MATH 300 Fall 2004
Advanced Boundary Value Problems I
Solutions to Assignment 2
Due: Friday October 8, 2004

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Question 1. [p 77, #26]

Solve the initial value problem

$$y'' + 9y = F(t)$$
$$y(0) = 0$$
$$y'(0) = 0$$

where $F(t)$ is the 2π-periodic input function given by its Fourier series $F(t) = \sum_{n=1}^{\infty} \left[\frac{\cos nt}{n^2} + (-1)^n \frac{\sin nt}{n} \right]$.

Solution: Recall that since the differential equation is a linear equation with constant coefficients, then the general solution to the nonhomogeneous equation is given by

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is the general solution to the corresponding homogeneous equation and $y_p(t)$ is *any* particular solution to the nonhomogeneous equation.

In order to solve the homogeneous equation

$$y'' + 9y = 0,$$

we try a solution of the form $y_h(t) = e^{\lambda t}$, substituting this into the equation, we obtain

$$e^{\lambda t}(\lambda^2 + 9) = 0,$$

and since $e^{\lambda t} \neq 0$, then the auxiliary equation is $\lambda^2 + 9 = 0$, which has roots $\lambda_1 = 3i$ and $\lambda_2 = -3i$. The general solution to the homogeneous equation is therefore

$$y_h(t) = c_1 \cos 3t + c_2 \sin 3t$$

where c_1 and c_2 are arbitrary constants.

In order to find a particular solution to the nonhomogeneous equation, we use the method of Fourier series and solve the equation

$$y''(t) + 9y(t) = a_n \cos nt + b_n \sin nt$$

for $n \geq 0$, where a_n and b_n are the Fourier coefficients of the driving force $F(t)$.

Note that for $n \neq 3$, from the method of undetermined coefficients, the n^{th} normal mode of vibration is

$$y_n(t) = \alpha_n \cos nt + \beta_n \sin nt$$

where the constants α_n and β_n are determined from the Fourier coefficients of $F(t)$ to be

$$\alpha_0 = 0, \quad \alpha_n = \frac{1}{n^2(9-n^2)}, \quad \beta_n = \frac{(-1)^n}{n(9-n^2)}$$

for $n \geq 1,\ n \neq 3$.

While for $n = 3$, the term in the driving force has the same frequency as the natural frequency of the system, and we have to solve the nonhomogeneous equation

$$y_3''(t) + 9y_3(t) = a_3 \cos 3t + b_3 \sin 3t.$$

In this case the method of undetermined coefficients suggests a solution of the form

$$y_3(t) = t(\alpha_3 \cos 3t + \beta_3 \sin 3t).$$

In order to determine the constants α_3 and β_3, we substitute this expression into the differential equation

$$y_3'' + 9y_3 = a_3 \cos 3t + b_3 \sin 3t$$

to obtain

$$\alpha_3 = -\frac{b_3}{6} \qquad \text{and} \qquad \beta_3 = \frac{a_3}{6}.$$

The particular solution to the nonhomogeneous equation can then be written as

$$y_p(t) = \sum_{\substack{n=1 \\ n \neq 3}}^{\infty} \left(\frac{1}{n^2(9-n^2)} \cos nt + \frac{(-1)^n}{n(9-n^2)} \sin nt \right) + \frac{t}{6}\left(-\frac{1}{3} \cos 3t + \frac{1}{3^2} \sin 3t \right),$$

and the general solution to the nonhomogeneous equation is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t + \sum_{\substack{n=1 \\ n \neq 3}}^{\infty} \left(\frac{1}{n^2(9-n^2)} \cos nt + \frac{(-1)^n}{n(9-n^2)} \sin nt \right) + \frac{t}{6}\left(-\frac{1}{3} \cos 3t + \frac{1}{3^2} \sin 3t \right)$$

and the constants c_1 and c_2 can now be evaluated using the initial conditions $y(0) = y'(0) = 0$. Applying the initial conditions, we find

$$c_1 = -\sum_{\substack{n=1 \\ n \neq 3}}^{\infty} \frac{1}{n^2(9-n^2)} \qquad \text{and} \qquad c_2 = \frac{1}{3^2 \cdot 6} - \frac{1}{3} \sum_{\substack{n=1 \\ n \neq 3}}^{\infty} \frac{(-1)^n}{9-n^2}.$$

Note: The solution with driving force

$$F(t) = \sum_{n=1}^{\infty} \frac{\sin nt}{n^2}$$

is also acceptable.

