find good approximations for w^{α_k} as well.

Lemma 3.3. 1. *If $w^{i_1\dots i_k}$ depends on some moments of p it is a linear function of these.*

2. *System* (9), (11) *and* (12) *is k-quasi closed if and only if the moments of $T(v,v')$ are linear in v'^{α_k}, i.e. for each $v \in V$ there exists a linear mapping $R_{\alpha_k\times\beta_k} : \mathbb{R}^{N_k} \to \mathbb{R}^{N_k}$ such that*

$$\int v^{\alpha_k} T(v,v')dv = R_{\alpha_k\times\beta_k} v'^{\beta_k}. \tag{13}$$

Proof.

1. We write

$$w^{i_1\dots i_k} = \int Q^{i_1\dots i_k}(v')p(v')dv', \quad \text{with} \quad Q^{i_1\dots i_k}(v) = \int v^{i_1}\cdots v^{i_k} T(v,v')dv'.$$

Now assume $w^{\alpha_k} = w^{\alpha_k}(m^{\alpha_j})$ for some $j \in \mathbb{N}$. Then for two functions $p, q \in L^2(V)$ and $c_1 \in \mathbb{R}$ we have

$$\begin{aligned} w^{\alpha_k}\left(c_1 m_p^{\alpha_j} + m_q^{\alpha_j}\right) &= \int Q^{\alpha_k}(v')\left(c_1 p(v') + q(v')\right)dv' \\ &= c_1 w^{\alpha_k}\left(m_p^{\alpha_j}\right) + w^{\alpha_k}\left(m_q^{\alpha_j}\right). \end{aligned}$$

2. We assume that the moment system is k-quasi closed. Since w^{α_k} is a linear function in m^{α_k}, we can find a linear map $R_{\alpha_k\times\beta_k} : \mathbb{R}^{N_k} \to \mathbb{R}^{N_k}$ with

$$w^{\alpha_k} = R_{\alpha_k\times\beta_k} m^{\beta_k} = \int R_{\alpha_k\times\beta_k} v'^{\beta_k} p(v')dv'. \tag{14}$$

On the other hand

$$w^{\alpha_k} = \int\int v^{\alpha_k} T(v,v')p(v')dv',$$

which equals (14) if and only if

$$\int \left[R_{\alpha_k\times\beta_k} v'^{\beta_k} - \int v^{\alpha_k} T(v,v')dv\right] p(v')dv' = 0, \quad \text{for all} \quad p \in L^2(V).$$

This is true only if

$$R_{\alpha_k\times\beta_k}v'^{\beta_k} = \int v^{\alpha_k}T(v,v')dv.$$

□

Example 3.4. *Besides the examples shown above we get a k-quasi closed moment system if T has the form*

$$T(v,v') = a_0(v) + a_i(v)v'^i + \cdots + a_{i_1\ldots i_l}(v)v'^{i_1}\ldots v'^{i_l} \tag{15}$$

for some $l \le k$ and bounded integrable coefficients $a_{\alpha_k}(v)$.

Note. If a system of moments is l-quasi closed it does not need to be k-quasi closed for $k > l$.

3.1. **Minimizing the L^2-Norm.** First we show that the negative L^2-norm is an entropy for the transport model (1). We denote the $L^2(V)$-norm by

$$E(u) := \int \frac{u^2}{2}dv$$

and the corresponding flux by

$$F(u) := \int v\,\frac{u^2}{2}dv.$$

Theorem 3.5. (H-Theorem) *Assume* (T1)-(T4). *Solutions $p(t,x,v)\in\mathcal{X}$ of the linear transport equation* (1) *satisfy*

$$\frac{d}{dt}E(p) + \partial_j F^j(p) \le 0.$$

Proof.

$$\frac{d}{dt}E(p) = \int p(-v^j\partial_j p + \mathcal{L}p)dv = -\partial_j F^j(p) + \int p\mathcal{L}p\,dv.$$

In Proposition 2.1 it has been shown that on $\langle 1\rangle^\perp$ the operator $\mathcal{L}_0$ satisfies

$$\int p\mathcal{L}p\,dv \le -\mu_2\|p\|_2^2.$$

For $p(t,x,.)\in\langle 1\rangle$ we have $\int p\mathcal{L}p\,dv = 0$. Hence the entropy estimate follows. □

For now we fix (t,x) as a parameter and consider the dependence on v. For functions in $L^2(V)$ we aim to minimize the functional $E(u)$ with constraints of given moments m^{α_k} of order less than or equal to k:

$$G(u) = 0, \qquad \text{with} \quad G(u) = \int v^{\alpha_k}u(v)dv - m^{\alpha_k}.$$

Note that α_k defines a multi-index such that $G : L^2(V)\to\mathbb{R}^{N_k}$.

For minimization of E under the constraint $G = 0$ we use the framework of Lagrangian multipliers as presented e.g. in Zeidler [17]. If u_0 is a minimizer, then

$$E'(u_0) : L^2(V)\to\mathbb{R} : \qquad h\mapsto \int u_0(v)h(v)dv$$

$$G'(u_0) : L^2(V)\to\mathbb{R}^{N_k} : \qquad h\mapsto \int v^{\alpha_k}h(v)dv.$$

Theorem 3.6. *Assume u_{min} is a minimizer, then there exist Lagrangian multipliers $\Lambda_{\alpha_k} \in \mathbb{R}^{N_k}$ such that all $\phi \in L^2(V)$ satisfy*

$$E'(u_{min})\phi + \Lambda_{\alpha_k} G'(u_{min})\phi = 0,$$

where the summation convention is applied for $\Lambda_{\alpha_k} G'(u_{min})\phi$, since $G'(u_{min})\phi \in \mathbb{R}^{N_k}$.

