Math 209

Assignment 8 – Solutions

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1. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

(a) $\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy$, C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Solution:

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy = \int\int_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}})\right] dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2-1)dx\,dy = \int_0^1 (\sqrt{y} - y^2)dy = \frac{1}{3}.$$

(b) $\int_C \sin y\,dx + x\cos y\,dy$, C is the ellipse $x^2 + xy + y^2 = 1$.

Solution:

$$\int_C \sin y\,dx + x\cos y\,dy = \int\int_D \left[\frac{\partial}{\partial x}(x\cos y) - \frac{\partial}{\partial y}(\sin y)\right] dA = \int\int_D (\cos y - \cos y)dA = 0.$$

2. If f is a harmonic function, that is $\nabla^2 f = 0$, show that the line integral $\int f_y dx - f_x dy$ is independent of path in any simple region D.

Solution:

$\nabla^2 f = 0$ means that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\int_C \mathbf{F}.dr = \int_C f_y dx - f_x dy = \int\int_D \left[\frac{\partial}{\partial x}(-f_x) - \frac{\partial}{\partial y}(f_y)\right] dA$$

$$= -\int\int_D (f_{xx} + f_{yy})dA = 0.$$

3. Find the area enclosed by the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

Solution:

The astroid has parametric equations $x = \cos^3 t,\ y = \sin^3 t$, where $0 \le t \le 2\pi$.

$$A = \frac{1}{2}\int_C xdy - ydx = \frac{1}{2}\int_0^{2\pi} \cos^3 t \cdot (3\cos t \sin^2 t)dt - \sin^3 t \cdot (-3\sin t\cos^2 t)dt$$

$$= \frac{1}{2}\int_0^{2\pi} (3\cos^4 t \sin^2 t + 3\sin^4 t\cos^2 t)dt = \frac{1}{2}\int_0^{2\pi} 3\cos^2 t \sin^2 t\,dt$$

$$= \frac{3}{4}\int_0^{2\pi} \sin^2 2t\,dt = \frac{3}{4}\int_0^{2\pi} \frac{1-\cos 4t}{2}dt = \frac{3\pi}{4}.$$

4. Let

$$I = \int_C \frac{ydx - xdy}{x^2+y^2}$$

where C is a circle oriented counterclockwise.

(a) Show that $I = 0$ if C does not contain the origin.

Solution:

Let $P = \frac{y}{x^2+y^2}$, $Q = \frac{-x}{x^2+y^2}$ and let D be the region bounded by C. P and Q have continuous partial derivatives on an open region that contains region D. By Green's Theorem,

$$I = \int_C \frac{ydx - xdy}{x^2+y^2} = \int_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$= \int\int_D \left[\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2}\right] dxdy = 0\,.$$

(b) What is I if C contain the origin?

Solution:

The functions $P = \frac{y}{x^2+y^2}$ and $Q = \frac{-x}{x^2+y^2}$ are discontinuous at $(0,0)$, so we can not apply the Green's Theorem to the circle C and the region inside it. We use the definition of $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\int_C Pdx + Qdy = \int_{C_r} Pdx + Qdy = \int_0^{2\pi} \frac{r\sin t(-r\sin t) + (-r\cos t)(r\cos t)}{r^2\cos^2 t + r^2\sin^2 t} dt$$

$$= \int_0^{2\pi} -dt = -2\pi\,.$$

5. Find the curl and the divergence of the vector field $\mathbf{F} = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + z\,\mathbf{k}$. Is $\mathbf{F}$ conservative?

Solution:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^x \cos y & z \end{vmatrix}$$

$$= (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (e^x \sin y - e^x \sin y)\,\mathbf{k} = 0\,.$$

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y) + \frac{\partial}{\partial z}(z) = e^x \sin y - e^x \sin y + 1 = 1\,.$$

Since $\operatorname{curl}\mathbf{F} = 0$ and the domain of $\mathbf{F}$ is R^3 and its components have continuous partial derivatives, $\mathbf{F}$ is a conservative vector field.