Question 2. [p 107, #8]
Verify that the function

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

is a solution to the three dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

SOLUTION: By symmetry, we need only calculate the derivatives with respect to one of the variables, say x, and obtain the other derivatives by permuting the variables. For example,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}} \right) = \frac{-x}{(x^2+y^2+z^2)^{3/2}},$$

so that

$$\frac{\partial u}{\partial y} = \frac{-y}{(x^2+y^2+z^2)^{3/2}} \qquad \text{and} \qquad \frac{\partial u}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{3/2}}.$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}},$$

so that

$$\frac{\partial^2 u}{\partial y^2} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}.$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{(2x^2-y^2-z^2)+(2y^2-x^2-z^2)+(2z^2-x^2-y^2)}{(x^2+y^2+z^2)^{5/2}} = 0,$$

that is, u satisfies Laplace's equation $\nabla^2 u = 0$.

Question 3. [p 123, #2]

Solve the one dimensional wave equation with the boundary conditions and initial conditions as given below

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}, \quad 0<x<1,\ t>0 \\ u(0,t) &= 0, \quad t>0 \\ u(1,t) &= 0, \quad t>0 \\ u(x,0) &= \sin\pi x\cos\pi x, \quad 0<x<1, \\ \frac{\partial u}{\partial t}(x,0) &= 0, \quad 0<x<1, \end{aligned}$$

using the Method of Separation of Variables.

Solution: As in class, we assume a solution of the form $u(x,t) = X(x)T(t)$ and plug this expression in the differential equation to get

$$X\cdot T'' = \frac{1}{\pi^2}X''\cdot T,$$

and now separate the variables by dividing by $X\cdot T$ to get

$$\frac{T''}{T} = \frac{X''}{\pi^2 X} = -\lambda.$$

Since x and t are independent variables, then λ is a constant, and we have two ordinary differential equations to solve:

$$X''+\lambda\pi^2 X = 0 \qquad \text{and} \qquad T''+\lambda T = 0.$$

We can satisfy the two boundary conditions by requiring that $X(0)=0$ and $X(1)=0$, and X must satisfy the ordinary boundary value problem:

$$\begin{aligned} X''+\lambda\pi^2 X &= 0, \quad 0<x<1 \\ X(0) &= 0 \\ X(1) &= 0. \end{aligned}$$

The cases $\lambda=0$ and $\lambda<0$ both result in a solution $X(x)=0$ for all $x\in[0,1]$, and the only nontrivial solution arises when $\lambda>0$, say $\lambda=\mu^2$, where $\mu\neq 0$. In this case we have to solve the boundary value problem

$$\begin{aligned} X''+\mu^2\pi^2 X &= 0, \quad 0<x<1 \\ X(0) &= 0 \\ X(1) &= 0. \end{aligned}$$

The general solution to this differential equation is

$$X(x) = A\cos\mu\pi x + B\sin\mu\pi x,$$

and applying the first boundary condition, we see that $X(0) = A = 0$, and the solution is

$$X(x) = B\sin\mu\pi x.$$

Applying the second boundary condition, we see that $X(1) = B\sin\mu\pi = 0$, and in order to get a nontrivial solution we must have $\sin\mu\pi = 0$, but this can only happen if $\mu\pi = n\pi$, where n is an integer. For each $n \geq 1$ the solution is

$$X_n(x) = \sin n\pi x.$$

For each integer $n \geq 1$, we can solve the corresponding equation

$$T'' + n^2 T = 0$$

to get

$$T_n(t) = b_n \cos nt + b_n^* \sin nt$$

for $n \geq 1$.

Now, for each integer $n \geq 1$, the function

$$u_n(x,t) = X_n(x) \cdot T_n(t) = \sin n\pi x\,(b_n \cos nt + b_n^* \sin nt)$$

satisfies the wave equation and the two boundary conditions:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(1,t) &= 0, \quad t > 0. \end{aligned}$$

Since the partial differential equation and the two boundary conditions are linear and homogeneous, by the superposition principle, any linear combination of these solutions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin n\pi x\,(b_n \cos nt + b_n^* \sin nt)$$

is also a solution to the partial differential equation and the boundary conditions.