Proof. For the existence of Lagrangian multipliers we have to check two conditions (see Zeidler [17])

: (i) For each $h \in L^2(V)$ with $G'(u_{\min})h = 0$ there exists a curve $\tilde{u}(s)$ such that $\tilde{u}'(0) = h$ and $\tilde{u}$ is admissible, which means that $\tilde{u}$ is differentiable at $s = 0$ and $G(u(s)) = 0$ for $s \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

: (ii) The range $R(G'(u_{\min}))$ is closed.

We first check (i): Consider $h \in L^2(V)$ with $G'(u_{\min})h = 0$. Then

$$\int v^{\alpha_k} h(v) dv = 0^{\alpha_k}, \tag{16}$$

which means that the first k moments of h vanish identically (here 0^{α_k} denotes the zero of $\mathbb{R}^{N_k}$.) We define a curve

$$\tilde{u}(v, s) := p(v) + h(v)s$$

which satisfies

$$\frac{\partial}{\partial s}\tilde{u}(v, 0) = h(v)$$

and

$$G(\tilde{u}(v, s)) = \int v^{\alpha_k} p(v) dv + \int v^{\alpha_k} h(v) dv\ s - m^{\alpha_k} \ = 0^{\alpha_k},$$

with use of (16). Then $\tilde{u}(v, s)$ is admissible, i.e. It is tangential to relative minima of the functional E. Then indeed for each $h \in L^2(V)$ with $G'(u_{\min})h = 0$ there is an admissible curve $\tilde{u}$ and condition (i) is satisfied.

Condition (ii) is immediate in this case. Since $G'(u_{\min})$ is a linear mapping into a finite dimensional space, its range is closed. □

From Theorem 3.6 it follows that for all $\phi \in L^2(V)$ we get

$$\int u_{\min}(v)\phi(v)dv + \Lambda_{\alpha_k} \int v^{\alpha_k} \phi(v)\ dv = 0.$$

Hence the integrand vanishes pointwise and the minimizer satisfies:

$$u_{\min} = -\Lambda_{\alpha_k} v^{\alpha_k}. \tag{17}$$

The first k moments of the minimizer $u_{\min}$ are given by the constraints $G(u_{\min}) = 0$, hence we obtain for $l \le k$ $(i_1, \dots, i_l) \in \{1, \dots, n\}^l$

$$m^{i_1 \dots i_l} = -\int v^{i_1} \cdots v^{i_l} \Lambda_{\alpha_k} v^{\alpha_k}\ dv. \tag{18}$$

This is a linear system for the Lagrangian multiplier Λ_{α_k}. Since from Theorem 3.6 we know that this multiplier exists it must be a linear function of the first k moments. Hence there is a $N_k \times N_k$-matrix $\mathcal{B}_{\alpha_k \times \beta_k}$ with

$$\Lambda_{\alpha_k} = \mathcal{B}_{\alpha_k \times \beta_k} m^{\beta_k}, \tag{19}$$

hence

$$u_{\min} = -v^{\alpha_k} \mathcal{B}_{\alpha_k \times \beta_k} m^{\beta_k}. \tag{20}$$

With use of the notation of the velocity tensor $\bar{v}^{i_1\cdots i_k}$ introduced above we can write the linear system (18) in explicit form.

$$\begin{aligned} m^0 &= -\Lambda_0\bar{v}^0 - \Lambda_j\bar{v}^j - \cdots - \Lambda_{j_1\ldots j_k}\bar{v}^{j_1\cdots j_k} \\ m^i &= -\Lambda_0\bar{v}^i - \Lambda_j\bar{v}^{ij} - \cdots - \Lambda_{j_1\ldots j_k}\bar{v}^{ij_1\cdots j_k} \\ &\vdots \\ m^{i_1\cdots i_k} &= -\Lambda_0\bar{v}^{i_1\cdots i_k} - \cdots - \Lambda_{j_1\ldots j_k}\bar{v}^{i_1\cdots i_k j_1\cdots j_k} \end{aligned} \tag{21}$$

In case of $V = sS^{n-1}$ the odd velocity tensors vanish identically (see Lemma 2.2), and the system de-couples into two independent systems for odd and even multipliers.

If $k \in \mathbb{N}$ is even and $V = sS^{n-1}$ then we obtain for the even indices

$$\begin{aligned} m^0 &= -\Lambda_0\bar{v}^0 - \Lambda_{j_1j_2}\bar{v}^{j_1j_2} - \cdots - \Lambda_{j_1\ldots j_k}\bar{v}^{j_1\cdots j_k} \\ m^{i_1i_2} &= -\Lambda_0\bar{v}^{i_1i_2} - \Lambda_{j_1j_2}\bar{v}^{i_1i_2j_1j_2} - \cdots - \Lambda_{j_1\ldots j_k}\bar{v}^{i_1i_2j_1\cdots j_k} \\ &\vdots \\ m^{i_1\cdots i_k} &= -\Lambda_0\bar{v}^{i_1\cdots i_k} - \cdots - \Lambda_{j_1\ldots j_k}\bar{v}^{i_1\cdots i_k j_1\cdots j_k} \end{aligned} \tag{22}$$

and for the odd indices

$$\begin{aligned} m^i &= -\Lambda_j\bar{v}^{ij} - \cdots - \Lambda_{j_1\ldots j_{k-1}}\bar{v}^{ij_1\cdots j_{k-1}} \\ &\vdots \\ m^{i_1\cdots i_{k-1}} &= -\Lambda_j\bar{v}^{i_1\cdots i_{k-1}j} - \cdots - \Lambda_{j_1\ldots j_{k-1}}\bar{v}^{i_1\cdots i_{k-1}j_1\cdots j_{k-1}}. \end{aligned} \tag{23}$$

