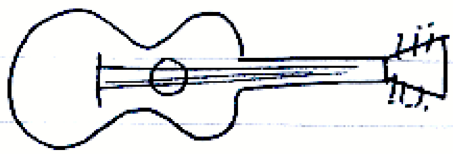
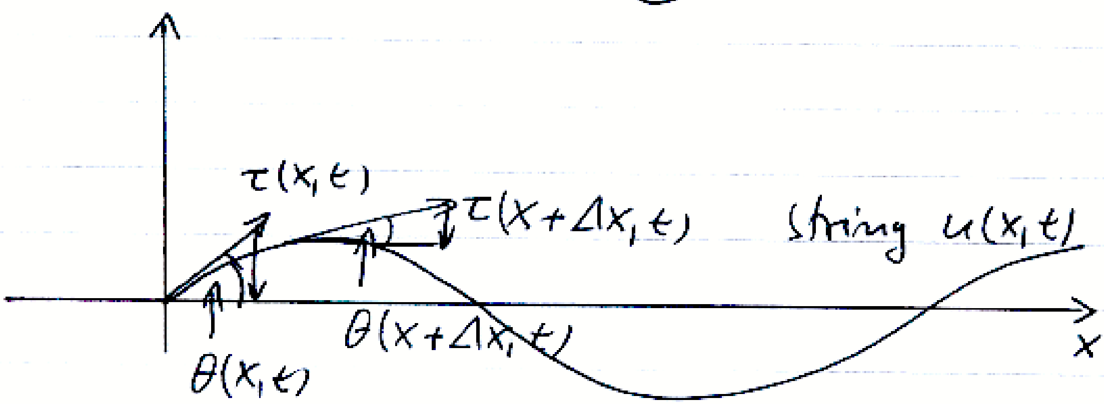
3 The one-dimensional wave equation

(3.1) Applications in acoustics, water waves, electrical circuits, telegraph equation, quantum mechanics, resonances etc... string.

(3.2) Derivation of the 1-D wave equation

Example : Guitar - string





$u(x,t)$: vertical displacement of the string

$\tau(x,t)$: tangential force from tension of string

m : mass density

friction in air : $\alpha \dfrac{du}{dt}$

restoring force βu

Newton's law

$$m\Delta x \frac{d^2u}{dt^2} = \tau(x+\Delta x,t)\,\sin\big(\theta(x+\Delta x,t)\big) - \tau(x,t)\,\sin\big(\theta(x,t)\big) - \alpha\,\Delta x \frac{du}{dt} - \beta\,\Delta x\, u$$

Force acting on the part of the string between x and $x+\Delta x$

Initial conditions

Initial translocation $\qquad u(x,0) = f(x)$

initial velocity $\qquad \frac{\partial u}{\partial t}(x,0) = g(x)$

Why two conditions?

$$\frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2}$$

Introduce a new function $v(x,t) := \frac{\partial u(x,t)}{\partial t}$

Then the wave equation reads

$$\left.\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= v(x,t) \\ \frac{\partial v(x,t)}{\partial t} &= \omega^2 \frac{\partial^2}{\partial x^2} u(x,t) \end{aligned}\right\}$$

Hence it is natural to have one initial condition for u and one for $v(x,0) = \frac{\partial}{\partial t} u(x,0)$.

(3.3) Boundary conditions

a) Dirichlet conditions (first kind)

$$u(0,t) = g_1(t), \qquad u(L,t) = g_2(t)$$

homogeneous Dirichlet conditions

$$u(0,t) = 0, \qquad u(L,t) = 0$$

Example: Guitarr

b) Neumann conditions (second kind)

$$\frac{\partial u}{\partial x}(0,t) = \psi_1(t), \qquad \frac{\partial u}{\partial x}(L,t) = \psi_2(t)$$

Example: Trumpet

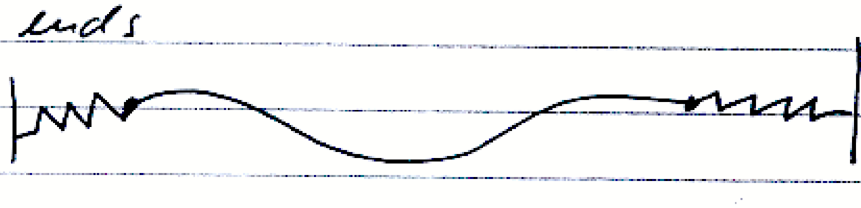
homogeneous Neumann b.c.

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0$$

c) Robin conditions (third kind)

$$\frac{\partial u}{\partial x}(0,t) = k\, u(0,t) \qquad \frac{\partial u}{\partial x}(L,t) = -k\, u(L,t)$$

example: elastic ends



Example: The initial-boundary value problem for the wave equation on $[0, L]$ with given translocation $g_1(t)$ at $x=0$, with homogeneous Neumann boundary conditions at $x=L$, with initial translocation $f(x)$ and with initial velocity 0 reads:

$$\left.\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(x,t) = \omega^2 \frac{\partial^2 u}{\partial x^2}(x,t) \\
&u(x,0) = f(x) \\
&\frac{\partial u}{\partial t}(x,0) = 0 \\
&u(0,t) = g_1(t) \\
&\frac{\partial u}{\partial x}(L,t) = 0
\end{aligned}\right\}$$