MATH 300 Fall 2004
Advanced Boundary Value Problems I
Solutions to Assignment 5
Due: Monday December 6, 2004

Department of Mathematical and Statistical Sciences
University of Alberta

Question 1. [p 395, #3]

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 - \cos x & \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

SOLUTION: The Fourier integral representation of $f(x)$ is given by

$$f(x) \sim \int_0^\infty \left(A(\omega) \cos \omega x + B(\omega) \sin \omega x \right) \, d\omega,$$

where

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt \qquad \text{and} \qquad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt.$$

Since $f(x)$ is an even function, then $B(\omega) = 0$ for all ω.

Also, since $f(x)$ is even and $f(x) = 0$ for $|x| \geq \frac{\pi}{2}$, then for all $\omega \neq 0$ and $\omega \neq \pm 1$, we have

$$\begin{aligned}
A(\omega) &= \frac{2}{\pi} \int_0^{\pi/2} (1 - \cos t) \cos \omega t \, dt \\
&= \frac{2}{\pi} \int_0^{\pi/2} \cos \omega t \, dt - \frac{2}{\pi} \int_0^{\pi/2} \cos t \cos \omega t \, dt \\
&= \frac{2}{\pi} \frac{\sin(\omega\pi/2)}{\omega} - \frac{1}{\pi} \int_0^{\pi/2} \left[\cos(1-\omega)t + \cos(1+\omega)t \right] \, dt \\
&= \frac{2}{\pi} \frac{\sin(\omega\pi/2)}{\omega} - \frac{1}{\pi} \frac{\sin(1-\omega)t}{1-\omega} \Bigg|_0^{\pi/2} - \frac{1}{\pi} \frac{\sin(1+\omega)t}{1+\omega} \Bigg|_0^{\pi/2} \\
&= \frac{2}{\pi} \frac{\sin(\omega\pi/2)}{\omega} - \frac{1}{\pi} \frac{\sin((1-\omega)\pi/2)}{1-\omega} - \frac{1}{\pi} \frac{\sin((1+\omega)\pi/2)}{1+\omega} \\
&= \frac{2}{\pi} \frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{\pi} \left[\frac{1}{1-\omega} + \frac{1}{1+\omega} \right] \\
&= \frac{2}{\pi} \left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2} \right],
\end{aligned}$$

so that

$$A(\omega) = \frac{2}{\pi} \left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2} \right]$$

for $\omega \neq 0, \pm 1$.

If $\omega = 0$, then

$$A(0) = \frac{2}{\pi}\int_0^{\pi/2}(1-\cos t)\,dt = \frac{2}{\pi}\left[\frac{\pi}{2} - \sin(\pi/2)\right] = 1 - \frac{2}{\pi}.$$

If $\omega = \pm 1$, then

$$A(\pm 1) = \frac{2}{\pi}\frac{\sin(\pm\pi/2)}{\pm 1} - \frac{2}{\pi}\int_0^{\pi/2}\cos^2 t\,dt = \frac{2}{\pi} - \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{1+\cos 2t}{2}\right)dt = \frac{2}{\pi} - \frac{1}{2}.$$

Note that $A(w)$ is continuous for all ω.

From Dirichlet's theorem, the integral

$$\frac{2}{\pi}\int_0^{\infty}\left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right]\cos\omega x\,d\omega$$

converges to $1-\cos x$ for all $|x| < \frac{\pi}{2}$, converges to 0 for all $|x| > \frac{\pi}{2}$, and converges to $\frac{1}{2}$ for $x = \pm\frac{\pi}{2}$.

