MATH 300 Fall 2004
Advanced Boundary Value Problems I
Solutions to Midterm Examinations
Wednesday October 27, 2004

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Question 1. [15 pts]

Find the values of λ^2 for which the boundary value problem

$$\frac{d^2u}{dx^2} + \lambda^2 u = 0, \quad 0 < x < \frac{\pi}{2}$$

$$u(0) = 0$$

$$\int_0^{\frac{\pi}{2}} u(t)\,dt = 0$$

has non-trivial solutions.

SOLUTION: We consider two cases:

case (i): $\lambda = 0$

In this case, the general solution to $\dfrac{d^2u}{dx^2} = 0$ is given by $u(x) = Ax + B$, and $u(0) = 0$ implies that $B = 0$, so that $u(x) = Ax$.

The condition $\int_0^{\frac{\pi}{2}} u(t)\,dt = 0$ implies that

$$\int_0^{\frac{\pi}{2}} A\,t\,dt = A\,\frac{t^2}{2}\bigg|_0^{\frac{\pi}{2}} = A\,\frac{\pi^2}{8} = 0,$$

which implies that $A = 0$, and the boundary value problem has only the trivial solution in this case.

case (ii): $\lambda \neq 0$

In this case, the general solution to $\dfrac{d^2u}{dx^2} + \lambda^2 u = 0$ is given by $u(x) = A\cos\lambda x + B\sin\lambda x$, and $u(0) = 0$ implies that $A = 0$ so that $u(x) = B\sin\lambda x$.

The condition $\int_0^{\frac{\pi}{2}} u(t)\,dt = 0$ implies that

$$\int_0^{\frac{\pi}{2}} B\sin\lambda t\,dt = -\frac{B}{\lambda}\cos\lambda t\bigg|_0^{\frac{\pi}{2}} = \frac{B}{\lambda}\left(1 - \cos\tfrac{\lambda\pi}{2}\right) = 0,$$

and so either $B = 0$ or $\cos\frac{\lambda\pi}{2} = 1$.

Therefore, a nontrivial solution exists if and only if we have $\cos\frac{\lambda\pi}{2} = 1$, that is, $\frac{\lambda\pi}{2} = 2\pi n$, where $n \neq 0$ is an integer. The values of λ^2 for which the boundary value problem has non-trivial solutions are

$$\lambda_n^2 = 16n^2,$$

for $n = 1, 2, 3, \ldots$.

Question 2. [20 pts]

Let $f(x) = \cos^2 x, \quad 0 \le x \le \pi$, and $f(x + 2\pi) = f(x)$ otherwise.

(a) Find the Fourier sine series for f on the interval $[0, \pi]$.

Hint: For $n \ge 1$, $\displaystyle\int \cos^2 x \sin nx\, dx = -\frac{1}{2n}\cos nx + \frac{1}{4}\int [\sin(n+2)x + \sin(n-2)x]\, dx.$

(b) Find the Fourier cosine series for f on the interval $[0, \pi]$.

(c) For which values of x in $[0, \pi]$ do the series in (a) and (b) converge to $f(x)$?

Solution:

(a) (White) Writing $f(x) = \cos^2 x \sim \sum_{n=1}^{\infty} b_n \sin nx$, the coefficients b_n in the Fourier sine series are computed as follows:

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \cos^2 x \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)\sin nx\, dx \\ &= \frac{1}{\pi}\left(-\frac{1}{n}\cos nx\Big|_0^{\pi}\right) + \frac{1}{\pi}\int_0^{\pi}\cos 2x \sin nx\, dx \\ &= \frac{1}{n\pi}(1 - (-1)^n) + \frac{1}{2\pi}\int_0^{\pi}[\sin(n-2)x + \sin(n+2)x]\, dx \\ &= \frac{1-(-1)^n}{2\pi}\left(\frac{2}{n} + \frac{1}{n-2} + \frac{1}{n+2}\right) = 0 \end{aligned}$$

if $n \ne 2$ is even, while if $n = 2$,

$$\begin{aligned} b_2 &= \frac{2}{\pi}\int_0^{\pi}\cos^2 x \sin 2x\, dx = \frac{4}{\pi}\int_0^{\pi}\sin x \cos^3 x\, dx \\ &= -\frac{4}{\pi}\cos^4 x\Big|_0^{\pi} = 0. \end{aligned}$$

Therefore, $b_n = 0$ for all even $n \ge 2$.

