Math 209

Solutions to assignment 7

Due: 12 Noon on Thursday, November 3, 2005

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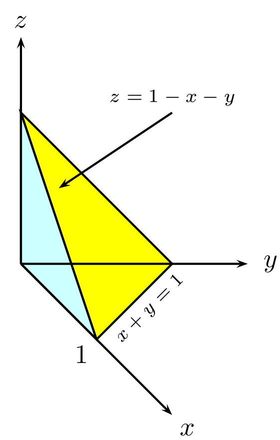
1. By direct integration, we obtain

$$\begin{aligned}\int_0^a\int_0^x\int_0^{xy} x^3y^2z\,dzdydx &= \int_0^a\int_0^x \frac{1}{2}x^3y^2z^2\Big|_0^{xy}\,dydx\\ &= \frac{1}{2}\int_0^a\int_0^x x^5y^4dydx = \frac{1}{10}\int_0^a x^5y^5\Big|_0^x dx = \frac{1}{10}\int_0^a x^{10}dx\\ &= \frac{1}{110}x^{11}\Big|_0^a = \frac{a^{11}}{110}.\end{aligned}$$

2. The tetrahedron Ω is illustrated below:

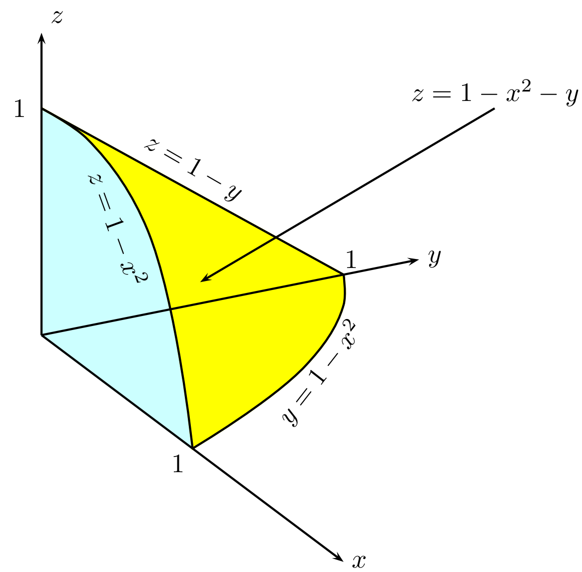


Consequently, we obtain

$$\begin{aligned}\iiint_\Omega \frac{1}{(x+y+z+1)^3}dV &= \int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dz\,dy\\ &= -\frac{1}{2}\int_0^1\int_0^{1-x} 1/(x+y+z+1)^2\Big|_0^{1-x-y}dy\,dx\\ &= \frac{1}{2}\int_0^1\int_0^{1-x}\left(\frac{1}{(x+y+1)^2}-\frac{1}{4}\right)dy\,dx\\ &= \frac{1}{2}\int_0^1\left(\frac{-1}{x+y+1}-\frac{1}{4}y\right)\Big|_0^{1-x}dx\\ &= \frac{1}{2}\int_0^1\left[\frac{1}{x+1}-\frac{3}{4}+\frac{1}{4}\right]dx = \frac{1}{2}\left[\ln(x+1)-\frac{3}{4}x+\frac{1}{8}x^2\right]_0^1\\ &= \frac{1}{2}\left[\ln 2-\frac{5}{8}\right]\end{aligned}$$

3. (a) One can easily recognize that D is the region contained in the first octant, which is bounded

by the surface $z = 1 - x^2 - y$ and the plane $z = 0$, see the picture below:



(b): We have the following equivalent to (1) iterated integrals

$$\int_0^1 \int_0^{1-z} \int_0^{\sqrt{1-y-z}} f(x,y,z)\,dx\,dy\,dz,$$
$$\int_0^1 \int_0^{1-y} \int_0^{\sqrt{1-y-z}} f(x,y,z)\,dx\,dz\,dy,$$
$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x^2-z} f(x,y,z)\,dy\,dx\,dz,$$
$$\int_0^1 \int_0^{1-y} \int_0^{\sqrt{1-x^2-z}} f(x,y,z)\,dy\,dz\,dx,$$
$$\int_0^1 \int_0^{\sqrt{1-y}} \int_0^{1-x^2-y} f(x,y,z)\,dz\,dx\,dy,$$

