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If we assume, by contradiction, that $\dim N > 1$, the operator $\frac{\partial}{\partial t}$ has at least one non-zero eigenvector $\psi$, $u = \psi$ and $u' = \lambda \psi$. Consequently $u = e^{\lambda t} \psi$ where $\psi = \frac{\partial u}{\partial u} = 0$ on $\Gamma_0$, an open subset of $\Gamma$, one has $\psi = 0$ on an open subset $\omega$ of $\Omega$ provided $\Gamma$ is piecewise analytic. If $\Omega$ is connected we conclude that $u = 0$, a contradiction.

Then if (7) were not true this would imply $\dim N > 1$. From (1') and (7) we deduce (1) and exact controllability.

6. Conclusion
The conclusion is as follows. The four assumptions
1. $Dm + Dm^* = 0$ on $\Gamma$
2. $\Gamma$ is connected and $\Gamma$ is piecewise analytic (or even $C^2$)
3. $m, \psi = 0$ at crack tips,
4. $m, \lambda > 0$ at crack tips
imply exact controllability. If in addition $m, \psi = 0$ along the cracks, the control $\psi$ vanishes on the cracks as one would reasonably expect.

The extra flexibility allowed by these possible choices of $m$ yields more general distribution of the cracks than the very particular choice of $m$ considered in the above §3. Various examples of such multipliers $m$ are given in Tripiliani (1988).

Differential Equations on Branched Manifolds

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Summary: Vector fields on two-dimensional branched manifolds can be seen as caricatures of three-dimensional problems. The corresponding semiflows are easy to visualize locally because of dimension two, on the other hand they are not continuous. With suitable transversality conditions one can obtain information on qualitative behavior and limit sets of trajectories. Particular attention is given to the fact that limit sets, in general, need not be invariant. Typical bifurcations are studied. The phenomena and applications are illustrated by graphical and numerical examples.

Introduction

The qualitative behavior of differential equations on two-dimensional compact manifolds is relatively well understood (see, e.g., Pailis and di Melo [11]) whereas the behavior of differential equations in three dimensions can be very complicated and difficult to visualize. Therefore branched manifolds have been used as two-dimensional caricatures of three-dimensional systems. Williams [16] has developed a general concept of branched manifolds and he has studied caricatures of the Lorenz attractor ([16, [17]), see also Guckenheimer and Holmes [5] and Sparrow [13]). One can make a similar construction for the Rössler attractor ([12], see also Jetchev [7]). Hadeler and Shonkwiler [6] have used branched manifolds in an epidemiological model. There is a rather close connection between differential equations on branched manifolds and differential equations with reset conditions or, in other terminology, differential equations with discontinuous right hand sides. Filippov [4] has developed a theory of such differential equations and has collected a vast bibliography. His view is essentially restricted to local qualitative analysis whereas we shall try to take a global view. In an abstract setting differential equations with discontinuous right hand sides can also be seen as differential inclusions (see, e.g., Aubin and Cellina [2]) or systems for set-valued functions.

Mostly a branched manifold has been seen as a set of planes (with boundaries) joined together along certain edges to form a geometric object on which vector fields are studied. At the edges transition conditions have been defined ad hoc ("if the trajectory ... "). In this view the connectedness and, in particular, the embedding in three dimensional space play a major role. Here we follow a more abstract approach which provides a rigorous definition of transition conditions. The appropriate construction requires some effort. Later we shall return to the familiar object by way of identifications.
Definition of a branched manifold

The purpose of branched manifolds is, of course, to study vector fields and their trajectories. In our approach (as in Williams [15]) we define a branched manifold as a geometric object independent of any vector fields or flows. Roughly speaking, a branched manifold is a collection $\mathcal{M}$ of smooth manifolds $M_i$ with transition conditions. Trajectories "run" on one manifold, "arrive" at some submanifold, and "continue" on another manifold.

If the transition conditions are used to identify points on the manifolds $M_i$ one obtains an object $\mathcal{M}$ which heuristically can be seen as a set of surfaces glued together along lines. Then the transition conditions ensure that at each point of $\mathcal{M}$ there is a well-defined tangent space.

In this approach one can, on a fixed branched manifold, study vector fields depending on parameters and related bifurcations. In another view, used in [6], one can start from given vector fields on the $M_i$ and study changes in the transition conditions.

In applications the constituting $M_i$ will be mostly spheres $S^1$, and the $N_{ij}$ will be spheres $S^1$.

In realistic applications the manifolds will be simply connected planar domains, and the submanifolds will be line segments. However, these manifolds are topologically equivalent to spheres. As usual in differential equations the assumption of compact manifolds without boundary merely leads to some simplification of the representation and some unification, it makes a coherent theory possible.

Let $M_i$, $i = 1, \ldots, r$, be two-dimensional compact $C^1$ manifolds without boundary.

Let $N_{ij}$, $j = 1, \ldots, s_i$, be one-dimensional (compact) $C^1$ submanifolds of $M_i$.

Assume

$$N_{ij} \cap N_{ik} = \emptyset \quad \text{for} \quad j \neq k.$$  \hspace{1cm} (1)

We define

$$M_i' = M_i \setminus \bigcup_{j=1}^{s_i} N_{ij}.$$  \hspace{1cm} (2)

Now we define transition conditions. It is useful to introduce an index set $J = \{(i,j) : i = 1, \ldots, r; j = 1, \ldots, s_i\}$. One can visualize $J$ as a matrix with rows of different lengths. Furthermore denote by $L$ the set $L = \{1, \ldots, r\}$. Let $V : J \to L$ be a function.

