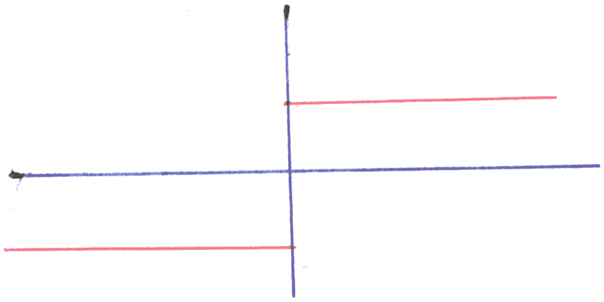
(4.3) PWS: the space of piecewise smooth functions

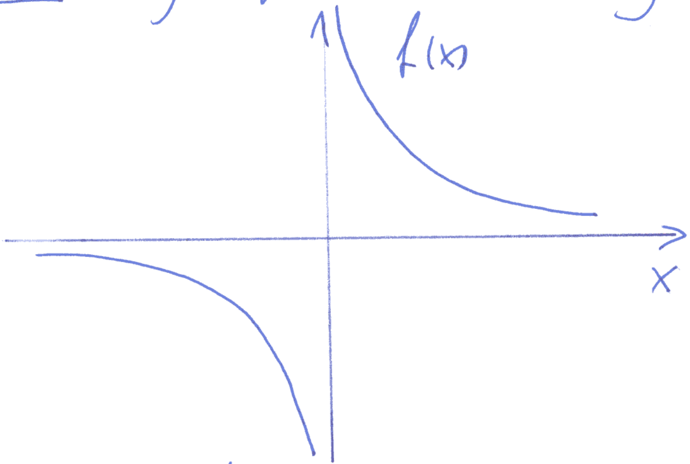
(i) Def 18: A function $f(x)$ has a jump-discontinuity at x_0, if the right and left sided limits exist and $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$

Example 1: Heaviside function $H(x) = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases}$



Example 2: Not a jump discontinuity at $x_0 = 0$:

$f(x) = \frac{1}{x}$



$\lim_{x \to 0^-} f(x) = -\infty$, $\lim_{x \to 0^+} f(x) = +\infty$

Def 19: A function $f(x)$ on an interval $[a,b]$ is called piecewise smooth, if $[a,b]$ can be broken in a finite number of subintervals where f is C^1. Moreover $f(x)$ has only jump discontinuities in $[a,b]$.

The set of all piecewise smooth functions on $[a,b]$ is denoted by $PWS[a,b]$.

$PWS[a,b]$ - functions can be added and multiplied by a scalar. Hence $PWS[a,b]$ is a vector space.

(ii) We define an <u>inner product</u>: $f, g \in PWS[a,b]$

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$$

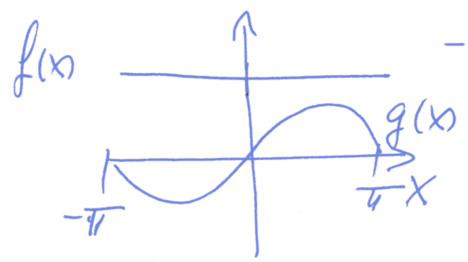
It is straightforward to check the requirements on $\langle \, , \rangle$ (see p. 93).

<u>Of particular interest</u> $PWS[-L, L]$.

What is orthogonality? $\langle f, g \rangle = \int_{-L}^{L} f(x) g(x) \, dx = 0$

<u>Example 3</u>: The functions $f(x) = 3$, $g(x) = \sin x$ are orthogonal in $PWS[-\pi, \pi]$:

$$\langle 3, \sin x \rangle = \int_{-\pi}^{\pi} 3 \sin x \, dx = -3 \left[\cos \pi - \cos(-\pi) \right]$$
$$= 0$$



(iii) Now we use a differential equation to find an appropriate basis for PWS$[-L,L]$.