Thus, if we redefine $f(\pm\pi/2) = \frac{1}{2}$, then the Fourier integral representation of $f(x)$ is given by

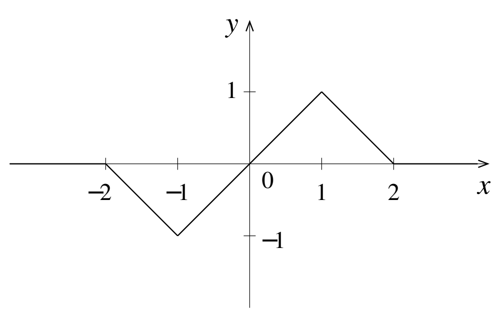
$$\frac{2}{\pi}\int_0^{\infty}\left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right]\cos\omega x\,d\omega = f(x) = \begin{cases} 1-\cos x & \text{for } \ |x| < \frac{\pi}{2} \\ 0 & \text{for } \ |x| > \frac{\pi}{2} \\ \frac{1}{2} & \text{for } \ x = \pm\frac{\pi}{2}. \end{cases}$$

Question 2. [p 395, #9]

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} x & \text{if } \ -1 < x < 1, \\ 2-x & \text{if } \ \ 1 < x < 2, \\ -2-x & \text{if } \ -2 < x < -1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: The graph of $f(x)$ is shown below and it is easy to see that the function $f(x)$ is an odd function.



Therefore, $A(\omega) = 0$ for all ω, and

$$B(\omega) = \frac{2}{\pi}\int_0^2 f(t)\sin\omega t\,dt = \frac{2}{\pi}\int_0^1 t\sin\omega t\,dt + \frac{2}{\pi}\int_1^2 (2-t)\sin\omega t\,dt.$$

Therefore, integrating by parts, we have

$$
\begin{aligned}
B(\omega) &= \frac{2}{\pi}\left[\frac{-t}{\omega}\cos\omega t\Big|_0^1 + \int_0^1 \frac{\cos\omega t}{\omega}\right] + \frac{2}{\pi}\left[\frac{-2+t}{\omega}\cos\omega t\Big|_1^2 - \int_1^2 \frac{\cos\omega t}{\omega}\,dt\right] \\
&= \frac{2}{\pi}\left[-\frac{\cos\omega}{\omega} + \frac{\sin\omega t}{\omega^2}\Big|_0^1\right] + \frac{2}{\pi}\left[\frac{\cos\omega}{\omega} - \frac{\sin\omega t}{\omega^2}\Big|_1^2\right] \\
&= \frac{2}{\pi}\left[\frac{2\sin\omega}{\omega^2} - \frac{\sin 2\omega}{\omega^2}\right] \\
&= \frac{2}{\pi}\left(\frac{2\sin\omega - \sin 2\omega}{\omega^2}\right),
\end{aligned}
$$

that is,

$$
B(\omega) = \frac{2}{\pi}\left(\frac{2\sin\omega - \sin 2\omega}{\omega^2}\right)
$$

for all $\omega \neq 0$.

If $\omega = 0$, then

$$
B(0) = \frac{2}{\pi}\int_0^2 f(t)\sin(0\cdot t)\,dt = 0.
$$

Since $f(x)$ is continuous everywhere, from Dirichlet's theroem, the Fourier sine integral converges to $f(x)$ for all x, and therefore

$$
\frac{2}{\pi}\int_0^\infty \left(\frac{2\sin\omega - \sin 2\omega}{\omega^2}\right)\sin\omega x\,d\omega = f(x)
$$

for all $x \in \mathbb{R}$.

Question 3. [p 407, #4]

Let

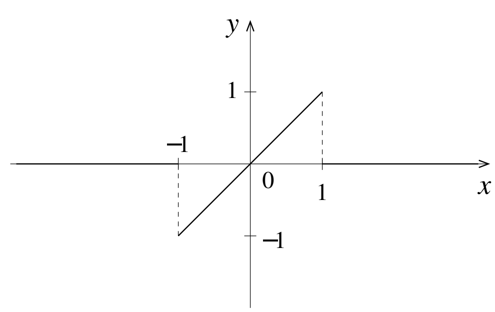
$$
f(x) = \begin{cases} x & \text{if } \ |x| < 1, \\ 0 & \text{otherwise.} \end{cases}
$$

(a) Plot the function $f(x)$ and find its Fourier transform.

(b) If $\widehat{f}$ is real valued, plot it; otherwise plot $|\widehat{f}|$.