In case of $k \in \mathbb{N}$ is odd and $V = sS^{n-1}$ then we obtain the following two de-coupled systems. For the even indices:

$$\begin{aligned} m^0 &= -\Lambda_0\bar{v}^0 - \cdots - \Lambda_{j_1\ldots j_{k-1}}\bar{v}^{j_1\cdots j_{k-1}} \\ &\vdots \\ m^{i_1\cdots i_{k-1}} &= -\Lambda_0\bar{v}^{i_1\cdots i_{k-1}} - \cdots - \Lambda_{j_1\ldots j_{k-1}}\bar{v}^{i_1\cdots i_{k-1}j_1\cdots j_{k-1}} \end{aligned} \tag{24}$$

and for the odd indices

$$\begin{aligned} m^i &= -\Lambda_j\bar{v}^{ij} - \cdots - \Lambda_{j_1\ldots j_k}\bar{v}^{ij_1\cdots j_k} \\ &\vdots \\ m^{i_1\cdots i_k} &= -\Lambda_j\bar{v}^{i_1\cdots i_k j} - \cdots - \Lambda_{j_1\ldots j_k}\bar{v}^{i_1\cdots i_k j_1\cdots j_k}. \end{aligned} \tag{25}$$

We will use these equations to consider explicit examples later.

The above systems of equations are invariant under exchange of pairs of indices. Hence it follows that

Lemma 3.7. *The Lagrangian multipliers $\Lambda^{i_1\cdots i_k}$ are symmetric with respect to exchange of indices.*

Now we proceed with the general notion of (18) to find the general moment closure.

3.2. **Moment Closure.** We consider the unknown $(k+1)$-st moment of $u_{\min}$. Using (20), we get

$$\int v^{i_1}\cdots v^{i_{k+1}} u_{\min}(v)dv = -\int v^{i_1}\cdots v^{i_{k+1}} v^{\alpha_k}\mathcal{B}_{\alpha_k\times\beta_k} dv\; m^{\beta_k}.$$

Hence the $(k+1)$-st moment of $u_{\min}$ is a linear combination of the lower-order moments of the form

$$m^{i_1\dots i_{k+1}} = \mathcal{A}^{i_1\dots i_{k+1}}_{\beta_k} m^{\beta_k}, \tag{26}$$

with mappings $\mathcal{A}^{i_1\dots i_{k+1}}_{\beta_k} : \mathbb{R}^{N_k}\to\mathbb{R}$ given by

$$\mathcal{A}^{i_1\dots i_{k+1}}_{\beta_k} := \int v^{i_1}\cdots v^{i_{k+1}} v^{\alpha_k}\mathcal{B}_{\alpha_k\times\beta_k}\, dv. \tag{27}$$

The next step to obtain the moment closure is to assume that the highest moment $m^{i_1\dots i_{k+1}}$ of $p(t,x,v)$ has approximately the same relation to the lower order moments as $u_{\min}$ has, and to replace $m^{i_1\dots i_k j}$ in (12) with (26). Since this is an approximation we switch notation to capital letters $M^{i_1\dots i_l}$ to distinguish from the original (exact) values $m^{i_1\dots i_l}$.

In cases where the system (9), (11) and (12) is k-quasi closed (see Def. 3.2 and Lemma 3.3) we obtain the following closed system:

$$\begin{array}{rcl}
M^0_t + \partial_j M^j &=& 0\\
M^i_t + \partial_j M^{ij} &=& \mu(w^i - M^i)\\
&\vdots&\\
M^{i_1\dots i_l}_t + \partial_j M^{i_1\dots i_l j} &=& \mu(w^{i_1\dots i_l} - M^{i_1\dots i_l})\\
&\vdots&\\
M^{i_1\dots i_k}_t + \partial_j\left(\mathcal{A}^{i_1\dots i_k j}_{\alpha_k} M^{\alpha_k}\right) &=& \mu(w^{i_1\dots i_k} - M^{i_1\dots i_k}),
\end{array} \tag{28}$$

with $w^{\alpha_k} = w^{\alpha_k}(M^{\alpha_k})$ as given in Lemma 3.3.