For $(i,j) \in J$ let $g_{ij}$ be a $C^1$ mapping $g_{ij} : N_{ij} \to M_i$ where $l = V(i,j)$. By construction the image $g_{ij}(N_{ij}) \subset M_l$ is compact.

The main hypothesis on the various manifolds is the following nonintersection property. For $l \in L$ we require

$$\left(\bigcup_{V(i,j) = l} g_{ij}(N_{ij})\right) \cap \left(\bigcup_{j=1}^{s_i} N_{ij}\right) = \emptyset.$$  \hspace{1cm} (3)

---

**Fig.1:** The reset problem as a branched manifold

**Example 1:** $r = 1$, $s = 1$. Thus $V$ maps $(i,j) = (1,1)$ into $l = 1$. There is only one manifold $M = M_1 = S^2$, and only one submanifold $N = N_{11}$, and $g : N \to M$.

Condition (3) reduces to $g(N) \cap N = \emptyset$. This example corresponds to the classical reset problem. The trajectory runs on $M$ until it meets $N$. Then it is reset to $g(N)$ and starts again (see Fig.1a). The caricature of the dynamics of the Rössler attractor as given by Jetsche [7] fits into this scheme (Fig.1b).

**Fig.2:** The Lorenz attractor

**Example 2:** $r = 1$, $s = 2$, $M_1 = S^2$, $V(1,1) = 1$, $V(1,2) = 1$. $N_1$, $N_2$ are disjoint sets $S^1$, and the images $g_1(N_1)$, $g_2(N_2)$ coincide (as sets). $g_1(N_1) = g_2(N_2)$ is a sphere $S_1$ disjoint from $N_1 \cup N_2$ (Fig.2a)\textcolor{red}{.} The Fig.2b shows a flow on $M$. The shaded area in Fig.2b is equivalent to William's caricature of the Lorenz attractor (Fig.2c).

**Example 3:** $r = 2$, $s_1 = s_2 = 1$, $M_1 = S^2$, $M_2 = S^2$, $V(1,1) = 2$, $V(2,1) = 1$. This setting is the common situation in problems of pest control or epidemic control ([6]) where there is a switch between strategies. Fig.2a presents the general situation for spheres. In the case studied in [6] one cap of each sphere can be
each point of this curve all surfaces have the same tangent plane. In the present context the appropriate visualization is given by Figs.1 and 2. Here the uniqueness of the tangent space is a consequence of the construction as it will be shown in the next proposition.

**Proposition 2:** In each equivalence class \( x \in \mathcal{M} \) there is one and only one point \( \tilde{x} \) with the property:

\[
\left\{ \tilde{y} \in \tilde{\mathcal{M}}, f \in \mathcal{G}, f(\tilde{z}) = \tilde{y} \right\} \Rightarrow f = \text{id}.
\]

Proof: If \( x \) contains only one point \( \tilde{x} \) then \( \tilde{x} \) has property (4). Suppose \( x \) contains two points \( \tilde{y}, \tilde{z} \in \tilde{\mathcal{M}} \) with \( \tilde{y} \neq \tilde{z} \). Then there are \( f, g \in \mathcal{G} \) such that \( g(\tilde{y}) = h(\tilde{z}) \). Put \( \tilde{x} = g(\tilde{y}) \). Then \( \tilde{x} \in x \) because \( \text{id} \in \mathcal{G} \). The point \( \tilde{x} \) has property (4) in view of condition (3). Now assume there are \( \tilde{x}_1, \tilde{x}_2 \in x \) with property (4). Then there are functions \( h_1, h_2 \in \mathcal{G} \) such that \( h_1(\tilde{x}_1) = h_2(\tilde{x}_2) \). Then, again by (4), \( h_1 = h_2 = \text{id} \) and \( \tilde{x}_1 = \tilde{x}_2 \).

**Definition:** For each equivalence class \( x \in \mathcal{M} \) the point \( \tilde{x} \in \mathcal{M} \) with property (4) is called the actual point \( \alpha(x) \). The tangent space at \( x \in \mathcal{M} \) is given by \( T_x\mathcal{M} = T_{\alpha(x)}\mathcal{M} \).

Thus \( \alpha : \mathcal{M} \to \tilde{\mathcal{M}} \) is the map which attributes to each point on \( \mathcal{M} \) the actual point \( \alpha(x) \). An equivalence class \( \tilde{x} \in \tilde{\mathcal{M}} \) is called trivial if it contains only one point of \( \mathcal{M} \). Otherwise the point is called a branch point. The set of trivial equivalence classes is open in \( \tilde{\mathcal{M}} \). The function \( \alpha \) is continuous on this open set. It should be underlined that \( \pi \) does not induce a natural projection of \( T\mathcal{M} \) to \( T\tilde{\mathcal{M}} \).

**Flows**

Suppose that on each manifold \( M_i, i = 1, \ldots, r \), there is a \( C^1 \) vector field \( f_i \). This collection of vector fields defines a \( C^1 \) vector field \( f \) on \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \). By integrating the vector fields \( f_i \) we obtain flows \( \Phi(t, x), t = 1, \ldots, r \) which exist for all \( t \in \mathbb{R} \).