4. The moment M_{xy} of E about the coordinate plane $0xy$ is

$$M_{xy} = \iiint\limits_V z\,dV,$$

and we have

$$M_{xy} = \iiint\limits_V z\,dV = \int_{-b}^{b} \int_0^{\frac{c}{b}\sqrt{b^2-y^2}} \int_{-\frac{a}{bc}\sqrt{b^2c^2-c^2y^2-b^2z^2}}^{\frac{a}{bc}\sqrt{b^2c^2-c^2y^2-b^2z^2}} z\,dx\,dz\,dy$$
$$= \frac{2a}{bc} \int_{-b}^{b} \int_0^{\frac{c}{b}\sqrt{b^2-y^2}} z\sqrt{b^2c^2 - c^2y^2 - b^2z^2}\,dz\,dy$$

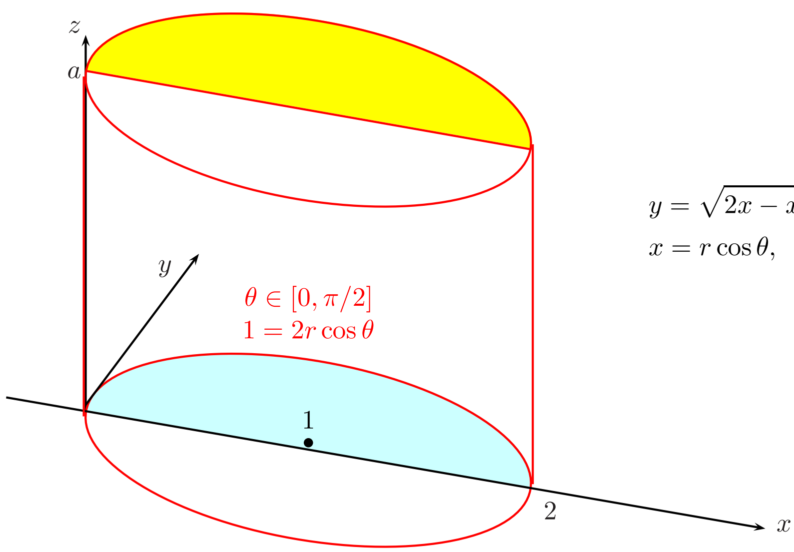
We apply the substitution $u = b^2c^2 - c^2y^2 - b^2z^2$, i.e. $-2b^2zdz = du$, we obtain

$$\begin{aligned}
M_{xy} &= \frac{2a}{bc}\int_{-b}^{b}\int_{b^2c^2-c^2y^2}^{0} \frac{-1}{b^2}\sqrt{u}\,du\,dy = \frac{a}{b^3c}\int_{-b}^{b}\int_{0}^{b^2c^2-c^2y^2} \sqrt{u}\,du\,dy \\
&= \frac{2a}{3b^3c}\int_{-b}^{b}(b^2c^2 - c^2y^2)^{\frac{3}{2}}dy = \frac{4ac^2}{3b^3}\int_0^b (b^2-y^2)\sqrt{b^2-y^2}\,dy \\
&= \frac{4ac^2}{3b^3}\left[b^2\int_0^b \sqrt{b^2-y^2}dy - \int_0^b y^2\sqrt{b^2-y^2}\,dy\right] \\
&= \frac{4ac^2}{3b^3}\left[b^2\frac{\pi b^2}{4} - \int_0^b y^2\sqrt{b^2-y^2}\,dy\right].
\end{aligned}$$

By applying the substitution $y = b\sin\theta$ we finally obtain

$$\begin{aligned}
M_{xy} &= \frac{4ac^2}{3b^3}\left[\frac{\pi b^4}{4} - \int_0^{\frac{\pi}{2}} b^4\sin^2\theta\,\cos^2\theta\,d\theta\right] \\
&= \frac{4ac^2}{3b^3}\left[\frac{\pi b^4}{4} - \frac{b^4}{8}\int_0^{\frac{\pi}{2}}(1-\cos 4\theta)\,d\theta\right] \\
&= \frac{4ac^2}{3b^3}\left[\frac{\pi b^4}{4} - \frac{b^4}{8}\frac{\pi}{2}\right] = \frac{\pi abc^2}{4}.
\end{aligned}$$