If the moment system is not k-quasi closed, then the terms $w^{i_1\dots i_k}$ in (12) depend on the original distribution as well. Hence we also assume that they are appropriately approximated by using the minimizer $u_{\min}$ instead of p. This way they will depend on moments of order less than or equal to k. We carry out this approximation in equation (12) and obtain a closed system for approximations to the first k moments:

$$\begin{array}{rcl}
M^0_t + \partial_j M^j &=& 0\\
M^i_t + \partial_j M^{ij} &=& \mu(W^i - M^i)\\
&\vdots&\\
M^{i_1\dots i_l}_t + \partial_j M^{i_1\dots i_l j} &=& \mu(W^{i_1\dots i_l} - M^{i_1\dots i_l})\\
&\vdots&\\
M^{i_1\dots i_k}_t + \partial_j\left(\mathcal{A}^{i_1\dots i_k j}_{\alpha_k} M^{\alpha_k}\right) &=& \mu(W^{i_1\dots i_k} - M^{i_1\dots i_k}),
\end{array} \tag{29}$$

where for $1\le l\le k$ we have *approximate T-modulated moments*

$$W^{i_1\dots i_l} := \int_V\int_V v^{i_1}\cdots v^{i_l} T(v,v')U(t,x,v')dv'dv \tag{30}$$

with an *approximate minimizer*

$$U(t,x,v) := -v^{\alpha_k}\mathcal{B}_{\alpha_k\times\beta_k}M^{\beta_k}, \tag{31}$$

and $\mathcal{B}_{\alpha_k\times\beta_k}$ is given by (19).

Note that the system (29)-(31) indeed defines a closed system for M^{α_k}.

4. **Examples.**

4.1. **The Cattaneo Approximation.**

To obtain the Cattaneo model as a second-order moment approximation we recall the arguments from Part I [6]. We study a transport equation with fixed speed s, and constant turn-angle distribution $T(v,v') = \frac{1}{\omega}$. In this case $V = sS^{n-1}$ with $s > 0$ and we denote $\omega = |V| = s^{n-1}\omega_0$, where $\omega_0 = |S^{n-1}|$.

Then the linear transport equation (1) reads

$$p_t + v\cdot\nabla p = \mu\left(\frac{m^0}{\omega} - p\right). \tag{32}$$

The system for the first two moments is

$$\begin{aligned} m_t^0 + \partial_j m^j &= 0,\\ m_t^i + \partial_j m^{ij} &= -\mu m^i. \end{aligned}$$

The entropy maximizer can be explicitly calculated as

$$u_{\min}(t,x,v) = \frac{1}{\omega}\left(m^0(t,x) + \frac{n}{s^2}(v_i m^i(t,x))\right), \tag{33}$$

where the Lagrange multipliers are given by

$$\Lambda_0 = \frac{1}{\omega}m^0, \quad \Lambda_i = \frac{n}{\omega s^2}m^i, \quad \text{for } i = 1,2,3.$$

The second moment of the above maximizer is

$$m^{ij}(u_{\min}) = \frac{s^2}{n}m^0\delta^{ij},$$

with transition matrices

$$\mathcal{A}_{\beta_2}^{ij} = \frac{s^2}{n}\delta^{ij}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since T is constant the moment system is 2-quasi closed. Hence the two-moment closure is given by a linear Cattaneo system

$$\begin{aligned} M_t^0 + \partial_j M^j &= 0,\\ M_t^i + \tfrac{s^2}{n}\partial_i M^0 &= -\mu M^i. \end{aligned} \tag{34}$$

In Part I we also consider nonlinear terms and drift terms and we prove approximation properties for the two-moment closure. For details we refer to Part I [6].

4.2. **The Three-Moment Equations.** In case of $k=3$ and $n=2$ and $V=sS^1$ we study the above procedure explicitly to find a closed system for the first three moments M^0, M^i, M^{i_1,i_2}, $i, i_1, i_2 \in \{1,2\}$. The 3-moment system reads:

$$\begin{array}{rcll} m_t^0 + \partial_j m^j &=& 0 & \\ m_t^i + \partial_j m^{ij} &=& \mu(w^i - m^i), & i=1,2 \\ m_t^{i_1 i_2} + \partial_j m^{i_1 i_2 j} &=& \mu(w^{i_1 i_2} - m^{i_1 i_2}) & i_1, i_2 = 1,2. \end{array} \tag{35}$$

We use systems (24) and (25) to find expressions for the Lagrangian multipliers $\Lambda_0, \Lambda_i, \Lambda_{i_1 i_2}$. In the present case system (25) for odd indices is

$$\begin{pmatrix} m^1 \\ m^2 \end{pmatrix} = -\begin{pmatrix} \bar{v}^{11} & \bar{v}^{12} \\ \bar{v}^{21} & \bar{v}^{22} \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}. \tag{36}$$

Now, with use of Lemma 2.2, we obtain, with $\omega_0 = |S^1| = 2\pi$,

$$\bar{v}^{11} = \bar{v}^{22} = s^3\pi, \qquad \bar{v}^{12} = \bar{v}^{21} = 0.$$

Then (36) is immediately solved with

$$\Lambda_i = -\frac{1}{\pi s^3} m^i, \qquad \text{for } i = 1,2. \tag{37}$$

The system (24) for the even indices reads in this case:

$$\begin{pmatrix} m^0 \\ m^{11} \\ m^{12} \\ m^{21} \\ m^{22} \end{pmatrix} = -\begin{pmatrix} \bar{v}^0 & \bar{v}^{11} & \bar{v}^{12} & \bar{v}^{21} & \bar{v}^{22} \\ \bar{v}^{11} & \bar{v}^{1111} & \bar{v}^{1112} & \bar{v}^{1121} & \bar{v}^{1122} \\ \bar{v}^{12} & \bar{v}^{1211} & \bar{v}^{1212} & \bar{v}^{1221} & \bar{v}^{1222} \\ \bar{v}^{21} & \bar{v}^{2111} & \bar{v}^{2112} & \bar{v}^{2121} & \bar{v}^{2122} \\ \bar{v}^{22} & \bar{v}^{2211} & \bar{v}^{2212} & \bar{v}^{2221} & \bar{v}^{2222} \end{pmatrix} \begin{pmatrix} \Lambda_0 \\ \Lambda_{11} \\ \Lambda_{12} \\ \Lambda_{21} \\ \Lambda_{22} \end{pmatrix}. \tag{38}$$