The function \( \Phi(t, x_0) \) is the solution of

\[
\dot{x}(t) = f_i(x(t)), \quad x(0) = x_0,
\]

i.e.,

\[
\frac{\partial\Phi(t, x)}{\partial t} = f_i(\Phi(t, x)),
\]

\[
\Phi(0, x) = x.
\]

We want to construct a semiflow \( \tilde{\Phi} \) on \( \tilde{\mathcal{M}} \) which for small \( t > 0 \) has the property

\[
\tilde{\Phi}(t, \tilde{x}) = \begin{cases} 
\Phi(t, x) & \text{if } \tilde{x} \in \tilde{M}_i, \\
\Phi(t, \tilde{y}) & \text{if } \tilde{x} \in \tilde{N}_i, \ V(i, j) = l.
\end{cases}
\]
Theorem 3:
1) There is a unique semifold \( \hat{\Phi} \) on \( \hat{M} \) which has the property (7).
2) The function
\[
\Phi(t, x) = \tau(t, \alpha(x))
\]
defines a semifold \( \Phi \) on \( M \) for which \( \hat{\Phi}(t, \alpha(x)) = \alpha(\Phi(t, x)) \) has the property (7).

Proof: Define, for \( \hat{x} \in \hat{M} \),
\[
\tau_i(\hat{x}) = \inf \left\{ t \geq 0 : \Phi_i(t, \hat{x}) \in \bigcup_{t \in \mathbb{R}} N_{i,j} \right\}.
\]
Furthermore define
\[
\delta = \inf_{i \in L} \left( \inf \left\{ \tau_i(\hat{z}) : \hat{z} \in \bigcup_{i \in N} N_{i,j} \right\} \right) .
\]
The number \( \delta \) is positive in view of (3) and compactness. For \( 0 \leq t < \delta \) define a local semifold by
\[
\hat{\Phi}(t, \hat{x}) = \begin{cases} 
\Phi_i(t, \hat{x}) & \text{if } \hat{x} \in M_i \text{ and } 0 \leq t \leq \tau_i(\hat{x}), \\
\Phi_i(t - \tau_i(\hat{x}), \phi_i(\Phi_i(t, \hat{x}), \hat{x})) & \text{if } \hat{x} \in M_i \text{ and } \Phi_i(t, \hat{x}), \hat{x} \in N_{i,j}, \\
V(i, j), l, \text{ and } \tau_i(\hat{x}) < t < \delta.
\end{cases}
\]
The function \( \hat{\Phi} \) is well-defined and satisfies
\[
\hat{\Phi}(t + s, \hat{x}) = \hat{\Phi}(s, \hat{\Phi}(t, \hat{x})), \quad \hat{\Phi}(0, \hat{x}) = \hat{x}
\]
for \( t, s \geq 0 \) with \( t + s < \delta \). Since \( \delta \) is uniform and \( \hat{M} \) is compact, this local semifold can be continued to a semifold.
These properties carry over to \( \Phi \) as defined by (8).

Corollary 4: The trajectory \( t \mapsto \Phi(t, x) \) is continuous (in the topology of \( M \)). The trajectory \( t \mapsto \hat{\Phi}(t, \hat{x}) \) has at most one discontinuity (in the given topology in \( M \)) in any given time interval of length \( \delta \).

The flow \( \hat{\Phi} \) can be recovered from \( \Phi \) as
\[
\hat{\Phi}(t, \hat{x}) = \lim_{\epsilon \to 0^+} \alpha(\Phi(\epsilon, x)), \quad \hat{x} \in \hat{x}, \quad \text{for } t > 0,
\]
and
\[
\hat{\Phi}(0, \hat{x}) = \hat{x}.
\]

It is evident that both \( \hat{\Phi}(t, \hat{x}) \) and \( \Phi(t, x) \) are not continuous in \( \hat{x} \) or \( x \), respectively. There is no sensible way to make these functions continuous. In some sense this is the sacrifice one has to make in replacing smooth three-dimensional vector fields by vector fields on branched manifolds. Some continuity can be recovered under transversality assumptions.

The idea of the construction can be explained as follows. For a given nontrivial point \( \hat{x} \in \hat{M} \) there are several points \( \hat{x} \in M_i \) for appropriate \( i \in L \). Thus there are several candidates \( \Phi_i(\hat{z}) \) for a tangent vector. By the principle of the actual point one of these tangent vectors is selected. Thus at every point of \( M \) there is a unique tangent vector. This vector field is piecewise smooth though it may not be smooth if \( M \) is embedded into some space of higher dimension.

The construction of Williams is somewhat different. Geometrically speaking, his construction assumes that the different manifolds glued together, embedded into some space of higher dimension, are tangent to each other.

Transversality
Suppose \( N \) is a closed \( C^1 \) curve in \( M_i \). The vector field \( f_i \), called transversal to \( N \) at \( \hat{x} \in N \) if \( \hat{x} \) is a simple point of \( N \) and \( f_i(\hat{x}) \) is not tangent to \( N \) at \( \hat{x} \). The vector field \( f_i \) is called transversal to \( N \) if it is transversal to \( N \) at every point of \( N \).

A point \( \hat{x} \in \hat{M} \) is called transversal if either \( \hat{x} \in M_i \) or \( \hat{x} = N_{i,j} \), and \( f_i \) is transversal to \( N_{i,j} \) at \( \hat{x} \). The vector field \( f \) is called transversal if all points of \( M \) are transversal.