5. In order to compute the integral (2), first, we identify the region of integration (see the picture below):

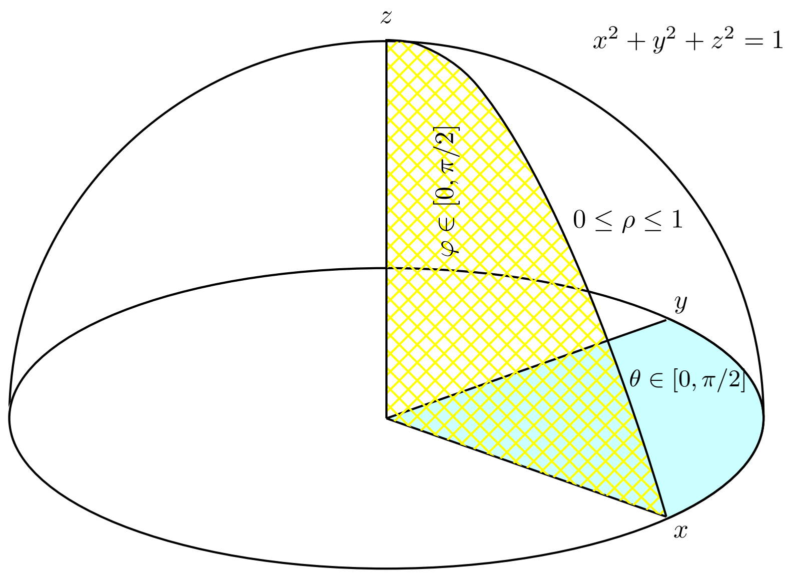


$$\begin{aligned}
y &= \sqrt{2x - x^2} \quad \Rightarrow \quad x^2 + y^2 = 2x \\
x &= r\cos\theta, \quad y = r\sin\theta \quad \Rightarrow \quad r = 2\cos\theta
\end{aligned}$$

One can easily observe that it is appropriate to use cylindrical coordinates, i.e. we obtain

$$\begin{aligned}
\int_0^2\int_0^{\sqrt{2x-x^2}}\int_0^a z\sqrt{x^2+y^2}\,dz\,dy\,dx &= \int_0^a\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} zr^2\,dr\;d\theta\,dz \\
&= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \frac{r^3}{3}\Big|_0^{2\cos\theta} d\theta = \frac{4a^2}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta\,d\theta \\
&= \frac{4a^2}{3}\int_0^{\frac{\pi}{2}}\cos(1-\sin^2\theta)\,d\theta = \frac{8a^2}{9}.
\end{aligned}$$

6. The region of integration is identified on the picture below:



In order to compute (3) we apply the change to spherical coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx = \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sqrt{\rho^2}\rho^2 \sin\varphi\, d\varphi\, d\theta\, d\rho$$

$$= \frac{\pi}{8} \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi = \frac{\pi}{8}.$$