In order to satisfy the initial conditions, we need

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin n\pi x, \tag{1}$$

and

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} n b_n^* \sin n\pi x, \tag{2}$$

and it is clear then that these are just the half-range expansions of the odd periodic extensions of $u(x,0)$ and $\dfrac{\partial u}{\partial t}(x,0)$.

Therefore, from (1) we have

$$b_n = 2\int_0^1 u(x,0)\sin n\pi x\,dx$$

and

$$nb_n^* = 2\int_0^1 \frac{\partial u}{\partial t}(x,0)\sin n\pi x\,dx$$

for $n \geq 1$.

Note that $b_n^* = 0$ for all $n \geq 1$, since $\dfrac{\partial u}{\partial t}(x,0) = 0$ for $0 < x < 1$.

Also, we have

$$u(x,0) = \sin \pi x \cos \pi x = \frac{1}{2}\sin 2\pi x,$$

so that $u(x,0)$ is its own Fourier sine series, and

$$b_n = \begin{cases} \dfrac{1}{2} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2. \end{cases}$$

Therefore, the solution is

$$u(x,t) = \frac{1}{2}\sin 2\pi x \cos 2t$$

for $0 \leq x \leq 1,\ t \geq 0$.

Question 4. [p 123, #4]

Solve the one dimensional wave equation with the boundary conditions and initial conditions as given below

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \\
u(0,t) &= 0, \quad t > 0 \\
u(1,t) &= 0, \quad t > 0 \\
u(x,0) &= \sin \pi x + \tfrac{1}{2}\sin 3\pi x + 3\sin 7\pi x, \quad 0 < x < 1, \\
\frac{\partial u}{\partial t}(x,0) &= \sin 2\pi x, \quad 0 < x < 1,
\end{aligned}$$

using the Method of Separation of Variables.

Solution: As in the previous problem, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \left(b_n \cos n\pi t + b_n^* \sin n\pi t\right),$$

where the coefficients are to be determined using the initial conditions. Differentiating, we have

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \left(-n\pi b_n \sin n\pi t + n\pi b_n^* \cos n\pi t\right),$$

and setting $t = 0$, we get

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} n\pi b_n^* \sin n\pi x,$$

and again these are just the Fourier sine series of $f(x)$ and $g(x)$, the initial displacement and initial velocity.

From the first initial condition

$$u(x,0) = \sin \pi x + \tfrac{1}{2}\sin 3\pi x + 3\sin 7\pi x,$$

we see that

$$b_1 = 1, \quad b_3 = \frac{1}{2}, \quad b_7 = 3,$$

and $b_n = 0$ for all other values of n.

From the second initial condition

$$\frac{\partial u}{\partial t}(x,0) = \sin 2\pi x,$$

so that

$$b_n^* = \begin{cases} \dfrac{1}{2\pi} & \text{if } n = 2, \\ 0 & \text{if } n \neq 2. \end{cases}$$

Therefore, the solution is

$$u(x,t) = \sin \pi x \cos \pi t + \frac{1}{2\pi}\sin 2\pi x \sin 2\pi t + \frac{1}{2}\sin 3\pi x \cos 3\pi t + 3\sin 7\pi x \cos 7\pi t$$

for $0 < x < 1,\ t > 0$.

Question 5. [p 124, #12]

Damped vibrations of a string. In the presence of resistance proportional to velocity, the one dimensional wave equation becomes

$$\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad 0 < x < L,\ t > 0.$$

Solve this equation subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \quad \text{for all} \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for} \quad 0 < x < L.$$

Solution:

(a) Assume a product solution of the form $u(x,t) = X(x)T(t)$, and plug it into the equation to get

$$XT'' + 2kXT' = c^2X''T,$$

and now separate the variables by dividing by c^2XT to get

$$\frac{T''}{c^2T} + \frac{2kT'}{c^2T} = \frac{X''}{X}.$$

Since x and t are independent variables and the left hand side depends only on t, while the right hand side depends only on x, then both sides must be constant, so that

$$\frac{T''}{c^2T} + \frac{2kT'}{c^2T} = \lambda \quad \text{and} \quad \frac{X''}{X} = \lambda,$$

so that x and T must satisfy the following ordinary differential equations

$$X'' - \lambda X = 0$$
$$T'' + 2kT' - \lambda c^2 T = 0.$$

Now, we can satisfy the boundary conditions by requiring that $X(0) = X(L) = 0$, so that X must satisfy the boundary value problem

$$X'' - \lambda X = 0$$
$$X(0) = 0$$
$$X(L) = 0.$$

As in the previous problems, we only get a nontrivial solution if the separation constant λ is negative, say $\lambda = -\mu^2$ where $\mu \neq 0$, and in this case, the equations for X and T become

$$X'' + \mu^2 X = 0, \quad X(0) = 0, \quad X(L) = 0,$$
$$T'' + 2kT' + (\mu c)^2 T = 0,$$

where $\mu \neq 0$ is the separation constant.

(b) The general solution to the equation

$$X'' + \mu^2 X = 0$$

is given by

$$X(x) = A\cos\mu x + B\sin\mu x,$$

where the constants are determined from the boundary conditions. Since $X(0) = 0$, then we must have $A = 0$; and since $X(L) = 0$, the only nontrivial solutions arise when $\sin\mu L = 0$, and this happens if and only if $\mu L = n\pi$, where n is an integer.

Therefore, the only nontrivial solutions to the boundary value problem for X occur for

$$\mu = \mu_n = \frac{n\pi}{L}$$

and the solutions are

$$X = X_n = \sin(n\pi x/L)$$

for $n = 1, 2, \ldots$.

(c) For each integer $n \geq 1$, the corresponding equation for T is

$$T'' + 2kT' + (n\pi c/L)^2\, T = 0,$$

a second order, linear, homogeneous, constant coefficient equation which we know how to solve. Assuming a solution of the form $T(t) = e^{\lambda t}$, and plugging this into the differential equation we get the characteristic equation

$$\lambda^2 + 2k\lambda + \frac{n^2\pi^2c^2}{L^2} = 0,$$

and the roots of this quadratic equation are

$$\lambda_{n,1} = -k + \sqrt{k^2 - \frac{n^2\pi^2c^2}{L^2}} \qquad \text{and} \qquad \lambda_{n,2} = -k - \sqrt{k^2 - \frac{n^2\pi^2c^2}{L^2}}.$$

In order to find the corresponding solutions $T_n(t)$, we need to consider three cases, according to whether $\sqrt{k^2 - \dfrac{n^2\pi^2c^2}{L^2}}$ is zero, positive or negative.

Case 1: $k^2 - \dfrac{n^2\pi^2c^2}{L^2} = 0$. In this case, we have equal real roots, and the solution is

$$T_n(t) = e^{-kt}\left(a_n + b_n t\right)$$

where $k = \dfrac{n\pi c}{L} > 0$.

Case 2: $k^2 - \dfrac{n^2\pi^2c^2}{L^2} > 0$. In this case, we have two distinct real roots, and the solution is

$$T_n(t) = e^{-kt}\left(a_n \cosh\lambda_n t + b_n \sinh\lambda_n t\right)$$

where $\lambda_n = \sqrt{k^2 - \dfrac{n^2\pi^2c^2}{L^2}}$.

Case 3: $k^2 - \dfrac{n^2\pi^2c^2}{L^2} < 0$. In this case, we have two distinct imaginary roots, and the solution is

$$T_n(t) = e^{-kt}\left(a_n \cos\lambda_n t + b_n \sin\lambda_n t\right)$$

where $\lambda_n = \sqrt{\dfrac{n^2\pi^2c^2 - k^2}{L^2}}$.

(d) Since the partial differential equation and the boundary conditions are linear and homogeneous, then we can use the superposition principle to write the solution $u(x,t)$ as a linear combination of the solutions $u_n(x,t) = X_n(x)\cdot T_n(t)$ that we found in part (c).