A point \( x \in M \) is called transversal if all \( \hat{x} \in x \) are transversal.

Suppose \( \Phi(t, x) \), \( t \geq 0 \), is a trajectory in \( \hat{M} \). Suppose \( \Phi(0, x) \) is transversal for some \( t_0 > 0 \). Then \( \Phi(t, x) \) is transversal for \( t_0 \leq t < t_0 + \delta \) where \( \delta \) is defined by (9).

Theorem 5: Suppose \( x \in M \) is trivial and \( \Phi(t, x) \) is transversal for all \( t \geq 0 \). Then for every \( T > 0 \) there is a neighborhood \( \mathcal{U}_T \subset M \) of \( x \) such that \( \Phi \) is continuous in \( [0, T] \times U_T \).

Proof: The orbit \( \Phi(t, x) \) through \( x \) has only countably many transitions \( \tau_1 < \tau_2 < \cdots \). By assumption \( \alpha(x) \in M_i \). Hence there is a neighborhood \( U_0 \) of \( x \) and a \( \delta_0 > 0 \), \( \delta_0 < \delta \), such that \( \Phi(t, y) \) contains only one point for \( (t, y) \in [0, \delta_0] \times U_0 \), and consequently \( \Phi \) is continuous in this set.

Now suppose it has been shown that there is a \( \delta_k \), \( 0 < \delta_k < \delta \), and a neighborhood \( U_k \) of \( x \) such that \( \Phi(t, y) \) is continuous in \( [0, \tau_k] \times U_k \). Since \( \Phi(\tau_k - \epsilon, x) \) is transversal for small \( \epsilon > 0 \) by assumption, by Ważewski’s theorem [14] and assumption (3) there is a \( \delta_{k+1} \), \( 0 < \delta_{k+1} < \delta \), and a neighborhood \( U_{k+1} \subset U_k \) such that \( \Phi(t, y) \) is continuous in \( [0, \tau_{k+1} + \delta_{k+1}] \times U_{k+1} \). This argument can be repeated.

We shall need a similar assertion for the flow \( \hat{\Phi} \).

Corollary 6: Let \( \hat{x} \in \hat{M}, \hat{x} \in M_i \), and suppose that \( \hat{\Phi}(t, \hat{x}) \) is transversal for all \( t \geq 0 \). Then for every \( T > 0 \) there is a neighborhood \( \hat{U}_T \subset \hat{M} \), \( \hat{U}_T \hat{x} \), such that \( \hat{\Phi} \) is continuous in \( [0, T] \times \hat{U}_T \).
The proof is essentially the same as that of Theorem 5.

The assumption of transversality everywhere is much too restrictive. In order to have something concrete at hand we define a class of vector fields which have some generic properties.

Suppose \( x \in M \) is a branch point which is not transversal. Then there is \( \tilde{x} \in \mathcal{M} \) such that \( \tilde{x} \in N_{ij} \subseteq M_1 \) and \( f_1 \) is tangent to \( N_{ij} \). We call a branch point \( x \) a generic contact point if for all \( \tilde{x} \) in \( \tilde{x} \) the following is true: If \( \tilde{x} \in N_{ij} \) then the contact of \( \Phi_{ij}(t, \tilde{x}) \) and \( N_{ij} \) is only of first order. Then locally \( \Phi_{ij}(t, \tilde{x}) \) stays on one side of \( N_{ij} \) (see Fig.4).

![Fig.4: Generic contact point](image)

If \( x \) is a generic contact point then at each \( \tilde{x} \in x, \tilde{x} \in N_{ij} \), there is a neighborhood \( \tilde{U} \subseteq M_1 \) of \( \tilde{x} \) such that the trajectory \( \Phi_{ij}(t, \tilde{x}) \) and \( N_{ij} \) define three domains \( \tilde{U}_0, \tilde{U}_1, \tilde{U}_2 \) (see Fig.4). Trajectories of \( \tilde{\Phi} \) starting in \( \tilde{U}_0 \) stay in \( M_1 \) as long as they stay in \( \tilde{U} \). Trajectories in \( \tilde{U}_1 \) and \( \tilde{U}_2 \) leave \( M_1 \). That is why \( \tilde{\Phi} \) is not continuous at these points.

We call a vector field \( f \) on \( M \) a field of generic contact if there are only finitely many points which are not transversal and if all these points are generic contact points. The vector fields in [6] have this property.

At least in topologically simple cases strong transversality properties provide equivalences between branched manifolds and unbranched manifolds of known topological structure. In the following examples 4a, b, c we assume that the vector field \( f_1 \) is transversal to all curves \( N_{ij} \) and also to all curves \( g_{ii}(N_{ij}) \) with \( V(j, l) = i \). We say that the Poincaré-Bendixson property holds for a trajectory when the trajectory eventually remains in some \( S^3 \) (or disc).

Example 4a: \( M \) is a 2-sphere, \( N \) and \( g(N) \) are disjoint circles. These curves define an annulus \( A \) and two discs \( D_0 \) (bounded by \( N \)) and \( D_1 \) (bounded by \( g(N) \)). Since the vector field is transversal to \( N \) and \( g(N) \) there are just four qualitatively different situations which can be presented as follows.

a) \( D_0 \) and \( D_1 \) are positively invariant. Then a trajectory either stays in \( D_0 \) or \( D_1 \), or it stays in \( A \), or it leaves \( A \) to stay in \( D_1 \). Hence the Poincaré-Bendixson property holds.

b) Trajectories from the annulus never arrive at \( N \). There is no reset. The Poincaré-Bendixson property holds.

c) \( D_0 \) is positively invariant, but \( D_1 \) is negatively invariant. Then the two discs can be discarded. The two curves can be identified, a 2-torus remains.