Again we use Lemma 2.2 to obtain explicit values for the velocity tensors. Especially in (8) we explicitly calculated the four-velocity tensor. In the present case the relevant constant is $c_4 = \frac{\pi}{4}$. Then the matrix in (38) is given by

$$\begin{pmatrix} s2\pi & s^3\pi & 0 & 0 & s^3\pi \\ s^3\pi & 3\sigma & 0 & 0 & \sigma \\ 0 & 0 & \sigma & \sigma & 0 \\ 0 & 0 & \sigma & \sigma & 0 \\ s^3\pi & \sigma & 0 & 0 & 3\sigma \end{pmatrix} \qquad \text{with } \sigma = s^5\frac{\pi}{4}.$$

Hence the equations for the mixed indices de-couple and due to symmetry (see Lemmata 2.3, 3.7) we have $m^{12} = m^{21}$ and $\Lambda_{12} = \Lambda_{21}$. Then it follows from (38) that

$$\Lambda_{12} = \Lambda_{21} = -\frac{2}{s^5\pi} m^{12}. \tag{39}$$

The remaining system for Λ_0, Λ_{11} and Λ_{22} reads

$$\begin{pmatrix} m^0 \\ m^{11} \\ m^{22} \end{pmatrix} = -\begin{pmatrix} s2\pi & s^3\pi & s^3\pi \\ s^3\pi & 3\sigma & \sigma \\ s^3\pi & \sigma & 3\sigma \end{pmatrix} \begin{pmatrix} \Lambda_0 \\ \Lambda_{11} \\ \Lambda_{22} \end{pmatrix}.$$

We denote the above matrix by J and observe that

$$\det(J) = \frac{\pi}{4} s^{11} \neq 0.$$

Hence J is invertible and we get

$$\begin{pmatrix} \Lambda_0 \\ \Lambda_{11} \\ \Lambda_{22} \end{pmatrix} = -J^{-1} \begin{pmatrix} m^0 \\ m^{11} \\ m^{22} \end{pmatrix}. \tag{40}$$

When we denote $J^{-1} = (\sigma_{ij})_{i,j\in\{1,2,3\}}$ then formula (19) can be written explicitly as

$$\begin{pmatrix} \Lambda_0 \\ \Lambda_1 \\ \Lambda_2 \\ \Lambda_{11} \\ \Lambda_{12} \\ \Lambda_{21} \\ \Lambda_{22} \end{pmatrix} = -\mathcal{B}_{\alpha_2\times\beta_2} \begin{pmatrix} m^0 \\ m^1 \\ m^2 \\ m^{11} \\ m^{12} \\ m^{21} \\ m^{22} \end{pmatrix}, \tag{41}$$

where

$$\mathcal{B}_{\alpha_2\times\beta_2} = \begin{pmatrix} \sigma_{11} & 0 & 0 & \sigma_{12} & 0 & 0 & \sigma_{13} \\ 0 & (\pi s^3)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\pi s^3)^{-1} & 0 & 0 & 0 & 0 \\ \sigma_{21} & 0 & 0 & \sigma_{22} & 0 & 0 & \sigma_{23} \\ 0 & 0 & 0 & 0 & 2(\pi s^5)^{-1} & 2(\pi s^5)^{-1} & 0 \\ 0 & 0 & 0 & 0 & 2(\pi s^5)^{-1} & 2(\pi s^5)^{-1} & 0 \\ \sigma_{31} & 0 & 0 & \sigma_{32} & 0 & 0 & \sigma_{33} \end{pmatrix}.$$

Finally the minimizer $u_{\min}$ given in (17) reads

$$u_{\min} = -\Lambda_0 - \Lambda_j v^j - \Lambda_{j_1 j_2} v^{j_1} v^{j_2}. \tag{42}$$

4.3. **Closure of the 3-Moment Equations.** To close the system (35) for the first three moments m^0, m^i, m^{ij} we consider the third moment of the minimizer $u_{\min}$, given in (42). For $i_1, i_2, i_3 \in \{1,2\}$ we obtain, using the representation of $\bar{v}$:

$$\begin{aligned} m^{i_1 i_2 i_3}(u_{\min}) &= \int v^{i_1} v^{i_2} v^{i_3} u_{\min} dv \\ &= -\Lambda_0 \bar{v}^{i_1 i_2 i_3} - \Lambda_j \bar{v}^{i_1 i_2 i_3 j} - \Lambda_{j_1 j_2} v^{i_1 i_2 i_3 j_1 j_2} \\ &= \frac{1}{\pi s^3}\left(m^1 \bar{v}^{i_1 i_2 i_3 1} + m^2 \bar{v}^{i_1 i_2 i_3 2}\right). \end{aligned}$$

Then, with (8), we get

$$\begin{aligned} m^{111}(u_{\min}) &= \frac{1}{\pi s^3}\left(3\sigma m^1\right) = \frac{3}{4} s^2 m^1 \\ m^{112}(u_{\min}) = m^{121}(u_{\min}) = m^{211}(u_{\min}) &= \frac{s^2}{4} m^2 \\ m^{122}(u_{\min}) = m^{212}(u_{\min}) = m^{221}(u_{\min}) &= \frac{s^2}{4} m^1 \\ m^{222}(u_{\min}) &= \frac{3}{4} s^2 m^2. \end{aligned}$$