7. In order to identify the surface (4) we apply the spherical coordinates:

$$x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho \cos\varphi.$$

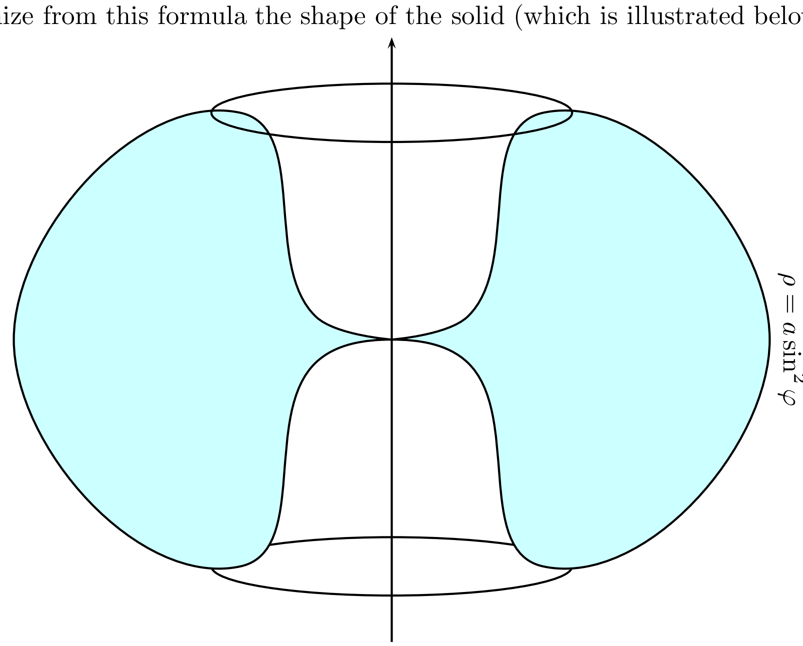
Since

$$\rho^2 = x^2 + y^2 + z^2, \ \ \rho^2 \sin^2\varphi = x^2 + y^2$$

the equation (4) can be written as

$$\rho^6 = a^2 \rho^4 \sin^4\varphi \quad \Leftrightarrow \quad \rho = a \sin^2\varphi. \tag{5}$$

One can recognize from this formula the shape of the solid (which is illustrated below):



Therefore, the region E bounded by the surface (4) is characterized by the equations

$$\theta \in [0, 2\pi], \ \ \varphi \in [0, \pi], \ \ \rho \le a \sin^2 \varphi,$$

thus we have

$$\begin{aligned}
V(E) = \iiint\limits_E dV &:= \int_0^{2\pi} \int_0^{\pi} \int_0^{a \sin^2 \varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\
&= 2\pi \int_0^{\pi} \left[\frac{\rho^3}{3} \right]_0^{a \sin^2 \varphi} d\varphi = \frac{2a^3\pi}{3} \int_0^{\pi} \sin^7 \varphi \, d\varphi \\
&= \frac{2a^3\pi}{3} \int_0^{\pi} (1 - \cos^2\varphi)^3 \sin\varphi \, d\varphi = \frac{2a^3\pi}{3} \int_{-1}^{1} (1-t^2)^3 \, dt \\
&= \frac{4a^3\pi}{3} \int_0^{1} (1 - 3t^2 + 3t^4 - t^6) dt = \frac{4a^3\pi}{3} \left(1 - 1 + \frac{3}{5} - \frac{1}{7} \right) \\
&= \frac{4a^3\pi}{3} \cdot \frac{16}{35} = \frac{64}{105} \pi a^3.
\end{aligned}$$

8. By switching to cylindrical coordinates we find out that

$$E := \{(r, \theta, z) : 0 \le \theta < \frac{\pi}{2}, \ 0 \le r \le 1, \ 0 \le z \le 1 - r^2\},$$

so, we obtain

$$\begin{aligned}
\iiint\limits_E (x^3 + xy^2) dV = \iiint\limits_E x(x^2 + y^2) dV \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} r\cos\theta (r^2 \cos^2\theta + r^2 \sin^2\theta) \, r \, dz \, dr \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} r^4 \cos\theta \, dz \, dr \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r^4 \cos\theta \, (z) \Big|_{z=0}^{z=1-r^2} dr \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r^4 (1 - r^2) \cos\theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \cos\theta \left[\frac{1}{5} r^5 - \frac{1}{7} r^7 \right]_{r=0}^{r=1} d\theta \\
&= \frac{2}{35} \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \frac{2}{35} [\sin\theta]_0^{\pi/2} = \frac{2}{35}.
\end{aligned}$$

9. We assume that the center of the sphere is located at the origin $(0, 0, 0)$ and that the diameter of the intersection is along the z-axis, with one of the planes being the xz-plane, while the other being the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates this wedge E is described by the equations:

$$E := \left\{ (\rho, \theta, \varphi) : 0 \le \rho \le a, \ 0 \le \theta \le \frac{\pi}{6}, \ 0 \le \varphi \le \pi \right\}.$$

Consequently, in spherical coordinates the volume of E is given by

$$\begin{aligned}
V(E) &= \int_0^{\frac{\pi}{6}} \int_0^{\pi} \int_0^{a} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\
&= \int_0^{\frac{\pi}{6}} d\theta \int_0^{\pi} \sin\theta \, d\theta \int_0^{a} \rho^2 \, d\rho = \frac{1}{9} \pi a^3.
\end{aligned}$$

10. By switching to the polar coordinates, we obtain

$$\begin{aligned}
\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)dy\,dx &= \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x(x^2 + y^2)dy\,dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 (r\cos\theta)(r^2)\, r\, dr\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^3 r^4\, dr \\
&= [\sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{5}r^5\right]_0^3 = \frac{486}{5}.
\end{aligned}$$