Example 4b: \( M_1, M_2 \) are 2-spheres, \( N_{11}, N_{12} \) and their images are disjoint circles. On \( M_1 \) these curves define an annulus and two discs. There are 16 qualitatively different cases. If we exchange \( M_1 \) and \( M_2 \) then ten cases remain. Among these all cases are trivial where either \( N_{11} \) or \( N_{12} \) cannot be reached from the interior of the annulus. In these cases every trajectory has at most one transition from one sphere to the other. Then four cases remain. In three of these cases one can disregard the four discs, and connect the two annuli to a 2-torus. Thus we conclude that the dynamics of a transversal vector field on this branched manifold can be essentially represented on a 2-sphere or on a 2-torus.

Example 4c: \( M_1, M_2 \) are 2-spheres with \( s_1 = 2 \) and \( s_2 = 1 \). The six resulting curves and the direction of the transversal vector field are shown in Fig.5. This structure cannot be reduced to a smooth (classical) manifold. A similar observation holds for the Lorenz manifold of Williams [16], [17].

![Fig.5: Illustration of Example 4c](image)

**Limit sets**

As usual the limit set (\( \omega \) limit set) of a trajectory \( \Phi(t, x) \) is defined as

\[
\omega(x) = \{ z : \exists t_1 < t_2 < \cdots, t_k \rightarrow \infty, \Phi(t_k, z) \rightarrow z \}.
\]

As in the classical case one proves the following proposition.

**Proposition 7:** The set \( \omega(x) \) is

i) nonempty,

ii) compact in the topology of \( \mathcal{M} \),

iii) connected in the topology of \( \mathcal{M} \).

Proof:

i) Choose any sequence \( t_k. \mathcal{M} \) is compact. Hence there is an accumulation point.
Let \( y \in \mathcal{M}, y \notin \omega(x) \). Then there is a neighborhood \( U \) of \( y \) and a \( T > 0 \) such that \( U \cap \{ \Phi(t,x) : t \geq T \} = \emptyset \). Hence \( \omega(x) \) is closed and thus compact.

iii) Suppose \( x \notin \omega(x) \) is not connected in the topology of \( \mathcal{M} \). Then there are open sets \( U \) and \( V \) such that \( U \cap V = \emptyset \), \( \omega(x) \subseteq U \cup V \), and \( \omega(x) \cap U \neq \emptyset \), \( \omega(x) \cap V \neq \emptyset \). Define \( K = \mathcal{M} \setminus (U \cup V) \). \( K \) is compact. For every \( T > 0 \) there are \( t_1, t_2 > T \) such that \( \Phi(t_1, x) \in U \), \( \Phi(t_2, x) \in V \), hence also \( t > T \) such that \( \Phi(t, x) \in K \). Hence there is a sequence \( t_k \to \infty \) such that \( \Phi(t_k, x) \to y \) for some point \( y \in K \). Hence \( y \in \omega(x) \). This gives a contradiction.

Limit sets on branched manifolds need not be positively invariant. A simple counterexample: In Example 1 above assume that a trajectory in \( \mathcal{M} \setminus N \) approaches a stationary point on \( N \). Then the stationary point is the limit set, but if the stationary point is chosen as an initial condition then the trajectory continues on \( g(N) \).

Even if the vector fields \( f_i \) are structurally stable (see, e.g., [10]) and all critical elements are transversal to the \( N_{ij} \) there may be limit sets which are not positively invariant. This fact is shown by Example 5b. However, one can prove the following result.

**Theorem 8**: Assume that all points of \( \omega(x) \) are transversal. Then \( \omega(x) \) is positively invariant.

**Proof**: Let \( y \in \omega(x) \). By assumption there is a sequence \( t_k \to \infty \) such that \( \Phi(t_k, x) \to y \). We have to show \( \Phi(s, y) \in \omega(x) \) for \( s \geq 0 \). We can use the flow property

\[
\Phi(t_k + s, x) = \Phi(s, \Phi(t_k, x)).
\]

If \( \Phi \) were continuous in \( s \) then \( \Phi(t_k + s, x) \to \Phi(s, y) \) would follow immediately. Since \( \Phi \) is not continuous in general, we have to consider the problem in more detail.

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**Case 1**: \( y \) is trivial. In view of Theorem 5, for every \( T > 0 \), there is a neighborhood \( U_T \subset \mathcal{M}, U_T \ni y \), such that \( \Phi \) is continuous in \( [0, T) \times U_T \). Choose \( T > s \). For large \( T \) we have \( \Phi(t_k, x) \in U_T \). We use

\[
\Phi(s + t_k, x) = \Phi(s, \Phi(t_k, x)).
\]

For \( k \to \infty \) on the right hand side has the limit \( \Phi(s, y) \) whereas the left hand side, by definition of the limit set, covers some point in \( \omega(x) \).

