

Math 525, Ordinary Differential Equations II
Winter 2007

Assignment 2, due Mo. Feb. 09, 2007, 9 AM

Exercise 5: (All those functions) (5)

1. Find a function $f \in C_c^\infty(\mathbb{R})$ with $\text{supp } f \subset [a, b]$, where $a < b \in \mathbb{R}$.
2. Let $\Omega \subset \mathbb{R}^n$ be bounded. Show that if $f \in L^2(\Omega)$ then it follows that $f \in L^1(\Omega)$.
3. If Ω is unbounded the above statement is not true. Show that $\rho(x) = \frac{1}{1+x}$ is contained in $L^2([0, \infty))$ but not in $L^1([0, \infty))$.
4. Show that $\rho(x) = e^{-x}x^{-\frac{2}{3}}$ is contained in $L^1([0, \infty))$ but not in $L^2([0, \infty))$.
5. Find a value $\gamma^* \in [0, 1]$ such that the function $f(x) = x^{\frac{3}{2}}$ is element of the Hölder space $C^{1,\gamma}([-1, 1])$ for $\gamma \leq \gamma^*$ and $f(x)$ is not contained in $C^{1,\gamma}([-1, 1])$ for $\gamma > \gamma^*$.

Exercise 6: (Mollifier) (4)

1. The mollification of a function can be written in two ways. Show that

$$\frac{1}{h^n} \int_{\mathbb{R}^n} \rho\left(\frac{x-z}{h}\right) u(z) dz = \frac{1}{h^n} \int_{\mathbb{R}^n} \rho\left(\frac{z}{h}\right) u(x-z) dz$$

2. Assume a Lipschitz continuous function $u \in C^{0,1}(\mathbb{R}^n)$ is uniformly Lipschitz continuous with constant K :

$$|u(x) - u(y)| \leq K|x - y|.$$

Show that each mollification $u_h = \rho_h * u$ is uniformly Lipschitz continuous with the same constant K .

Exercise 7: (Interpolation Inequality) (3)

Use Hölder's inequality to show the *interpolation inequality*: Assume $1 \leq p \leq q \leq r < \infty$ and consider $\lambda \in (0, 1)$ such that $\frac{1}{q} = \lambda \frac{1}{p} + (1 - \lambda) \frac{1}{r}$. Show

$$\|u\|_{L^q} \leq \|u\|_{L^p}^\lambda \|u\|_{L^r}^{(1-\lambda)}.$$

Exercise 8: (Energy Method) (8)

We use the *energy method* to show that all solutions of the following reaction-diffusion equation approach 0 as $t \rightarrow \infty$: On $\Omega = [0, 1]$ we consider

$$\begin{aligned} u_t &= 2u_{xx} - 3u \\ u_x(0, t) &= 0, \quad u_x(1, t) = 0, \end{aligned}$$

where lower case indices denote the partial derivative with respect to that variable, e.g. $u_t = \frac{\partial}{\partial t} u(x, t)$.

1. Use Young's inequality (with $p = q = 2$) to show that the *energy*

$$E[u, u_x](t) := \frac{1}{2} \int_0^1 (|u|^2 + |u_x|^2) dx$$

satisfies the differential inequality

$$\frac{\partial}{\partial t} E(t) \leq -\frac{1}{2} \int_0^1 |u_{xx}|^2 dx - E(t).$$

2. Use Gronwall's inequality to show that $E(t) \rightarrow 0$ as $t \rightarrow \infty$.
3. Argue that $u(t)$ converges in $H^1([0, 1])$ to 0 as $t \rightarrow \infty$. Does $u(t)$ also converge in $C^0([0, 1])$?