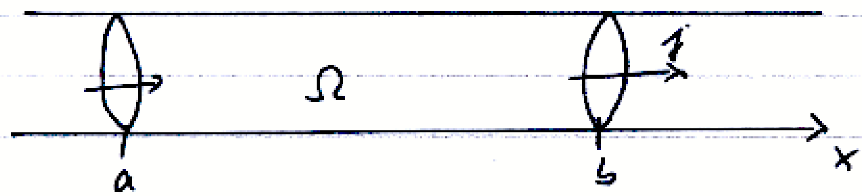
2 The one-dimensional heat-equation and diffusion-equation

(2.1) The heat equation is derived in the textbook. We will derive the same equation in the context of population dynamics.

(2.2) Diffusion equations for populations

Assume bacteria swim in a cylindrical tank



$u(t,x)$: bacterial density at time t at location x

$j(t,x)$: bacterial flux,

Balance equation:

Change of u in Ω = in or outflux through a or b

$$\frac{\partial}{\partial t}\int_a^b u(t,x)\,dx = j(t,a) - j(t,b)$$

First Fickian law: $j(t,x) = -k\frac{\partial}{\partial x}u(t,x)$

Then

$$\frac{\partial}{\partial t}\int_a^b u(t,x)\,dx = -k\frac{\partial}{\partial x}u(t,a) + k\frac{\partial}{\partial x}u(t,b)$$

$$= +k\int_a^b \frac{\partial^2}{\partial x^2}u(t,x)\,dx$$

We assume that $\frac{\partial}{\partial t}u(t,x)$ is continuous, then

we use Leibniz's formula: $\frac{\partial}{\partial t}\int_a^b u(t,x)\,dx = \int_a^b \frac{\partial}{\partial t}u(t,x)\,dx$.

Hence

$$\int_a^b \frac{\partial}{\partial t}u(t,x)\,dx = \int_a^b +k\frac{\partial^2}{\partial x^2}u(t,x)\,dx.$$

Which must be true for all choices of $a \leq b$!

Hence it is true pointwise (almost everywhere)

$$\boxed{\frac{\partial}{\partial t}u(t,x) = k\frac{\partial^2}{\partial x^2}u(t,x)}$$

one-dimensional diffusion-equation (2^{nd} Fickian law)
One-dimensional heat-equation

Heat: $u(t,x)$: temperature distribution
$j(t,x)$: heat flux $\qquad j = -k\frac{\partial}{\partial x}u$: Fourier's law.

Initial conditions: What are the conditions as the experiment / observation is started?

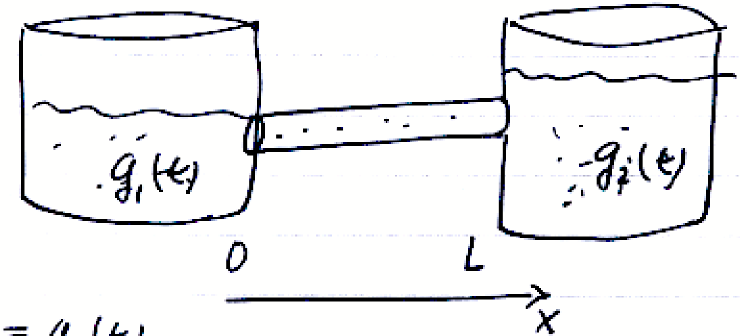
$$u(0,x) = f(x) \qquad \text{a given function } f(x).$$

(2.3) Boundary conditions

What happens at the ends of this cylinder?

a) Dirichlet conditions, (first kind)

$\hat{=}$ open cylinder:

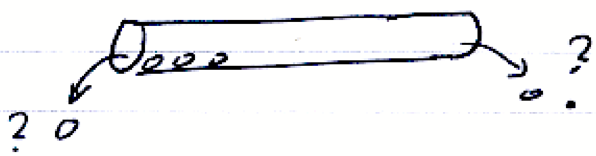


$$u(t,0) = g_1(t)$$

$$u(t,L) = g_2(t)$$

homogeneous Dirichlet conditions

$$u(t,0) = 0$$
$$u(t,L) = 0$$



b) <u>Neumann conditions</u> (second kind)

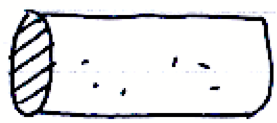
= "closed cylinder"

the particle flux $j(t,x)$ is given at $x=0$, $x=L$:

$$\frac{\partial}{\partial x} u(t,0) = \Psi_1(t) \qquad \frac{\partial}{\partial x} u(t,L) = \Psi_2(t)$$

<u>homogeneous Neumann conditions</u>

$$\frac{\partial}{\partial x} u(t,0) = 0 \,, \qquad \frac{\partial}{\partial x} u(t,L) = 0$$



c) <u>Robin conditions</u> (third kind)

combination of Dirichlet and Neumann conditions

$$\frac{\partial}{\partial x} u(t,0) = h\left(u(t,0) - g_1(t)\right)$$

for $h \to 0$: Neumann condition

$h \to \infty$: Dirichlet condition

$$\frac{\partial}{\partial x} u(t,L) = h\left(u(t,L) - g_2(t)\right).$$

Example: Write down a model for particles which diffuse in a cylinder of length $L = 10$, with initially distribution $f(x)$, when the cylinder has one open end $(x=0)$ and one closed end $(x=L)$.

$$\left.\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} = D \frac{\partial^2}{\partial x^2} u(t,x) \\
&u(0,x) = f(x) \\
&u(t,0) = 0 \\
&\frac{\partial}{\partial x} u(t,L) = 0
\end{aligned}\right\}$$

Initial boundary value problem for the diffusion equation.

Roughly: For each time derivative you need one initial condition!

for each spatial derivative you need one boundary condition!

Example: Wave equation:

$$\left.\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(t,x) = \omega^2 \frac{\partial^2 u}{\partial x^2}(t,x) \\
&u(0,x) = f(x), \quad \frac{\partial}{\partial t} u(0,x) = g(x) \\
&u(t,0) = 0, \quad u(t,L) = 0
\end{aligned}\right\}$$