SOLUTION:

(a) The graph of the function $f(x)$ is plotted below.



The Fourier transform of $f(x)$ is computed as

$$
\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} t e^{-i\omega t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{t}{i\omega} e^{-i\omega t} \Big|_{-1}^{1} + \frac{1}{i\omega} \int_{-1}^{1} e^{-i\omega t}\, dt \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{i\omega} \left(e^{-i\omega} + e^{i\omega} \right) - \frac{1}{(i\omega)^2} e^{-i\omega t} \Big|_{-1}^{1} \right] \\
&= \frac{2i}{\sqrt{2\pi}} \left[\left(\frac{e^{i\omega} + e^{-i\omega}}{2\omega} \right) - \left(\frac{e^{i\omega} - e^{-i\omega}}{2i\omega^2} \right) \right] \\
&= \frac{2i}{\sqrt{2\pi}} \left(\frac{\omega \cos\omega - \sin\omega}{w^2} \right),
\end{aligned}
$$

so that

$$
\widehat{f}(\omega) = i\sqrt{\frac{2}{\pi}} \left(\frac{\omega \cos\omega - \sin\omega}{w^2} \right)
$$

for all $\omega \neq 0$.

If $\omega = 0$, then

$$
\widehat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} t\, dt = \frac{1}{\sqrt{2\pi}} \frac{t^2}{2} \Big|_{-1}^{1} = 0,
$$

and from L'Hospital's rule, we see that $\lim_{\omega \to 0} \widehat{f}(\omega) = 0$ also, so that $\widehat{f}(\omega)$ is continuous at each ω.

(b) Since

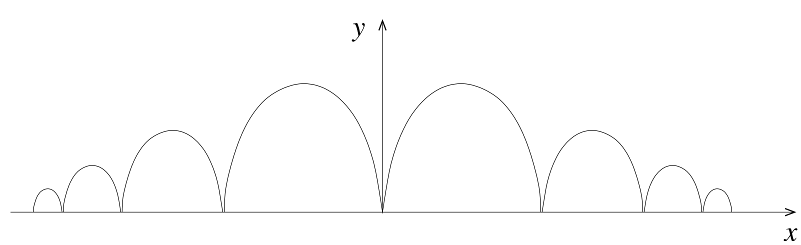
$$
\widehat{f}(\omega) = i\sqrt{\frac{2}{\pi}} \left(\frac{\omega \cos\omega - \sin\omega}{w^2} \right),
$$

then

$$
\left| \widehat{f}(\omega) \right| = \sqrt{\frac{2}{\pi}} \left| \frac{\sin\omega - \omega\cos\omega}{\omega^2} \right|
$$

for all ω.

Note that the zeros of the function $g(\omega) = \sin\omega - \omega\cos\omega$ are precisely the roots of the equation $\tan\omega = \omega$, so the graph of $\left|\widehat{f}(\omega)\right|$ looks something like the figure below.



Question 4. **[p 407, #10] Reciprocity relation for the Fourier transform.**

(a) From the definition of transforms, explain why

$$\mathcal{F}(f)(x) = \mathcal{F}^{-1}(f)(-x).$$

(b) Use (a) to derive the **reciprocity relation**

$$\mathcal{F}^2(f)(x) = f(-x),$$

where $\mathcal{F}^2(f) = \mathcal{F}\left(\mathcal{F}(f)\right).$

(c) Conclude the following: f is even if and only if $\mathcal{F}^2(f)(x) = f(x)$;

f is odd if and only if $\mathcal{F}^2(f)(x) = -f(x)$.

(d) Show that for any f, $\mathcal{F}^4(f) = f$.

SOLUTION:

(a) Note that the Fourier transform of f is

$$\mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt,$$

and evaluating this transorm at $\omega = x$, and making a change of variables, we get

$$\begin{aligned}\mathcal{F}(f)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ixt}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i(-x)t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{i\omega(-x)}\, d\omega = \mathcal{F}^{-1}(f)(-x),\end{aligned}$$

that is,

$$\mathcal{F}(f)(x) = \mathcal{F}^{-1}(f)(-x)$$

for all $x \in \mathbb{R}$.