If $\frac{kL}{\pi c}$ is not a positive integer, then

$$k^2 - \frac{n^2\pi^2c^2}{L^2} \neq 0,$$

and either $1 \le n < \frac{kL}{\pi c}$, or $n > \frac{kL}{\pi c}$, so we are in Case 2 or Case 3, and the solution is

$$\begin{aligned} u(x,t) = e^{-kt} &\sum_{1\le n<kL/\pi c} \sin\left(n\pi x/L\right)\left(a_n\cosh\lambda_n t + b_n \sinh\lambda_n t\right) \\ + e^{-kt} &\sum_{kL/\pi c<n<\infty} \sin\left(n\pi x/L\right)\left(a_n\cos\lambda_n t + b_n \sin\lambda_n t\right) \end{aligned}$$

where these sums run over integers only, and $\lambda_n = \sqrt{\left|k^2 - (n\pi c/L)^2\right|}$.

Also, to satisfy the initial conditions, the a_n are the Fourier sine coefficients for the odd periodic extension of $f(x)$, that is,

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\pi x/L\right)\,dx$$

for $n = 1, 2, \ldots$.

If we differentiate this expression for $u(x,t)$ with respect to t, and set $t = 0$, then we see that $-ka_n + \lambda_n b_n$ are just the Fourier sine coefficients of the odd periodic extension of $g(x)$, that is,

$$-ka_n + \lambda_n b_n = \frac{2}{L}\int_0^L g(x)\sin\left(n\pi x/L\right)\,dx$$

for $n = 1, 2, \ldots$.

(e) In $\frac{kL}{\pi c}$ is a positive integer, then we have to add the corresponding term in the sum when the index n is equal to $\frac{kL}{\pi c}$. In this case, if $n_0 = \frac{kL}{\pi c}$, the solution is as in (d) with the one additional term

$$\sin(kx/c)\left(a_{kL/\pi c}e^{-kt} + b_{kL/\pi c}te^{-kt}\right)$$

with a_n and b_n as in (d), except that $b_{kL/\pi c}$ is determined from the equation

$$-ka_{kL/\pi c} + b_{kL/\pi c} = \frac{2}{L}\int_0^L g(x)\sin(kx/c)\,dx.$$

Question 6. [p 133, #4]

Use D'Alembert's solution to solve the boundary value problem for the wave equation

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0\\
u(0,t) &= 0, \quad t > 0\\
u(1,t) &= 0, \quad t > 0\\
u(x,0) &= 0, \quad 0 < x < 1,\\
\frac{\partial u}{\partial t}(x,0) &= 1, \quad 0 < x < 1.
\end{aligned}$$

Solution: D'Alembert's solution to the wave equation is

$$u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds$$

where f^* and g^* are the the odd 2-periodic extensions of f and g.

For this problem, we have $c = 1$, and $f(x) = 0$ for $0 < x < 1$, so that $f^*(x) = 0$ for all $x \in \mathbb{R}$.

Also, we have $g(x) = 1$ for $0 < x < 1$, so that

$$g^*(x) = \begin{cases} 1 & \text{for} \quad 0 < x < 1\\ -1 & \text{for} \quad -1 < x < 0, \end{cases}$$

and $g^*(x+2) = g^*(x)$ otherwise.

An antiderivative of $g^*(x)$ on the interval $[-1,1]$ is given by

$$G(x) = \begin{cases} x & \text{for} \quad 0 < x < 1\\ -x & \text{for} \quad -1 < x < 0, \end{cases}$$

and $G(x+2) = G(x)$ otherwise.

Therefore, the solution is

$$u(x,t) = \frac{1}{2}\left[G(x+t) - G(x-t)\right]$$

where G is as above.

Question 7. [p 133, #8]

Use D'Alembert's solution to solve the boundary value problem for the wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(1,t) &= 0, \quad t > 0 \\ u(x,0) &= 0, \quad 0 < x < 1, \\ \frac{\partial u}{\partial t}(x,0) &= \sin \pi x, \quad 0 < x < 1. \end{aligned}$$

Solution: As in the previous problem d'Alembert's solution to the wave equation is

$$u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\, ds$$

where f^* and g^* are the the odd 2-periodic extensions of f and g.

Again, for this problem, we have $c = 1$, and $f(x) = 0$ for $0 < x < 1$, so that $f^*(x) = 0$ for all $x \in \mathbb{R}$.

Also, we have $g(x) = \sin \pi x$ for $0 < x < 1$, so that

$$g^*(x) = \sin \pi x$$

for $x \in \mathbb{R}$.