The linear forms $\mathcal{A}^{i_1 i_2 i_3}$ defined in (27) are given by

$$\begin{aligned} A^{111} &= (0, 3s^2/4, 0, 0, 0, 0, 0) \\ A^{112} = A^{121} = A^{211} &= (0, 0, s^2/4, 0, 0, 0, 0) \\ A^{122} = A^{212} = A^{221} &= (0, s^2/4, 0, 0, 0, 0, 0) \\ A^{222} &= (0, 0, 3s^2/4, 0, 0, 0, 0), \end{aligned}$$

which are linear forms for the vector $m^{\alpha_2} = (m^0, m^1, m^2, m^{11}, m^{12}, m^{21}, m^{22})^T$.

The crucial term in (35) is $\partial_j m^{i_1 i_2 j}$. For the moments of $u_{\min}$ we get

$$\begin{aligned} \partial_1 m^{111}(u_{\min}) + \partial_2 m^{112}(u_{\min}) &= \frac{s^2}{4}\left(3\partial_1 m^1 + \partial_2 m^2\right) \\ \partial_1 m^{121}(u_{\min}) + \partial_2 m^{122}(u_{\min}) &= \frac{s^2}{4}\left(\partial_1 m^2 + \partial_2 m^1\right) \\ \partial_1 m^{211}(u_{\min}) + \partial_2 m^{212}(u_{\min}) &= \frac{s^2}{4}\left(\partial_1 m^2 + \partial_2 m^1\right) \\ \partial_1 m^{221}(u_{\min}) + \partial_2 m^{222}(u_{\min}) &= \frac{s^2}{4}\left(\partial_1 m^1 + 3\partial_2 m^2\right). \end{aligned}$$

Again we choose capital letters M^0, M^i, M^{ij} to finally close the moment system

$$\begin{aligned} M_t^0 + \partial_j M^j &= 0 \\ M_t^1 + \partial_1 M^{11} + \partial_2 M^{12} &= \mu(W^1 - M^1) \\ M_t^2 + \partial_1 M^{21} + \partial_2 M^{22} &= \mu(W^2 - M^2) \\ M_t^{11} + \tfrac{s^2}{4}\left(3\partial_1 M^1 + \partial_2 M^2\right) &= \mu(W^{11} - M^{11}) \\ M_t^{12} + \tfrac{s^2}{4}\left(\partial_1 M^2 + \partial_2 M^1\right) &= \mu(W^{12} - M^{12}) \\ M_t^{21} + \tfrac{s^2}{4}\left(\partial_1 M^2 + \partial_2 M^1\right) &= \mu(W^{21} - M^{21}) \\ M_t^{22} + \tfrac{s^2}{4}\left(\partial_1 M^1 + 3\partial_2 M^2\right) &= \mu(W^{22} - M^{22}), \end{aligned} \tag{43}$$

with

$$W^{i_1 \dots i_l} := \int_V \int_V v^{i_1} \cdots v^{i_l} T(v, v') U(t, x, v') dv'.$$

The approximate minimizer is

$$U(t, x, v) := -\Lambda_0 - \Lambda_j v^j - \Lambda_{j_1 j_2} v^{j_1} v^{j_2} \tag{44}$$

and the approximate multipliers are given by (41) with capital M^{α_2} instead of m^{α_2}.

It is clear that if system (35) is 2-quasi closed then we obtain (43) with w^{α_2} instead of W^{α_2}.

4.4. **A Specific Example.** We assume for now that $T(v, v') = \frac{1}{\omega}$, with $\omega = |sS^1| = 2\pi s$. Then the moment system is 2-quasi closed (see Example 3.1) and we have

$$w^{\alpha_2} = \frac{\bar{v}^{i_1 \dots i_2}}{2\pi s} M^0.$$

Hence

$$\begin{aligned} w^0 &= M^0, & w^1 &= w^1 = 0, \\ w^{11} &= w^{22} = \frac{s^2}{2} M^0 & w^{12} &= w^{21} = 0. \end{aligned}$$

Then the closed moment system reads

$$\begin{array}{rcl} M_t^0 + \partial_j M^j &=& 0 \\ M_t^i + \partial_j M^{ij} &=& -\mu M^i, \qquad i = 1,2 \\ M_t^{11} + \frac{s^2}{4}\left(3\partial_1 M^1 + \partial_2 M^2\right) &=& \mu\left(\frac{s^2}{2}M^0 - M^{11}\right) \\ M_t^{12} + \frac{s^2}{4}\left(\partial_1 M^2 + \partial_2 M^1\right) &=& -\mu M^{12} \\ M_t^{21} + \frac{s^2}{4}\left(\partial_1 M^2 + \partial_2 M^1\right) &=& -\mu M^{21} \\ M_t^{22} + \frac{s^2}{4}\left(\partial_1 M^1 + 3\partial_2 M^2\right) &=& \mu\left(\frac{s^2}{2}M^0 - M^{22}\right). \end{array} \tag{45}$$

We consider a scaling limit for large turning rate $\mu \to \infty$ but finite speed $s < \infty$. Then formally the last four equations of (45) become

$$M^{11} = M^{22} = \frac{s^2}{2}M^0, \qquad M^{12} = M^{21} = 0. \tag{46}$$

The whole system (45) reduces to

$$\begin{array}{rcl} M_t^0 + \partial_j M^j &=& 0 \\ M_t^i + \dfrac{s^2}{2}\partial_i M^0 &=& -\mu M^i, \end{array}$$

which is exactly the two moment - or Cattaneo - approximation in 2-dimensions (34).