**Case 2**: \( y \) is not trivial. Let \( \alpha(y) = \tilde{y} \in M_t \) be the actual point. The sequence \( \Phi(t_k, x) \) may contain nontrivial points.

**Case 2a**: There is an infinite subsequence such that the actual points are in \( M_t \). We can assume that the given sequence has already this property, i.e., that \( \alpha(\Phi(t_k, x)) = \tilde{y}_k \in M_t \). By Corollary 6, for every \( T > 0 \) there is a neighborhood

\[
U_T \subset M_t, U_T \ni \tilde{y} \text{ such that } \Phi \text{ is continuous in } [0, T) \times U_T. \Choose T > s. \text{ For large } k \text{ we have } \Phi(t_k, \alpha(y)) \in U_T. \text{ Then }
\]

\[
\Phi(s + t_k, \alpha(y)) = \Phi(s, \Phi(t_k, \alpha(y))),
\]

and thus

\[
\Phi(s + t_k, \alpha(y)) = \Phi(s, \Phi(t_k, \alpha(y))).
\]

For \( k \to \infty \) we have \( \Phi(t_k, \alpha(y)) \to \tilde{y} \), thus \( \Phi(t_k, x) \to \tilde{y} \). Now we can continue as in Case 1.

**Case 2b**: There is no such sequence as in Case 2a. We choose, if necessary, a subsequence, and have the following situation. There is \( \tilde{y} \in y \), \( \tilde{y} \neq \alpha(y) \), \( \tilde{y} \in N_{ij} \subset M_t \). There is an open neighborhood \( U \subset M_t \) of \( \tilde{y} \) such that \( N_{ij} \cup U \) is homeomorphic to an interval, and \( N_{ij} \cup U \) separates \( U \) into two open sets \( U^- \) (where trajectories of \( \Phi \) leave \( M_t \)) and \( U_+ \) (where trajectories of \( \Phi \) stay in \( M_t \)) and \( U = U_+ \cup U_- \cup (N_{ij} \cup U) \). We can assume that \( \Phi \) is transversal to \( N_{ij} \) along \( N_{ij} \cup U \).

**Case 2b(a)**: There is an infinite subsequence such that the actual points are in \( U_+ \). We can assume that the original sequence has this property, i.e., \( \alpha(\Phi(t_k, x)) \in U_+ \).

**Case 2b(b)**: There is an infinite subsequence such that \( \alpha(\Phi(t_k, x)) \in U_- \). We can assume that the given sequence has this property. Choose \( \varepsilon > 0 \) such that \( \Phi(t_k, \tilde{y}) \in U_- \) for \( -\varepsilon < t < 0 \). Then for \( k \) sufficiently large there is \( s_k \) such that \( \Phi(t_k + s_k, \tilde{y}) = \Phi(t_k, \tilde{y}) \) uniformly in \( -\varepsilon < t < 0 \). Then \( \Phi(t_k + s_k, \tilde{y}) = \Phi(t_k, \tilde{y}) \) uniformly in \( -\varepsilon < t < 0 \) for \( k \to \infty \). Hence \( \Phi(t_k, \tilde{y}) \in \omega(x) \) for \( -\varepsilon < t < 0 \). Now choose any of these points and proceed as in Case 1.

We add some comments on attractors and basins. Consider a vector field on \( S^3 \) with finitely many critical elements. Then the basins of the attractors form finitely many open domains and their boundaries are formed by trajectories, hence are piecewise differentiable curves. These boundaries may contain repellers and saddle points. The basin boundaries are themselves invariant sets. On branched manifolds we have a different situation. In Example 5b there are finitely many stationary points and periodic orbits, every trajectory converges to one of these. The basin of each attractor contains an open set such that the closures of these sets cover the whole manifold. But the basins need not be open.

**Return mappings and global behavior**

The global behavior of the dynamical system defined by \( (\mathcal{M}, f) \) can be studied by return mappings (Poincaré mappings). The idea is that trajectories which have only finitely many transitions stay eventually in one of the \( M_t \) and hence are
"trivial". Thus one chooses a curve $L$ (preferably one of the curves $N_{ij}$ or $g_{ij}(N_{ij})$) and one follows all trajectories starting from $L$. For $x \in L$ (in the following we shall omit all tildes) define

$$\tau(x) = \inf\{t > 0 : \Phi(t, x) \in L\}.$$  

Then define

$$L_\infty = \{x \in L : \tau(x) = \infty\}.$$  

Introduce the symbol $\emptyset$ for "empty". Define a mapping $\varphi$ on $L \cup \{\emptyset\}$ by

$$\varphi(x) = \begin{cases} 
\Phi(\tau(x), x) & \text{if } x \in L \setminus L_\infty, \\
\emptyset & \text{if } x \in L_\infty, \\
\emptyset & \text{if } x = \emptyset.
\end{cases}$$  

The mapping $\varphi$ is called the return mapping. At least in cases where the number of the $N_{ij}$ small one can get a global view of the asymptotic behavior by studying $\varphi$ and its iterates.

If $x$ is such that $\varphi^k(x) = \emptyset$ for some $k$ then the trajectory $\Phi(t, x)$ meets $L$ only finitely often. Fixed points of $\varphi$ other than $\emptyset$ correspond to periodic orbits.

