(2.4) The Maximum principle and uniqueness

<u>Definition</u> An initial boundary value problem for a PDE is said to be <u>well posed</u> if there exists a unique solution and the solution depends continuously on the data of the problem.

For examples, the data of the above example are $\{f(x),\ 0,\ 0\}$.

The <u>Dirichlet problem</u> for the diffusion equation is well posed

$$\left.\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = D\frac{\partial^2}{\partial x^2}u(t,x) \\ &u(0,x) = f(x) \\ &u(t,0) = g_1(t), \qquad u(t,L) = g_2(t) \end{aligned}\right\}$$

The <u>Neumann problem</u> for the diffusion equation is well posed

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) \\ &u(0,x) = f(x) \\ &\frac{\partial}{\partial x}u(t,0) = \Psi_1(t), \qquad \frac{\partial}{\partial x}u(t,L) = \Psi_2(t) \end{aligned}\right\}$$

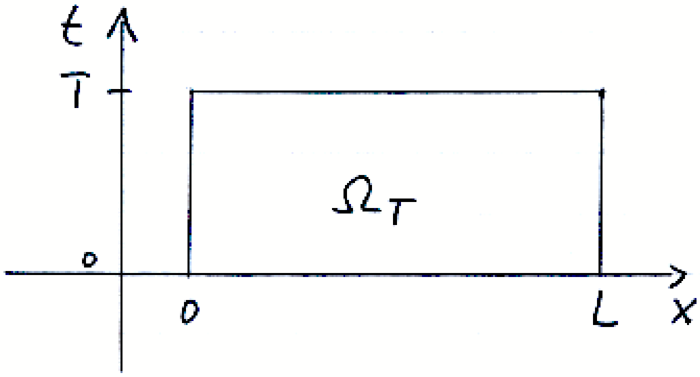
The <u>Robin problem</u> for the diffusion equation is also well posed

The Maximum principle

Given $T > 0$

The parabolic domain is defined as

$$\Omega_T := \{(t,x):\ 0 \le t \le T,\quad 0 \le x \le L\}$$



Let $u(t,x)$ be a solution of the diffusion equation

$$\frac{\partial}{\partial t} u(t,x) = D \frac{\partial^2}{\partial x^2} u(t,x) \quad \text{inside } \Omega_T.$$

Theorem (Maximum / Minimum principle)

$u(t,x)$ cannot attain its maximum or minimum inside Ω_T.

The maximum or minimum is attained either on the base-line $(t=0)$ or on one of the sides $(x=0$, or $x=L)$.

Corollary (Uniqueness for the Dirichlet problem)

There is at most one solution of the Dirichlet problem.

Proof: Assume two functions $u_1(t,x)$, $u_2(t,x)$ solve the Dirichlet problem

$$\left.\begin{aligned} &\frac{\partial}{\partial t} u(t,x) = D \frac{\partial^2}{\partial x^2} u(t,x) \\ &u(0,x) = f(x) \\ &u(t,0) = g_1(t), \quad u(t,L) = g_2(t) \end{aligned}\right\}$$

Then the difference $w(t,x) = u_1(t,x) - u_2(t,x)$ solves

$$\left.\begin{aligned} &\frac{\partial}{\partial t} w(t,x) = D \frac{\partial^2}{\partial x^2} w(t,x) \\ &w(0,x) = 0 \\ &w(t,0) = 0, \quad w(t,L) = 0 \end{aligned}\right\}$$

From the Max/Min principle it follows that

$w = 0$ inside Ω_T. Hence $w(t,x) = 0$

$$\Longrightarrow \quad u_1(t,x) = u_2(t,x) \quad \text{in } \Omega_T.$$

Polar and cylindrical coordinates

Diffusion equation in 2-D and 3-D.

$$\frac{\partial u(t,x,y)}{\partial t} = D\left(\frac{\partial^2}{\partial x^2} u(t,x,y) + \frac{\partial^2}{\partial y^2} u(t,x,y)\right)$$

$$\frac{\partial u(t,x,y,z)}{\partial t} = D\left(\frac{\partial^2}{\partial x^2} u(t,x,y,z) + \frac{\partial^2}{\partial y^2} u(t,x,y,z) + \frac{\partial^2}{\partial z^2} u(t,x,y,z)\right)$$

Example: Write the 2-D heat equation in polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$u(t,x,y) = u(t,\, x(r,\theta),\, y(r,\theta))$$

for later use: $\frac{\partial x}{\partial r} = \cos\theta, \ \frac{\partial x}{\partial \theta} = -r\sin\theta, \ \frac{\partial y}{\partial r} = \sin\theta, \ \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial}{\partial r} u(t,\, x(r,\theta), y(r,\theta)) = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial \theta} u = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial r}\left(r\cdot\frac{\partial}{\partial r}u\right) = \frac{\partial}{\partial r}\left(r\cos\theta\frac{\partial u}{\partial x} + r\sin\theta\frac{\partial u}{\partial y}\right)$$

$$= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} + r\cos\theta\frac{\partial}{\partial x}\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right)$$

$$+ r\sin\theta\frac{\partial}{\partial y}\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right)$$

$$= \cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y} + r\cos^2\theta \frac{\partial^2}{\partial x^2} u + 2r\cos\theta\sin\theta \frac{\partial^2}{\partial x \partial y} u$$
$$+ r\sin^2\theta \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta} u\right) = \frac{\partial}{\partial\theta}\left(-r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}\right)$$
$$= -r\cos\theta \frac{\partial u}{\partial x} + -r\sin\theta \frac{\partial u}{\partial y} - r\sin\theta \frac{\partial}{\partial x}\left(-r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}\right)$$
$$+ r\cos\theta \frac{\partial}{\partial y}\left(-r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}\right)$$
$$= -r\cos\theta \frac{\partial u}{\partial x} - r\sin\theta \frac{\partial u}{\partial y} + r^2\sin^2\theta \frac{\partial^2}{\partial x^2} u - 2r^2\cos\theta\sin\theta \frac{\partial^2}{\partial x \partial y} u$$
$$+ r^2\cos^2\theta \frac{\partial^2}{\partial y^2} u$$

Then:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial}{\partial r} u\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u.$$

$$\frac{\partial}{\partial t} u(t, r, \theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial}{\partial r} u\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} u$$
heat equation in polar coordinates.

In 2-D. $\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$