The wave equation (Ex 11.5.3)

$$\left.\begin{aligned} &u_{tt} = \omega^2 u_{xx} \qquad \text{on } -\infty < x < \infty \\ &u(x,0) = f(x) \\ &u_t(x,0) = g(x) \end{aligned}\right\}$$

$$\hat{u}(\theta,t) = \mathcal{F}\{u(x,t)\}$$

$$\hat{u}_{tt}(\theta,t) = -\theta^2\omega^2\,\hat{u}(\theta,t)$$

$$\Rightarrow \quad \hat{u}(\theta,t) = C_1(\theta)\cos(\theta\omega t) + C_2(\theta)\sin(\theta\omega t)$$

$$\hat{u}(\theta,0) = C_1(\theta) = \hat{f}(\theta)$$

$$\hat{u}_t(\theta,0) = \hat{g}(\theta) = \theta\omega\, C_2(\theta)$$

$$\Rightarrow \quad C_2(\theta) = \frac{\hat{g}(\theta)}{\theta\omega}$$

$$\Rightarrow \quad \hat{u}(\theta,t) = \hat{f}(\theta)\cos(\theta\omega t) + \hat{g}(\theta)\frac{\sin(\theta\omega t)}{\theta\omega}$$

Need inverse F-T:

$$u(x,t) = \mathcal{F}^{-1}\left(\hat{f}(\theta)\cos(\theta\omega t)\right) + \mathcal{F}^{-1}\left(\hat{g}(\theta)\frac{\sin(\theta\omega t)}{\theta\omega}\right)$$

$$F^{-1}\left(\hat{f}(\theta)\cos(\theta\omega t)\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\theta)\,\frac{1}{2}\left(e^{i\theta\omega t} + e^{-i\theta\omega t}\right) e^{i\theta x}\, d\theta$$

$$= \frac{1}{2}\,\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\theta)\left(e^{i\theta(x+\omega t)} + e^{i\theta(x-\omega t)}\right) d\theta$$

$$= \frac{1}{2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\theta)\, e^{i\theta(x+\omega t)}\, d\theta + \frac{1}{2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\theta)\, e^{+i\theta(x-\omega t)}\, d\theta$$

$$= \frac{1}{2}\, f(x+\omega t) + \frac{1}{2} f(x-\omega t)$$

$$= \frac{1}{2}\left(f(x+\omega t) + f(x-\omega t)\right) \qquad (*)$$

Now we write

$$\frac{\sin\theta\omega t}{\theta\omega} = \frac{1}{2}\left(\frac{\sin(\theta\omega t)}{\theta\omega} - \frac{\sin(-\theta\omega t)}{\theta\omega}\right)$$

$$= \frac{1}{2}\int_{-t}^{t} \cos(\theta\omega s)\, ds$$

$$\mathcal{F}^{-1}\left(\hat{g}(\theta)\,\frac{\sin(\theta\omega t)}{\theta\omega}\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{g}(\theta)\,\frac{1}{2}\int_{-t}^{t} \cos(\theta\omega s)\,ds\; e^{i\theta x}\, d\theta$$

$$= \frac{1}{2}\int_{-t}^{t} \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\theta)\cos(\theta\omega s)\, e^{i\theta x}\, d\theta\, ds$$

$$= \frac{1}{2}\int_{-t}^{t} \mathcal{F}^{-1}\left(\hat{g}(\theta)\cos(\theta\omega s)\right) ds$$

use (*)

$$= \frac{1}{2}\int_{-t}^{t} \frac{1}{2}\left(g(x+\omega s)+g(x-\omega s)\right) ds$$

$$= \frac{1}{2}\int_{-t}^{t} g(x+\omega s)\, ds$$

substitution

$y = x+\omega s$

$dy = \omega\, ds$

$$= \frac{1}{2\omega}\int_{x-\omega t}^{x+\omega t} g(y)\, dy$$

$s=-t \longrightarrow y = x-\omega t$

$s=t \longrightarrow y = x+\omega t$

Together we find

$$u(x,t) = \frac{1}{2}\left(f(x+\omega t) + f(x-\omega t) \right) + \frac{1}{2\omega}\int_{x-\omega t}^{x+\omega t} g(s)\,ds$$

D'Alemberts solution