(b) Let $\widehat{f}$ be the Fourier transform of f, from part (a) we have

$$\mathcal{F}\big(\hat{f}\big)(x) = \mathcal{F}^{-1}\big(\hat{f}\big)(-x) = f(-x),$$

and therefore

$$\mathcal{F}^2(f)(x) = f(-x)$$

for all $x \in \mathbb{R}$.

(c) The function f is even if and only if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, but from part (b), we have f is even if and only if

$$\mathcal{F}^2(f)(x) = \mathcal{F}\big(\hat{f}\big)(x) = f(-x) = f(x)$$

for all $x \in \mathbb{R}$. Similarly, f is odd if and only if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, but again from part (b), we have f is odd if and only if

$$\mathcal{F}^2(f)(x) = \mathcal{F}\big(\hat{f}\big)(x) = f(-x) = -f(x)$$

for all $x \in \mathbb{R}$.

(d) For any integrable f, we have

$$\mathcal{F}^4(f)(x) = \mathcal{F}^2\left(\mathcal{F}^2(f)\right)(x) = \mathcal{F}^2(f)(-x) = f(-(-x)) = f(x)$$

for all $x \in \mathbb{R}$.

Question 5. [p 410, #55] Basic Properties of Convolutions.

Establish the following properties of convolutions. (These properties can be derived directly from the definitions or by using the operational properties of the Fourier transform.)

(a) $f * g = g * f$ (commutativity).

(b) $f * (g * h) = (f * g) * h$ (associativity).

(c) Let a be a real number and let f_a denote the translate of f by a, that is,

$$f_a(x) = f(x - a).$$

Show that

$$(f_a) * g = f * (g_a) = (f * g)_a.$$

This important property says that convolutions commute with translations.

Solution: The most convenient way to prove these properties are true is to use the uniqueness of the Fourier transform, that is, if f and g are integrable and if $\widehat{f} = \widehat{g}$, then $f = g$. However, we will prove them directly from the definition of the convolution.

(a) Given absolutely integrable functions f and g, we make a simple substitution in the definition of the convolution to get

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - t)g(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)g(x - s)ds = g * f(x),$$

for all $x \in \mathbb{R}$, and therefore $f * g = g * f$.

(b) Let f, g, and h be absolutely integrable, then

$$\begin{aligned}
f * (g * h) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - t)(g * h)(t)\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - t) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t - s)h(s)\, ds \right) dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x - t)g(t - s)h(s)\, ds \right) dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(s)f(x - t)g(t - s)\, dt \right) ds \quad (v = x - s) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(x - v)f(x - t)g(t - (x - v))\, dt \right) dv \quad (u = x - t) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x - v)f(u)g(v - u)\, du \right) dv \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x - v) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)g(v - u)\, du \right) dv \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x - v)(g * f)(v)\, dv \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x - v)(f * g)(v)\, dv \\
&= h * (f * g) = (f * g) * h
\end{aligned}$$

(c) We use the shift theorem

$$\begin{aligned}
\mathcal{F}(f_a)(\omega) &= \mathcal{F}(f(x-a))(\omega) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-a)e^{-i\omega t}\,dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)e^{-i\omega(s+a)}\,dt \\
&= e^{-i\omega a}\mathcal{F}(f)(\omega),
\end{aligned}$$

for all ω, so that

$$\mathcal{F}(f_a) = e^{-i\omega a}\mathcal{F}(f).$$

We have

$$\begin{aligned}
\mathcal{F}\left((f*g)_a(x)\right) &= \mathcal{F}\left((f*g)(x-a)\right) \\
&= e^{-i\omega a}\mathcal{F}\left((f*g)(x)\right) \\
&= e^{-i\omega a}\mathcal{F}(f(x))\mathcal{F}(g(x)) \\
&= \mathcal{F}(f_a(x))\mathcal{F}(g(x)) \\
&= \mathcal{F}\left((f_a)*g)(x)\right),
\end{aligned}$$

and $\mathcal{F}\left((f*g)_a\right) = \mathcal{F}\left((f_a)*g)\right)$. Since the Fourier transform is unique, then $(f*g)_a = (f_a)*g$.