If n is odd,

$$\begin{aligned} b_n &= \frac{2}{n\pi} + \frac{1}{\pi}\left[\frac{1}{n-2} + \frac{1}{n+2}\right] \\ &= \frac{2}{n\pi} + \frac{1}{\pi}\frac{2n}{n^2-4}. \end{aligned}$$

The Fourier series for f is therefore

$$\cos^2 x \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{2n-1} + \frac{2n-1}{(2n-1)^2-4}\right\}\sin(2n-1)x$$

for $0 < x < \pi$.

(b) The Fourier cosine series for f is

$$\cos^2 x \sim \frac{1}{2} + \frac{1}{2}\cos 2x,$$

and this can be checked by integrating to find the a_n's.

(c) From Dirichlet's theorem, the Fourier sine series in part (a) converges to $\cos^2 x$ for all $x \in (0,\pi)$ and converges to 0 for $x = 0$ and $x = \pi$. The Fourier cosine series in part (b) converges to $\cos^2 x$ for all $x \in [0,\pi]$ since the series is actually finite.

(a) (BLUE) Writing $f(x) = \sin^2 x \sim \sum_{n=1}^{\infty} b_n \sin nx$, the coefficients b_n in the Fourier sine series are computed as follows:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\sin nx\,dx \\
&= \frac{1}{\pi}\left(-\frac{1}{n}\cos nx\Big|_0^{\pi}\right) - \frac{1}{\pi}\int_0^{\pi}\cos 2x \sin nx\,dx \\
&= \frac{1}{n\pi}\left(1-(-1)^n\right) - \frac{1}{2\pi}\int_0^{\pi}\left[\sin(n-2)x + \sin(n+2)x\right]\,dx \\
&= \frac{1-(-1)^n}{2\pi}\left(\frac{2}{n} - \frac{1}{n-2} - \frac{1}{n+2}\right) = 0
\end{aligned}$$

if $n \neq 2$ is even, while if $n = 2$,

$$\begin{aligned}
b_2 &= \frac{2}{\pi}\int_0^{\pi}\sin^2 x \sin 2x\,dx = \frac{4}{\pi}\int_0^{\pi}\sin^3 x\cos x\,dx \\
&= -\frac{4}{\pi}\sin^4 x\Big|_0^{\pi} = 0.
\end{aligned}$$

Therefore, $b_n = 0$ for all even $n \geq 2$.

If n is odd,

$$\begin{aligned}
b_n &= \frac{2}{n\pi} - \frac{1}{\pi}\left[\frac{1}{n-2} + \frac{1}{n+2}\right] \\
&= \frac{2}{n\pi} - \frac{1}{\pi}\frac{2n}{n^2-4}.
\end{aligned}$$

The Fourier series for f is therefore

$$\cos^2 x \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{2n-1} - \frac{2n-1}{(2n-1)^2-4}\right\}\sin(2n-1)x$$

for $0 < x < \pi$.

(b) The Fourier cosine series for f is

$$\sin^2 x \sim \frac{1}{2} - \frac{1}{2}\cos 2x,$$

and this can be checked by integrating to find the a_n's.

(c) From Dirichlet's theorem, the Fourier sine series in part (a) converges to $\sin^2 x$ for all $x \in [0,\pi]$. The Fourier cosine series in part (b) converges to $\sin^2 x$ for all $x \in [0,\pi]$ since the series is actually finite.

Question 3. [15 pts]

Let $v(x)$ be the steady-state solution to the initial boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + r = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 < x < a, \quad t > 0$$
$$u(0,t) = T_0, \quad t > 0$$
$$\frac{\partial u}{\partial x}(a,t) = 0, \quad t > 0$$

where r is a constant. Find and solve the boundary value problem for the steady-state solution $v(x)$.

SOLUTION: The steady-state solution $v(x)$ satisfies the boundary value problem

$$\frac{d^2 v}{dx^2} + r = 0, \quad 0 < x < a$$
$$v(0) = T_0$$
$$\frac{dv}{dx}(a) = 0,$$

and the general solution to the differential equation is

$$v(x) = -\frac{1}{2}rx^2 + Ax + B,$$

and

$$\frac{dv}{dx}(x) = -rx + A.$$

Therefore,

$$v(0) = T_0 \quad \text{implies} \quad B = T_0$$
$$\frac{dv}{dx}(a) = 0 \quad \text{implies} \quad -ra + A = 0,$$

so that

$$A = ra \qquad \text{and} \qquad B = T_0.$$

The steady-state solution is therefore

$$v(x) = -\frac{1}{2}rx^2 + rax + T_0$$

for $0 \leq x \leq a$.