Here we study a type of branched manifold closely related to Example 3. Let $M_1 = \mathbb{R}^2$, $M_2 = \mathbb{R}^2$, each endowed with the same cartesian coordinate system $(x, y)$. Let $N_{11} = \{z = b\}$ and $N_{21} = \{z = a\}$, and let $g_{11}(x) = z$, $g_{21}(x) = z$ (with respect to the identical coordinate system). For convenience we assume $a < b$. If we discard the sets $\{x \in M_1 : x > b\}$ and $\{x \in M_2 : x < a\}$ then we arrive at the manifold studied in [6].

As in [6] we introduce the sets $A_1 = \{(x, y) \in M_1 : a < y < b\}$, $A_2 = \{(x, y) \in M_2 : a < y < b\}$, $B = \{(x, y) \in M_2 : y > b\}$, $C = \{(x, y) \in M_1 : y < a\}$. Furthermore we introduce the lines $L_a = \{y = a\}$ and $L_b = \{y = b\}$ as subsets of $M_1$ and $M_2$. It is sufficient to consider trajectories which meet $N_{11}$ and $N_{21}$ infinitely often. Choose $L = L_b$. Define a mapping $\varphi_1$ as follows. For any point $x \in L_b \subset M_1$ define $\tau_1(x) = \inf\{t > 0 : \Phi(t, x) \in L_a \subset M_1\}$. Define $\varphi_1(x) = \Phi(\tau_1(x), x)$ if $\tau_1(x)$ is finite and $\varphi_1(x) = \emptyset$ otherwise. Extend the definition of $\varphi_1$ by putting $\varphi_1(\emptyset) = \emptyset$. Define $\tau_2$ and $\varphi_2$ using $M_2$ instead of $M_1$. Then $\varphi = \varphi_2 \circ \varphi_1$ is the return mapping.

To have something concrete at hand we assume that the trajectories in $A_1 \cup C$ look as in Fig.6a, whereas the trajectories in $A_2 \cup B$ look as in Fig.6b.

Apparently there are four interesting points called $P, Q, R, S$ as indicated. We assume that $P$ and $S$ do not coincide on $M_1$, neither do $Q$ and $R$. The points $P$ and $Q$ are generic contact points. The points $R$ and $S$ are just points where $f_i$ is not transversal to $g_{ij}(N_{ij})$.

![Fig.6: The flows on $M_1$ and $M_2$](image)

Then the function $\varphi_1$ is continuous everywhere, it is decreasing for $z < R$ and increasing for $z > R$. Hence it attains its minimum at $R$ (see Fig.7a). Let $S_1, S_2$ be the two preimages of $Q$ with respect to the flow in $A_2$. Then $\varphi_2$ increases for $x < S_1$, decreases for $x > S_2$, and $\varphi_2$ takes the interval $(S_1, S_2)$ to $\emptyset$ (see Fig.7b).

![Fig.7: The maps $\varphi_1$ and $\varphi_2$](image)

We have to study the composition $\varphi_2 \circ \varphi_1$. The behavior depends on the relative position of some interesting points. For $x < 0$ the function $\varphi_2 \circ \varphi_1$ is large and decreasing, thus $\varphi$ is small and increasing. For $x > 0$ the function $\varphi_1$ is large and increasing, thus $\varphi$ is large and decreasing. We assume that $\varphi$ is dissipative in the sense that $\varphi(x) < x$ for $x > 0$ and $\varphi(x) > x$ for $x < 0$.

The minimum $\varphi_1(R)$ of $\varphi$ may be located below $S_1$, between $S_1$ and $S_2$, or above $S_2$ (we do not discuss limit cases).

Case 1: $\varphi_1(R) > S_2$. Then the function $\varphi$ does not assume the value $\emptyset$, it is first increasing, then decreasing. The maximum is assumed at $R$. If $\varphi(R) < R$ then there is an odd number of fixed points, all located below $R$, these are alternatingly stable and unstable, every trajectory of $\varphi$ converges to one of these fixed point. If $\varphi(R) > R$ then there is an even number of fixed points in $(-\infty, R]$, and a
single fixed point in \([R, +\infty)\). The latter may be unstable and give rise to period doubling or other complex behavior.

Case 2: \(\varphi_1(R) \in (S_1, S_2)\). There are two values \(R_0\) and \(R'_1\) with \(R_0 < R < R'_1\) such that \(\varphi_1(R_0) = \varphi_1(R'_1) = S_2\). The function \(\varphi\) is increasing in \((-\infty, R_0]\), decreasing in \([R_0, +\infty)\), and it carries \((R_0, R'_1)\) into \(0\). Furthermore, \(\varphi(R_0) = \varphi(R'_1) = Q\). Now there are again three cases.

If \(Q < R_0\) then there is an odd number of fixed points in \((-\infty, R_0]\), and no fixed point in \([R_0, +\infty)\). Trajectories cannot enter the interval \((R_0, R'_1)\), all trajectories end up in \((-\infty, R_0]\), and approach one of the fixed points.

There is a largest fixed point in \((-\infty, R_0]\), and this fixed point is stable (in a generic situation, otherwise it is stable from above). This fixed point corresponds to a stable periodic orbit of \(\Phi\). The interval \((R_0, R'_1)\) corresponds to trajectories of \(\Phi\) which approach the attractor in \(A_2 \cup B\). Trajectories of \(\varphi\) cannot enter \((R_0, R'_1)\).

The trajectory of \(\Phi\) starting from \(R_0\) is the boundary between the basins of the stable periodic orbit and the attractor in \(A_2 \cup B\). The boundary itself approaches the attractor.