We study the following <u>eigenvalue problem</u>

$$\left.\begin{aligned} \varphi''(x) &= -\lambda\,\varphi(x) \quad (*) \\ \varphi(L) &= \varphi(-L) \\ \frac{\partial \varphi(L)}{\partial x} &= \frac{\partial \varphi(-L)}{\partial x} \end{aligned}\right\}$$

periodic boundary conditions on $[-L,L]$.

To solve this equation we study three cases:

<u>Case 1</u> $\lambda < 0$

The characteristic equation of $(*)$ is

$$\mu^2 = -\lambda \quad \Rightarrow \quad \mu = \pm\sqrt{-\lambda}$$

Hence the general solution of $(*)$ reads

$$\varphi(x) = c_1 e^{-\sqrt{-\lambda}\,x} + c_2 e^{\sqrt{-\lambda}\,x}$$

Now $\varphi(L) = c_1 e^{-\sqrt{-\lambda}\,L} + c_2 e^{\sqrt{-\lambda}\,L}$

$\quad\quad\ \ \|$

$\quad\ \varphi(-L) = c_1 e^{\sqrt{-\lambda}\,L} + c_2 e^{-\sqrt{-\lambda}\,L}$

$$\Rightarrow \quad c_1\left(e^{-\sqrt{-\lambda}\,L} - e^{\sqrt{-\lambda}\,L}\right) = c_2\left(e^{-\sqrt{-\lambda}\,L} - e^{\sqrt{-\lambda}\,L}\right)$$

$\Rightarrow \quad c_1 = c_2$ and

$$\varphi(x) = c_1\left(e^{-\sqrt{-\lambda}\,x} + e^{\sqrt{-\lambda}\,x}\right) = \tilde{c}_1 \cosh(\sqrt{-\lambda}\,x)$$

2^{nd} boundary condition: $\frac{d}{dx}\varphi(x) = \tilde{c}_1 \sqrt{-\lambda}\ \sinh(\sqrt{-\lambda}\, x)$

$$\frac{d}{dx}\varphi(L) = \tilde{c}_1 \sqrt{-\lambda}\ \sinh(\sqrt{-\lambda}\, L) = \tilde{c}_1 \sqrt{-\lambda}\ \sinh(-\sqrt{-\lambda}\, L)$$

$$= -\tilde{c}_1 \sqrt{-\lambda}\ \sinh(\sqrt{-\lambda}\, L)$$

$\Rightarrow \quad \tilde{c}_1 = 0$

together: $\varphi(x) = 0$ only the trivial solution.

Case 2 $\lambda = 0$: $\varphi'' = 0 \Rightarrow \varphi(x) = c_1 x + c_2$

~~φ to~~ $\varphi(-L) = -c_1 L + c_2 = \varphi(L) = c_1 L + c_2$

$\Rightarrow \quad c_1 = 0$

then $\frac{d}{dx}\varphi(x) = 0$ and $\frac{d\varphi}{dx}(L) = \frac{d\varphi}{dx}(-L) = 0$

Hence $\varphi(x) = c_2$ corresponds to $\lambda = 0$.

Case 3 $\lambda > 0$: characteristic equation:

$$\mu^2 = -\lambda \qquad \mu = \pm i\sqrt{\lambda}$$

general solution

$$\varphi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

Again, check the boundary conditions:

$$\varphi(-L) = C_1 \cos(\sqrt{\lambda}(-L)) + C_2 \sin(\sqrt{\lambda}(-L))$$
$$= C_1 \cos(\sqrt{\lambda} L) - C_2 \sin(\sqrt{\lambda} L)$$
$$\varphi(L) = C_1 \cos(\sqrt{\lambda} L) + C_2 \sin(\sqrt{\lambda} L)$$