It is important to investigate the classical parabolic limit. As shown earlier there are two ways to obtain the parabolic limit for transport equations. One is a parameter scaling of $s \to \infty, \mu \to \infty$ such that $\frac{s^2}{2\mu} \to D < \infty$, the other is to consider scaled space and time variables $\tau = \varepsilon^2 t$ and $\xi = \varepsilon x$. It is easily checked that the first limit is not appropriate for the study of (45), since an additional factor of s^2 appears in the equations for M^{11} and M^{22}. It is however useful to study the scaling of $\tau = \varepsilon^2 t$ and $\xi = \varepsilon x$. In these new coordinates the system (45) reads:

$$\begin{array}{rcl} \varepsilon^2 M_\tau^0 + \varepsilon\partial_j M^j &=& 0 \\ \varepsilon^2 M_\tau^i + \varepsilon\partial_j M^{ij} &=& -\mu M^i, \qquad i = 1,2 \\ \varepsilon^2 M_\tau^{11} + \varepsilon\frac{s^2}{4}\left(3\partial_1 M^1 + \partial_2 M^2\right) &=& \mu\left(\frac{s^2}{2}M^0 - M^{11}\right) \\ \varepsilon^2 M_\tau^{12} + \varepsilon\frac{s^2}{4}\left(\partial_1 M^2 + \partial_2 M^1\right) &=& -\mu M^{12} \\ \varepsilon^2 M_\tau^{21} + \varepsilon\frac{s^2}{4}\left(\partial_1 M^2 + \partial_2 M^1\right) &=& -\mu M^{21} \\ \varepsilon^2 M_\tau^{22} + \varepsilon\frac{s^2}{4}\left(\partial_1 M^1 + 3\partial_2 M^2\right) &=& \mu\left(\frac{s^2}{2}M^0 - M^{22}\right). \end{array} \tag{47}$$

We consider solutions of this system which can be written as a perturbation expansion

$$M^{\alpha_2} = M_{(0)}^{\alpha_2} + \varepsilon M_{(1)}^{\alpha_2} + \varepsilon^2 M_{(2)}^{\alpha_2}.$$

The order one terms of the above system (47) lead to

$$M_{(0)}^i = 0, \quad M_{(0)}^{12} = M_{(0)}^{21} = 0, \quad M_{(0)}^{11} = M_{(0)}^{22} = \frac{s^2}{2}M_{(0)}^0. \tag{48}$$

From the order ε system we only need the second equation of (47) which has the following order ε terms:

$$\partial_j M_{(0)}^{ij} = -\mu M_{(1)}^i. \tag{49}$$

The first equation of the order ε^2 system reads

$$M^0_{(0),\tau} + \partial_j M^j_{(1)} = 0. \tag{50}$$

Together with (49) and the last identity from (48) we obtain the diffusion limit of

$$M^0_{(0),\tau} = \frac{s^2}{2\mu}\partial_i\partial^i M^0_{(0)}. \tag{51}$$

5. **Steady States.** For dissipative processes steady states are typical candidates for limit sets. Moreover the study of steady states for different levels of moment closure helps to get insight into the relation of different closures. Here we consider the example of constant speed $V = s\cdot S^1$ in two dimensions with uniformly distributed velocities $T(v,v') = \frac{1}{\omega}$.

5.1. **Cattaneo-Approximation.** The system for steady states of the Cattaneo approximation (34) is

$$\partial_j M^j = 0, \qquad \frac{s^2}{2\mu}\partial_j M^0 = -M^j, \quad \text{for} \quad j = 1,2.$$

We introduce the second equation into the first and arrive at the Laplace equation

$$\frac{s^2}{2\mu}\Delta M^0 = 0, \quad \text{and} \quad M^j = -\frac{s^2}{2\mu}\partial_j M^0, \tag{52}$$

which describes exactly the steady states of the corresponding heat equation (51).

5.2. **The Three-Moment Closure.** The system for stationary solutions of (45) is

$$\begin{aligned}
\partial_1 M^1 + \partial_2 M^2 &= 0 && (53)\\
\partial_1 M^{11} + \partial_2 M^{12} &= -\mu M^1 && (54)\\
\partial_1 M^{21} + \partial_2 M^{22} &= -\mu M^2 && (55)\\
\frac{s^2}{4}(3\partial_1 M^1 + \partial_2 M^2) &= \mu\left(\frac{s^2}{2}M^0 - M^{11}\right) && (56)\\
\frac{s^2}{4}(\partial_1 M^2 + \partial_2 M^1) &= -\mu M^{12} = -\mu M^{21} && (57)\\
\frac{s^2}{4}(\partial_1 M^1 + 3\partial_2 M^2) &= \mu\left(\frac{s^2}{2}M^0 - M^{22}\right). && (58)
\end{aligned}$$

We solve (56)-(58) for $M^{ij}, i,j = 1,2$ and introduce these into (54) and (55), respectively.