If \(Q \in (R_0, R'_1)\) then there is an even number of fixed points in \((-\infty, R_0]\), no fixed point in \([R_0, +\infty)\). Some trajectories of \(\varphi\) can end up in \((R_0, R'_1)\) (all trajectories will end up in this interval, if \(\varphi\) has no fixed points.

If \(Q > R'_1\) then there is an even number of fixed points in \((-\infty, R_0]\), and exactly one fixed point in \([R_0, +\infty)\). The interval \((R_0, R'_1)\) will attract some trajectories.

Case 3: \(\varphi_1(R) \in S_1\). There are four values \(R_3 < R < R'_3 < R'_2 < R_4\) such that \(\varphi_1(R_3) = \varphi_1(R'_2) = S_2\), \(\varphi(R_3) = \varphi(R'_2) = S_1\). There are five cases depending on where the point \(Q\) is located in relation to these numbers. We shall list some essential features.

\(Q \in R_3\) an odd number of fixed points, no fixed points otherwise.

\(Q < R_3\) there is an even number of fixed points, no fixed points otherwise.

\(R_3 < Q < R_1\) an even number of fixed points in \((-\infty, R_0]\), an odd number of fixed points in \((R_1, R'_1)\), no fixed point in \([R'_2, +\infty)\).

An even number of fixed points in \((-\infty, R_0]\) and in \((R_1, R'_1)\), no other fixed points.

\(R'_3 < Q < R'_2\). An even number of fixed points in \((-\infty, R_0]\) and in \((R_1, R'_1)\), exactly one fixed point in \([R'_2, +\infty)\).

It should be underlined that the return map \(\varphi\) describes all trajectories \(\Phi(t, x)\) of the original system which meet the curve \(L_a\). All other trajectories have trivial behavior insofar as they stay eventually either in \(M_1\) or in \(M_2\).

**Theorem 9:** Under the assumptions stated above there are two types of limit sets: limit sets of the Poincaré–Bendixon type in \(A_0 \cup C\) and \(A_1 \cup B\), and periodic orbits which meet \(L_a\) and \(L_b\).

**Numerical examples**

On the manifolds \(M_i\) of the preceding section consider two vector fields \(f_i\) in polar coordinates,

\[
\begin{align*}
\theta &= r - R_i, \\
\phi &= 1.
\end{align*}
\]

where \(i = 1, 2\). Hence the problem depends on four parameters \(a, b, R_0, R_1\). In the numerical study we keep \(R_1, R_2\) fixed and vary \(a, b\). The numerical calculations indicate that all non-constant solutions converge to periodic solutions. The number of periodic solutions varies between one and two (and not between one and three, as one might think). The transition between the different cases shows the saddle-node bifurcation of periodic orbits which has been found in [6].

*Example 5:* Let \(r = 2, s_1 = 1, s_2 = 1, V(1, 1) = 2, V(2, 1) = 1, M_1 = S^2 = \mathbb{R}^2 \cup \{\infty\}, M_2 = S^2 = \mathbb{R}^2 \cup \{\infty\}.* Define the \(g_{ij}\) with the obvious identifications in cartesian coordinates

\[
\begin{align*}
N_{11} &= \{(x, y) : y = b\}, \\
N_{12} &= \{x : x = 0\}, \\
N_{21} &= \{x : x = a\}, \\
N_{22} &= \{y : y = 0\},
\end{align*}
\]

\(a) R_1 = 1, R_1 = 0.5, a = -0.4, b = 0.9.\) Then there are two unstable stationary points on \(M_1\), and the two circles are not periodic orbits on \(M\). Numerical evidence shows that there is a unique "large" periodic orbit, which is globally stable (with the exception of the stationary points). Fig.8 shows the case \(b = 0.4\).

Fig. 8: One attracting periodic orbit

b) \(R_1 = 1, R_2 = 0.3, b = 0.2, a = 0.3\). There are two unstable stationary points on \(M_1\), and the smaller circle is a locally stable periodic orbit on \(M\) for \(-0.4 \leq a \leq 0.3\).

For \(-0.4 \leq a \leq -0.3\) the smaller circle is a limit set. But for \(a = -0.3\) the smaller circle is not a periodic orbit. The trajectory starting at \(z = 0, y = -0.3\) leaves the limit set. Hence the limit set is not positively invariant.
Numerical evidence shows that the smaller circle is globally stable (with the exception of the stationary points) for \( a \) close to \(-0.4\). On the other hand, for \( a \in [-0.3, -0.1] \) we are in the situation of Example 5a, there is a single “large” periodic orbit which changes between \( M_1 \) and \( M_2 \).

However, for \( a \) in between something interesting happens. At \( a = \hat{a} \approx -0.302 \) there is a saddle-node bifurcation of periodic orbits which results in a “large” stable periodic orbit and a basin boundary. For \( a \in (\hat{a}, -0.3) \) the two stable orbits coexist. The boundary of the basins is defined by the trajectory of \( f_3 \) which is tangent to the line \( L_a \). The boundary trajectory itself approaches the large periodic orbit. Hence the basin is not open. Fig.9 shows the situation for \( a = -0.301 \).

c) \( R_1 = 1, R_2 = 0.2, a = 0.5, b = 0.7 \). There are two unstable stationary points and one stable periodic orbit which shows a complicated behavior (Fig.10).

**References**