We can also prove this directly, as follows.

$$\begin{aligned}
(f*g)_a(x) &= (f*g)(x-a) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a-t)g(t)\,dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_a(x-t)g(t)\,dt \\
&= ((f_a)*g)(x)
\end{aligned}$$

for all $x \in \mathbb{R}$, so that $(f*g)_a = (f_a)*g$.

Also, since $f*g = g*f$, we have

$$(f*g)_a = (g*f)_a = (g_a)*f = f*(g_a).$$

Question 6. [p 418, #3]

Determine the solution of the following initial boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = e^{-x^2}, \qquad -\infty < x < \infty.$$

Give your answer in the form of an inverse Fourier transform.

Solution: We hold t fixed and take the Fourier transform of the partial differential equation and the initial condition with respect to the space variable to get the initial value problem for $\widehat{u}(\omega,t) = \mathcal{F}(u(x,t))(\omega):$

$$\frac{d\widehat{u}}{dt}(\omega,t) = -\frac{\omega^2}{4}\widehat{u}(\omega,t),$$

$$\widehat{u}(\omega,0) = \mathcal{F}\big(e^{-x^2}\big)(\omega) = \frac{1}{\sqrt{2}}e^{-\frac{\omega^2}{4}}.$$

The general solution to this first-order linear equation is

$$\widehat{u}(\omega,t) = A(\omega)e^{-\frac{\omega^2}{4}t},$$

and we can determine the "constant" of integration $A(\omega)$ from the initial condition. Setting $t=0$, we get

$$\widehat{u}(\omega,0) = A(\omega) = \frac{1}{\sqrt{2}}e^{-\frac{\omega^2}{4}},$$

so that

$$\widehat{u}(\omega,t) = \frac{1}{\sqrt{2}}e^{-\frac{\omega^2}{4}}e^{-\frac{\omega^2}{4}t} = \frac{1}{\sqrt{2}}e^{-\frac{\omega^2}{4}(1+t)}.$$

Taking the inverse transform, the solution is

$$u(x,t) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4}(1+t)}e^{i\omega x}\,d\omega$$

for $-\infty < x < \infty$, $t \geq 0$.

Question 7. [p 418, #11]

Solve the following initial boundary value problem

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = f(x), \quad -\infty < x < \infty.$$

Assume that the function f has a Fourier transform.

Solution: Taking the Fourier transform of the partial differential equation and the initial condition with respect to x, we have

$$\frac{d\widehat{u}}{dt}(\omega,t) - i\omega\widehat{u}(\omega,t) = 0,$$

$$\widehat{u}(\omega,0) = \widehat{f}(\omega).$$

The general solution to this first-order linear equation is

$$\widehat{u}(\omega, t) = A(\omega)e^{i\omega t},$$

and we can determine the "constant" of integration $A(\omega)$ from the transformed initial condition

$$\widehat{u}(\omega, 0) = A(\omega) = \widehat{f}(\omega).$$

Therefore,

$$\widehat{u}(\omega, t) = \widehat{f}(\omega) \cdot e^{i\omega t},$$

and taking the inverse Fourier transform, we have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{i\omega t} e^{i\omega x}\, d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{i\omega(x+t)}\, d\omega \\ &= f(x+t), \end{aligned}$$

and the solution is

$$u(x,t) = f(x+t)$$

for $-\infty < x < \infty,\ t \geq 0$.

