MATH 300 Fall 2004
Advanced Boundary Value Problems I
Sample Final Exam
Friday December 3, 2004

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Question 1. Given the function

$$f(x) = \cos \tfrac{\pi}{a} x, \quad 0 \le x < a$$

find the Fourier sine series for f.

Question 2. Let

$$f(x) = \begin{cases} \cos x & |x| < \pi, \\ 0 & |x| > \pi. \end{cases}$$

(a) Find the Fourier integral of f.

(b) For which values of x does the integral converge to $f(x)$?

(c) Evaluate the integral

$$\int_0^\infty \frac{\lambda \sin \lambda\pi \cos \lambda x}{1-\lambda^2}\, d\lambda$$

for $-\infty < x < \infty$.

Question 3. Let $\mathcal{F}_c$ denote the Fourier cosine transform and $\mathcal{F}_s$ denote the Fourier sine transform. Assume that $f(x)$ and $xf(x)$ are both integrable.

(a) Show that

$$\mathcal{F}_c(xf(x)) = \frac{d}{d\omega}\mathcal{F}_s(f(x)).$$

(b) Show that

$$\mathcal{F}_s(xf(x)) = -\frac{d}{d\omega}\mathcal{F}_c(f(x)).$$

Question 4. Chebyshev's differential equation reads

$$\begin{aligned} (1-x^2)y'' - xy' + \lambda y = 0, &\qquad -1 < x < 1 \\ y(1) = 1, & \\ |y'(1)| < \infty & \end{aligned}$$

(a) Divide by $\sqrt{1-x^2}$ and bring the differential equation into Sturm-Liouville form. Decide if the resulting Sturm-Liouville problem is regular or singular.

(b) For $n \ge 0$, the Chebyshev polynomials are defined as follows:

$$T_n(x) = \cos(n \text{ arc} \cos x), \quad -1 \le x \le 1.$$

Show that $T_n(x)$ is an eigenfunction of this Sturm-Liouville problem and for each $n \ge 0$ find the corresponding eigenvalue.

Hint: If $v = \text{arc} \cos x$, then $\cos v = x$, and $v' = -\dfrac{1}{\sin v} = -\dfrac{1}{(1-x^2)^{1/2}}$.

(c) Show that

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{(1-x^2)^{1/2}}\, dx = 0$$

for $m \neq n$, so that these eigenfunctions are orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(x) = \dfrac{1}{(1-x^2)^{1/2}}$.

Question 5. Solve the following initial value problem for the damped wave equation

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \frac{1}{1+x^2},$$

$$\frac{\partial u}{\partial t}(x,0) = 1.$$

Hint: Do not use separation, instead consider $w(x,t) = e^t \cdot u(x,t)$.

Table of Integrals

$$\int \sin\lambda x \sin\mu x\, dx = \frac{\sin(\mu-\lambda)x}{2(\mu-\lambda)} - \frac{\sin(\mu+\lambda)x}{2(\mu+\lambda)} \quad (\lambda \neq \mu)$$

$$\int \sin\lambda x \cos\mu x\, dx = \frac{\cos(\mu-\lambda)x}{2(\mu-\lambda)} - \frac{\cos(\mu+\lambda)x}{2(\mu+\lambda)} \quad (\lambda \neq \mu)$$

$$\int \cos\lambda x \cos\mu x\, dx = \frac{\sin(\mu-\lambda)x}{2(\mu-\lambda)} + \frac{\sin(\mu+\lambda)x}{2(\mu+\lambda)} \quad (\lambda \neq \mu)$$

$$\int \sin^2\lambda x\, dx = \frac{x}{2} - \frac{\sin 2\lambda x}{4\lambda}$$

$$\int \sin\lambda x \cos\lambda x\, dx = \frac{\sin^2\lambda x}{2\lambda}$$

$$\int \cos^2\lambda x\, dx = \frac{x}{2} + \frac{\sin 2\lambda x}{4\lambda}$$

$$\int x\sin\lambda x\, dx = \frac{\sin\lambda x}{\lambda^2} - \frac{x\cos\lambda x}{\lambda}$$

$$\int x\cos\lambda x\, dx = \frac{\cos\lambda x}{\lambda^2} + \frac{x\sin\lambda x}{\lambda}$$

$$\int x^2\sin\lambda x\, dx = \frac{2\cos\lambda x}{\lambda^3} + \frac{2x\sin\lambda x}{\lambda^2} - \frac{x^2\cos\lambda x}{\lambda}$$

$$\int x^2\cos\lambda x\, dx = -\frac{2\sin\lambda x}{\lambda^3} + \frac{2x\cos\lambda x}{\lambda^2} + \frac{x^2\sin\lambda x}{\lambda}$$

$$\int e^{kx}\sin\lambda x\, dx = \frac{e^{kx}(k\sin\lambda x - \lambda\cos\lambda x)}{k^2+\lambda^2}$$

$$\int e^{kx}\cos\lambda x\, dx = \frac{e^{kx}(k\cos\lambda x + \lambda\sin\lambda x)}{k^2+\lambda^2}$$