$$\Rightarrow \boxed{2 C_2 \sin(\sqrt{\lambda} L) = 0}$$

$$\frac{d\varphi(-L)}{dx} = -\sqrt{\lambda} C_1 \sin(-\sqrt{\lambda} L) + C_2 \sqrt{\lambda} \cos(-\sqrt{\lambda} L)$$
$$= \sqrt{\lambda} C_1 \sin(\sqrt{\lambda} L) + \sqrt{\lambda} C_2 \cos(+\sqrt{\lambda} L)$$
$$\frac{d\varphi(L)}{dx} = -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda} L) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} L)$$

$$\Rightarrow \boxed{2\sqrt{\lambda} C_1 \sin(\sqrt{\lambda} L) = 0}$$

either $C_1 = C_2 = 0$, then $\varphi \equiv 0$

$\Rightarrow$ or $C_2 = 0$, $\lambda = 0$, then <u>case 2</u>

or $\sin(\sqrt{\lambda} L) = 0$

Then $\sqrt{\lambda} = \frac{n\pi}{L}$, $n \in \mathbb{N}$

or $\boxed{\lambda = \left(\frac{n\pi}{L}\right)^2}$ $\underline{n \in \mathbb{N}}$.

Hence the general solution is:

$$\varphi(x) = a \cos\left(\frac{n\pi x}{L}\right) + b \sin\left(\frac{n\pi x}{L}\right)$$

Summary: The second derivative $\frac{d^2}{dx^2}$ on $[-L, L]$ with periodic boundary conditions has the eigenvalues $0, \left(\frac{\pi x}{L}\right)^2, \left(\frac{2\pi x}{L}\right)^2, \dots, \left(\frac{n\pi x}{L}\right)^2, \dots$

with eigenfunctions

$$1, \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right),$$

$$\dots, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right), \dots$$

which we write as

$$\mathcal{S} := \left\{ \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, 3, \dots, \right\}$$

$$\mathcal{C} := \left\{ \cos\left(\frac{n\pi x}{L}\right), n = 0, 1, 2, \dots \right\}$$

Theorem $\mathcal{S} + \mathcal{C}$ forms an orthogonal basis of $PWS[-L, L]$.

(iv) Orthogonality To show

$$\left\langle 1, \sin\left(\frac{n\pi x}{L}\right)\right\rangle = 0 \quad , \quad \left\langle 1, \cos\left(\frac{n\pi x}{L}\right)\right\rangle = 0$$

$$\left\langle \sin\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}\right\rangle = 0$$

$$\left\langle \sin\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}\right\rangle = 0 \quad \text{if } n\neq m$$

$$\left\langle \cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}\right\rangle = 0 \quad \text{if } n\neq m.$$

We study for $n\neq m$

$$\left\langle \sin\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}\right\rangle = \int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

Using trigonometric identities we find

$$\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \frac{1}{2}\left(\sin\left(\frac{n\pi x}{L}+\frac{m\pi x}{L}\right)+\sin\left(\frac{n\pi x}{L}-\frac{m\pi x}{L}\right)\right)$$

$$= \frac{1}{2}\left(\sin\left((n+m)\frac{\pi x}{L}\right)+\sin\left((n-m)\frac{\pi x}{L}\right)\right)$$

Then $\left\langle \sin\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}\right\rangle$

$$= \frac{1}{2}\int_{-L}^{L} \sin\left((n+m)\frac{\pi x}{L}\right)+\sin\left((n-m)\frac{\pi x}{L}\right)dx$$

$$= \frac{1}{2}\left[\frac{-L}{(n+m)\pi}\cos\left((n+m)\frac{\pi x}{L}\right)+\frac{-L}{(n-m)\pi}\cos\left((n-m)\frac{\pi x}{L}\right)\right]_{-L}^{L}$$

$$= \frac{1}{2}\left(\frac{-L}{(n+m)\pi}\left(\cos((n+m)\pi) - \cos(-(n+m)\pi)\right)\right.$$

$$\left.+ \frac{-L}{(n-m)\pi}\left(\cos((n-m)\pi) - \cos(-(n-m)\pi)\right)\right)$$

$$= 0$$

The other cases are left as an exercise.