Question 8. [p 426, #2]

Use convolutions, the error function, and operational properties of the Fourier transform to solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{1}{100}\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \begin{cases} 100 & \text{if} \quad -2 < x < 0, \\ 50 & \text{if} \quad 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Transforming the heat equation and the initial conditions, we get the solution to the transformed problem

$$\widehat{u}(\omega, t) = \widehat{f}(\omega) e^{-\omega^2 t/100}.$$

Since this is the product of two Fourier transforms, we know that it is the Fourier transform of a convolution, and taking the inverse transorm, the solution is

$$u(x,t) = \frac{1}{10\sqrt{2t}}\, e^{-\frac{x^2}{400t}} * f(x) = \frac{1}{20\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{400t}}\, ds,$$

that is,

$$u(x,t) = \frac{1}{20\sqrt{\pi t}} \left[\int_{-2}^{0} 100 e^{-\frac{(x-s)^2}{400t}}\, ds + \int_{0}^{1} 50 e^{-\frac{(x-s)^2}{400t}}\, ds \right].$$

We can write the solution

$$u(x,t) = \frac{1}{20\sqrt{\pi t}}\left[\int_{-2}^{0} 100e^{-\frac{(x-s)^2}{400t}}\,ds + \int_{0}^{1} 50e^{-\frac{(x-s)^2}{400t}}\,ds\right]$$

in terms of the error function

$$\operatorname{erf}(w) = \frac{2}{\sqrt{\pi}}\int_0^w e^{-z^2}\,dz,$$

by letting $z = \dfrac{x-s}{20\sqrt{t}}$, so that $dz = -\dfrac{1}{20\sqrt{t}}ds$, then

$$u(x,t) = \frac{100}{\sqrt{\pi}}\int_{\frac{x}{20\sqrt{t}}}^{\frac{x+2}{20\sqrt{t}}} e^{-z^2}dz + \frac{50}{\sqrt{\pi}}\int_{\frac{x-1}{20\sqrt{t}}}^{\frac{x}{20\sqrt{t}}} e^{-z^2}dz,$$

that is,

$$u(x,t) = 50\left[\operatorname{erf}\left(\tfrac{x+2}{20\sqrt{t}}\right) - \operatorname{erf}\left(\tfrac{x}{20\sqrt{t}}\right)\right] + 25\left[\operatorname{erf}\left(\tfrac{x}{20\sqrt{t}}\right) - \operatorname{erf}\left(\tfrac{x-1}{20\sqrt{t}}\right)\right].$$

Question 9. [p 439, #6]

Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1-x & \text{if } \; 0 < x < 1, \\ 0 & \text{if } \; x \geq 1. \end{cases}$$

and write $f(x)$ as an inverse cosine transform. Use a known Fourier transform and the fact that if $f(x),\ x \geq 0$, is the restriction of an *even* function f_e, then

$$\mathcal{F}_c(f)(\omega) = \mathcal{F}(f_e)(\omega)$$

for all $\omega \geq 0$.

Solution: The Fourier cosine transform of the function f is given by

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\omega t\,dt = \sqrt{\frac{2}{\pi}}\int_0^1 (1-t)\cos\omega t\,dt,$$

and this is the same as the Fourier transform of the *even* extension f_e of f to the whole real line $\mathbb{R}$.

In this case however, we can evaluate the last integral directly by integration by parts:

$$\begin{aligned}
\int_0^1 (1-t)\cos\omega t\,dt &= \int_0^1 \cos\omega t\,dt - \int_0^1 t\cos\omega t\,dt \\
&= \frac{\sin\omega t}{\omega}\bigg|_0^1 - \left[t\cdot\frac{\sin\omega t}{\omega}\bigg|_0^1 - \frac{1}{\omega}\int_0^1 \sin\omega t\,dt\right] \\
&= \frac{\sin\omega}{\omega} - \frac{\sin\omega}{\omega} + \frac{1}{\omega}\left[-\frac{1}{\omega}\cos\omega t\bigg|_0^1\right] \\
&= \frac{1-\cos\omega}{\omega^2},
\end{aligned}$$

and therefore

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}}\cdot\frac{1-\cos\omega}{\omega^2}$$

for $\omega > 0$.

Knowing that f_c is absolutely integrable implies that $\widehat{f_c}$ is continuous at $\omega = 0$, and we have

$$\widehat{f_c}(0) = \lim_{\omega \to 0+} \sqrt{\frac{2}{\pi}} \cdot \frac{1 - \cos\omega}{\omega^2} = \sqrt{\frac{2}{\pi}} \cdot \lim_{\omega \to 0+} \frac{\sin\omega}{2\omega} = \frac{1}{\sqrt{2\pi}}$$

by L'Hospital's rule.