6. Is there a vector field $\mathbf{G}$ on R^3 such that $\text{curl}\,\mathbf{G} = xy^2\,\mathbf{i} + yz^2\,\mathbf{j} + zx^2\,\mathbf{k}$? Explain.

Solution:

No. Assume there is such a $\mathbf{G}$. Then $\text{div}(\text{curl}\mathbf{G}) = y^2 + z^2 + x^2 \neq 0$, which contradicts Theorem (If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on R^3 and P, Q and R have continuous second-order partial derivatives, then $\text{div}(\text{curl}\,\mathbf{F}) = 0$).

7. Identify the surface with the given vector equation.

(a) $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\,\mathbf{k}$

Solution:

$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x = u\cos v$, $y = u\sin v$ and $z = u^2$. For any point (x, y, z) on the surface, we have $x^2 + y^2 = u^2\cos^2 + u^2\sin^2 v = u^2 = z$. Since no restrictions are placed on the parameters, the surface is $z = x^2 + y^2$. Which we recognize as a circular paraboloid opening upward whose axis is the z-axis.

(b) $\mathbf{r}(x, \theta) = \langle x, x\cos\theta, x\sin\theta\rangle$

Solution:

$\mathbf{r}(x, \theta) = \langle x, x\cos\theta, x\sin\theta\rangle$, so the corresponding parametric equations for the surface are $x = x$, $y = x\cos\theta$ and $z = ux\sin\theta$. For any point (x, y, z) on the surface, we have $y^2 + z^2 = x\cos^2\theta + x\sin^2\theta = x^2$. Whit $x = x$ and no restrictions on the parameters, the surface is $y^2 + z^2 = x^2$, Which we recognize as a circular con opening whose axis is the x-axis.

8. Find a parametric representation for the surface.

(a) The part of elliptic paraboloid $x + y^2 + 2z^2 = 4$ that lies in front of the plane $x = 0$

Solution:

$x = 4 - y^2 - 2z^2$, $y = y$, $z = z$, where $y^2 + 2z^2 \leq 4$ since $x \geq 0$. Then the associated vector equation is $\mathbf{r}(y, z) = (4 - y^2 - 2z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

(b) The part of sphere $x^2 + y^2 + z^2 = 16$ that lies above the cone $z = \sqrt{x^2 + y^2}$

Solution:

Since the cone intersects the sphere in the circle $x^2 + y^2 = 8$, $z = 2\sqrt{2}$ and we want the portion of the sphere above this, we can parameterize the surface $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \leq 8$.

Alternate Solution: Using spherical coordinates, $x = 4\sin\phi\cos\theta$, $y = 4\sin\phi\cos\theta$, $z = 4\cos\phi$ where $0 \leq \phi \leq \frac{\pi}{4}$ and $0 \leq \theta \leq 2\pi$.

9. Find the area of the part of the surface $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution:

$z = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \int\!\!\int_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 \sqrt{1 + 4r^2}\, r\, dr$$

$$= [\theta]_0^{2\pi} \left[\frac{1}{12}(1 + 4r^2)^{\frac{3}{2}} \right]_1^2 = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})\,.$$

10. Find the area of the part of the surface $z = x^2 + 2y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

Solution:

$z = x^2 + 2y$ with $0 \le x \le 1$, $0 \le y \le 2x$. Then

$$A(S) = \int\!\!\int_D \sqrt{1 + 4x^2 + 4}\, dA = \int_0^1 \int_1^{2x} \sqrt{5 + 4x^2}\, dx\, dy = \int_0^1 2x\sqrt{5 + 4x^2}\, dx$$

$$= \frac{1}{4} \left[\frac{2}{3}(5 + 4x^2)^{\frac{3}{2}} \right]_0^1 = \frac{9}{2}\,.$$