An antiderivative of $g^*(x)$ is given by

$$G(x) = -\frac{1}{\pi}\cos \pi x$$

for $x \in \mathbb{R}$.

$$u(x,t) = \frac{1}{2\pi}\left[\sin \pi(x-t) - \sin \pi(x+t)\right] = -\frac{1}{\pi}\cos \pi x \sin \pi t.$$

Question 8. [p 134, #16]

D'Alembert's solution for zero initial velocity. Show that the solution to the wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(L,t) &= 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 < x < L, \\ \frac{\partial u}{\partial t}(x,0) &= 0, \quad 0 < x < L \end{aligned}$$

is given by

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} b_n \left[\sin\left(n\pi(x-ct)/L\right) + \sin\left(n\pi(x+ct)/L\right)\right]$$

where $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(n\pi x/L\right)\, dx,\ n = 1, 2, \ldots.$

SOLUTION: We showed in class that the solution to this problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x/L) \big(b_n \cos(n\pi ct/L) + b_n^* \sin(n\pi ct/L)\big)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx \qquad \text{and} \qquad b_n^* = \frac{2}{n\pi c}\int_0^L g(x)\sin(n\pi x/L)\,dx$$

for $n \geq 1$.

Since $g(x) = 0$ for $0 < x < L$, then $b_n^* = 0$ for all $n \geq 1$, and the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)\cos(n\pi ct/L),$$

and since

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right],$$

then

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} b_n \left[\sin(n\pi(x-ct)/L) + \sin(n\pi(x+ct)/L)\right] \qquad (*)$$

where $b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin(n\pi x/L)\,dx$.

Now, if f^* is the odd 2L-periodic extension of f, then the Fourier series for f converges to f^* at all points of continuity of f^*, so that

$$f^*(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L),$$

and therefore, from $(*)$ we have

$$u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right].$$

Question 9. [p 144, #2]

Solve the boundary value problem for the one dimensional heat equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi,\ t > 0\\ u(0,t) &= 0, \quad t > 0\\ u(\pi,t) &= 0, \quad t > 0\\ u(x,0) &= 30\sin x, \quad 0 < x < \pi,\end{aligned}$$

and give a brief physical explanation of the problem.

SOLUTION: Using separation of variables as in class, we obtain the solution ($c = 1$ here)

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin nx,$$

where $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin nx\,dx$ for $n \geq 1$.

Now,

$$u(x,0) = f(x) = 30 \sin x$$

for $0 < x < \pi$, that is, $f(x)$ is its own Fourier sine series, so that $b_1 = 30$, and $b_n = 0$ for all $n \geq 2$. The solution is

$$u(x,t) = 30e^{-t} \sin x,$$

and this gives the temperature in a bar whose sides are insulated and whose ends $x = 0$ and $x = \pi$ are kept at 0 temperature, with an initial temperature distribution given by $u(x,0) = 30 \sin x,\ 0 < x < \pi$.

Question 10. [p 144, #6]

Solve the boundary value problem for the one dimensional heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \\ u(0,t) &= 0, \quad t > 0 \\ u(1,t) &= 0, \quad t > 0 \\ u(x,0) &= e^{-x}, \quad 0 < x < 1, \end{aligned}$$

and give a brief physical explanation of the problem.

Solution: After separating variables, applying the initial conditions, and using the superposition principle, we obtain the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where

$$b_n = 2 \int_0^1 e^{-x} \sin n\pi x \, dx$$

for $n \geq 1$.

Integrating by parts, we get

$$\begin{aligned} \int_0^1 e^{-x} \sin n\pi x \, dx &= -\frac{e^{-x}}{1 + n^2\pi^2} (\sin n\pi x + n\pi \cos n\pi x) \Bigg|_0^1 \\ &= \frac{n\pi}{1 + n^2\pi^2} \left[1 + (-1)^{n+1} e^{-1}\right], \end{aligned}$$

so that

$$u(x,t) = 2\pi \sum_{n=1}^{\infty} \frac{n}{1 + n^2\pi^2} \left[1 + (-1)^{n+1} e^{-1}\right] e^{-n^2\pi^2 t} \sin n\pi x,$$

and this gives the temperature in a bar whose sides are insulated and whose ends $x = 0$ and $x = 1$ are kept at 0 temperature, with an initial temperature distribution given by $u(x,0) = e^{-x},\ 0 < x < 1$.