Therefore, we have

$$\widehat{f_c}(\omega) = \begin{cases} \sqrt{\dfrac{2}{\pi}} \cdot \dfrac{1 - \cos\omega}{\omega^2} & \text{for} \quad \omega > 0 \\[2ex] \dfrac{1}{\sqrt{2\pi}} & \text{for} \quad \omega = 0. \end{cases}$$

Since f_e is continuous for all $x \in \mathbb{R}$, from Dirichlet's theorem the inverse Fourier cosine transform of $\widehat{f_c}$ is given by

$$\frac{2}{\pi} \int_0^\infty \frac{1 - \cos\omega}{\omega^2} \cdot \cos\omega x \, d\omega = \begin{cases} 1 - x & \text{for} \quad 0 \le x < 1 \\ 0 & \text{for} \quad x \ge 1. \end{cases}$$

Question 10. [p 439, #12]

Find the Fourier sine transform of

$$f(x) = \frac{x}{1 + x^2}, \qquad x > 0,$$

and write $f(x)$ as an inverse sine transform. Use a known Fourier transform and the fact that if $f(x),\ x \ge 0$, is the restriction of an *odd* function f_o, then

$$\mathcal{F}_s(f)(\omega) = i\mathcal{F}(f_o)(\omega)$$

for all $\omega \ge 0$.

Solution: We can find the Fourier sine transform of the given function using the suggested method, or we can find it directly. To do this, we consider the function

$$g(x) = e^{-x}, \qquad x > 0$$

with Fourier sine transform given by

$$\widehat{g_s}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \sin\omega t \, dt$$

and we can evaluate this integral by integrating by parts:

$$\begin{aligned} \int_0^\infty e^{-t} \sin\omega t \, dt &= -\frac{e^{-t}}{\omega}\bigg|_0^\infty - \frac{1}{\omega} \int_0^\infty e^{-t} \cos\omega t \, dt \\ &= \frac{1}{\omega} - \frac{1}{\omega}\left[e^{-t} \cdot \frac{\sin\omega t}{\omega}\bigg|_0^\infty + \frac{1}{\omega} \int_0^\infty e^{-t} \sin\omega t \, dt \right] \\ &= \frac{1}{\omega} - \frac{1}{\omega^2} \int_0^\infty e^{-t} \sin\omega t \, dt \end{aligned}$$

so that

$$\left(1 + \frac{1}{\omega^2}\right) \int_0^\infty e^{-t} \sin\omega t \, dt = \frac{1}{\omega}.$$

Therefore,

$$\int_0^\infty e^{-t} \sin \omega t \, dt = \frac{\omega}{1+\omega^2}$$

for $\omega \geq 0$, so that

$$\widehat{g}_s(\omega) = \sqrt{\frac{2}{\pi}} \cdot \frac{\omega}{1+\omega^2}$$

for $\omega \geq 0$.

Taking the inverse Fourier sine transform of this, we have

$$g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{g}_s(\omega) \sin \omega x \, d\omega = \frac{2}{\pi} \int_0^\infty \frac{\omega}{1+\omega^2} \sin \omega x \, d\omega,$$

that is,

$$e^{-\omega} = g(\omega) = \frac{2}{\pi} \int_0^\infty \frac{x}{1+x^2} \sin \omega x \, dx,$$

and

$$\widehat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{1+x^2} \sin \omega x \, dx = \sqrt{\frac{\pi}{2}} \cdot g(\omega) = \sqrt{\frac{\pi}{2}} \cdot e^{-\omega}$$

for $\omega \geq 0$.

From the above, we can write $f(x)$ as an inverse Fourier sine transform:

$$f(x) = \frac{x}{1+x^2} = \int_0^\infty e^{-\omega} \sin \omega x \, d\omega$$

for $x > 0$.