$$\begin{aligned}
\partial_1\left(\frac{s^2}{2}M^0 - \frac{s^2}{4\mu}(3\partial_1 M^1 + \partial_2 M^2)\right) - \partial_2\left(\frac{s^2}{4\mu}(\partial_1 M^2 + \partial_2 M^1)\right) &= -\mu M^1\\
-\partial_1\left(\frac{s^2}{4\mu}(\partial_1 M^2 + \partial_2 M^1)\right) + \partial_2\left(\frac{s^2}{2}M^0 - \frac{s^2}{4\mu}(\partial_1 M^1 + 3\partial_2 M^2)\right) &= -\mu M^2.
\end{aligned}$$

Rearrangement leads to

$$\begin{aligned}
\tfrac{s^2}{4\mu}(3\partial_1\partial_1 + \partial_2\partial_2)M^1 + \tfrac{s^2}{4\mu}(\partial_1\partial_2 + \partial_2\partial_1)M^2 &= \mu M^1 + \tfrac{s^2}{2}\partial_1 M^0\\
\tfrac{s^2}{4\mu}(\partial_1\partial_2 + \partial_2\partial_1)M^1 + \tfrac{s^2}{4\mu}(\partial_1\partial_1 + 3\partial_2\partial_2)M^2 &= \mu M^2 + \tfrac{s^2}{2}\partial_2 M^0.
\end{aligned} \tag{59}$$

We differentiate the first equation with respect to x_1 and the second equation with respect to x_2 and we add the resulting equations. We obtain after rearrangements:

$$\frac{3s^2}{4\mu}((\partial_1^2 + \partial_2^2)(\partial_1 M^1 + \partial_2 M^2)) = \mu(\partial_1 M^1 + \partial_2 M^2) + \frac{s^2}{2}\Delta M^0.$$

In view of equation (53) the Laplace equation follows

$$\frac{s^2}{2\mu}\Delta M^0 = 0.$$

To find the corresponding first moments M^1, M^2 we use Fourier transformation. Let (ξ_1, ξ_2) denote the dual parameters of (x_1, x_2), then the transformed system of (59) reads, with for now $d = \frac{s^2}{4\mu}$

$$\begin{aligned} d(-3\xi_1^2 - \xi_2^2)\hat{M}^1 - 2d\xi_1\xi_2\hat{M}^2 &= \mu\hat{M}^1 + \frac{s^2}{2}(-i\xi_1)\hat{M}^0 \\ -2d\xi_1\xi_2\hat{M}^1 + d(-\xi_1^2 - 3\xi_2^2)\hat{M}^2 &= \mu\hat{M}^1 + \frac{s^2}{2}(-i\xi_2)\hat{M}^0. \end{aligned}$$

We write this as a linear equation

$$FL = -i\frac{s^2}{2}\hat{M}^0 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{60}$$

with $L = (\hat{M}^1, \hat{M}^2)^T$ and

$$F = \begin{pmatrix} -\mu - d(3\xi_1^2 + \xi_2^2) & -2d\xi_1\xi_2 \\ -2d\xi_1\xi_2 & -\mu - d(\xi_1^2 + 3\xi_2^2) \end{pmatrix}.$$

We find for the determinant that

$$\det F = \mu^2 + 4\mu d(\xi_1^2 + \xi_2^2) + 3d^2(\xi_1^2 + \xi_2^2)^2, \tag{61}$$

which is positive for each $(\xi_1, \xi_2) \in \mathbb{R}^2$ and $\mu > 0$. Hence (60) is uniquely solvable for each $(\xi_1, \xi_2) \in \mathbb{R}^2$. The solution is given by

$$\begin{pmatrix} \hat{M}^1 \\ \hat{M}^2 \end{pmatrix} = i\frac{s^2}{2}\frac{\mu + d(\xi_1^2 + \xi_2^2)}{\det F}\hat{M}^0 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{62}$$

Then (M^1, M^2) are given by (62) and we can finally calculate the remaining functions from (56), (57) and (58)

$$\begin{aligned} M^{11} &= \tfrac{s^2}{2}M^0 - \tfrac{s^2}{4\mu}(3\partial_1 M^1 + \partial_2 M^2) \\ M^{12} &= M^{21} = -\tfrac{s^2}{4\mu}(\partial_1 M^2 + \partial_2 M^1) \\ M^{22} &= \tfrac{s^2}{2}M^0 - \tfrac{s^2}{4\mu}(\partial_1 M^1 + 3\partial_2 M^2). \end{aligned} \tag{63}$$

Lemma 5.1. *The steady states of the three-moment problem for* $(M^0, M^i, M^{ij})_{i,j\in\{1,2\}}$ *are given as follows*

1. $M^0(x)$ *solves the Laplace equation* $\Delta M^0(x) = 0$ *on* $\mathbb{R}^2$.
2. (M^1, M^2) *are given from Fourier transformation of (62).*
3. $(M^{ij})_{i,j\in\{1,2\}}$ *are given by (63).*

5.3. **Steady States for the Nonlinear Case.** Of course the steady states for the two examples given above on $\mathbb{R}^2$ are identically zero. The method, however, carries over to the nonlinear problem with

$$M_t^0 + \partial_j M^j = f(M^0)$$

(see [6]). Then the steady states of the two- and three-moment systems are related to a semilinear elliptic problem of the form

$$c\Delta M^0 = f(M^0)$$

with an appropriate diffusion constant $c > 0$.

The author believes that, at any level of moment closure, the stationary solution can be constructed from the elliptic equation $\Delta M^0 = 0$ in the linear case and $c\Delta M^0 = f(M^0)$ in the nonlinear case. This, however, needs further exploration.

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