Math 209
Assignment 5 — Solutions

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1. Integrate $f(x,y) = \sin(\sqrt{x^2+y^2})$ over:

   (a) the closed unit disc;

   (b) the annular region $1 \leqslant x^2+y^2 \leqslant 4$.

   Solution

   (a) $\displaystyle\iint_\Omega \sin(\sqrt{x^2+y^2})\,dA = \int_0^{2\pi}\int_0^1 (\sin r)\,r\,dr\,d\theta = 2\pi(\sin(1)-\cos(1)).$

   (b) $\displaystyle\iint_\Omega \sin(\sqrt{x^2+y^2})\,dA = \int_0^{2\pi}\int_1^2 (\sin r)\,r\,dr\,d\theta = 2\pi(\cos(1)-2\cos(2)+\sin(2)-\sin(1)).$

2. Calculate the following integrals by changing to polar coordinates:

   (a) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx;$ (b) $\displaystyle\int_0^1\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$

   Solution

   (a) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta = \frac{4\pi}{3}.$

   (b) The region of integration Ω is inside the $(x-1/2)^2+y^2 = 1/4$, which has polar equation $r = \cos\theta$. The integral becomes:
   $$\int_0^1\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} r^2\,dr\,d\theta = \frac{1}{3}\int_{-\pi/2}^{\pi/2}\cos^3\theta\,d\theta = \frac{4}{9}.$$

3. Find the area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1+\cos\theta$.

   Solution
   $$A = \int_{-\pi/3}^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}[9\cos^2\theta-(1+\cos\theta)^2]\,d\theta = \left[\frac{3\theta}{2}+\sin 2\theta-\sin\theta\right]_{-\pi/3}^{\pi/3} = \pi.$$

4. Find the volume of the solid bounded above by $z = 1-(x^2+y^2)$, bounded below by the xy–plane, and bounded on the sides by the cylinder $x^2+y^2-x = 0$.

   Solution
   $$V = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}(1-r^2)r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{\cos^2\theta}{2}-\frac{\cos^4\theta}{2}\right]d\theta = \frac{5\pi}{32}.$$

5. Find the mass and centre of mass of the plate that occupies the given region Ω with the given density function λ.

   (a) $\Omega = \{(x,y)\in\mathbb{R}^2;\ 0\leqslant x\leqslant a,\ 0\leqslant y\leqslant\sqrt{a^2-x^2}\};\ \lambda(x,y) = xy.$

   (b) Ω is the region inside the circle $r = 2\sin\theta$ and outside the circle $r = 1$; $\lambda(x,y) = y$.

Solution

(a) $m = \iint_\Omega \lambda(x,y)\,dA = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = \int_0^a \frac{x}{2}(a^2-x^2)\,dx = \frac{a^4}{8}.$

$$\bar{x} = \frac{1}{m}\iint_\Omega x\,\lambda(x,y)\,dA = \frac{1}{m}\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\,dy\,dx = \frac{1}{m}\int_0^a \frac{x^2}{2}(a^2-x^2)\,dx = \frac{1}{m}\frac{a^5}{15} = \frac{8}{15}a.$$

$$\bar{y} = \frac{1}{m}\iint_\Omega y\,\lambda(x,y)\,dA = \frac{1}{m}\int_0^a \int_0^{\sqrt{a^2-x^2}} xy^2\,dy\,dx = \frac{1}{m}\int_0^a \frac{x}{3}(a^2-x^2)^{3/2}\,dx = \frac{1}{m}\frac{a^5}{15} = \frac{8}{15}a.$$

(b) $m = \iint_\Omega \lambda(x,y)\,dA = \int_{\pi/6}^{5\pi/6} \int_1^{\sqrt{2\sin\theta}} r\sin\theta\; r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \left(\frac{8}{3}\sin^4\theta - \frac{1}{3}\sin\theta\right)d\theta = \frac{2\pi}{3} - \frac{\sqrt{3}}{4}.$

$\bar{x} = 0$ by symmetry.

$$\bar{y} = \frac{1}{m}\iint_\Omega y\,\lambda(x,y)\,dA = \frac{1}{m}\int_{\pi/6}^{5\pi/6} \int_1^{\sqrt{2\sin\theta}} r^2\sin^2\theta\; r\,dr\,d\theta$$

$$= \frac{1}{m}\int_{\pi/6}^{5\pi/6} \left(4\sin^6\theta - \frac{1}{4}\sin^2\theta\right)d\theta = \frac{1}{m}\left(\frac{11\sqrt{3}}{16} - \frac{3\pi}{4}\right) = \frac{3(12\pi + 11\sqrt{3})}{4(8\pi + 3\sqrt{3})}.$$

6. Consider a square fan blade with sides of length 2 and the lower left corner placed at the origin. If the density of the blade is $\lambda(x,y) = 1 + x/10$, is it more difficult to rotate the blade about the x-axis or the y-axis?

Solution

We compare moments about the x and y axes:

$$I_x = \iint_D y^2\lambda(x,y)\,dA = \int_0^2 \int_0^2 y^2(1+\frac{x}{10})\,dy\,dx = \frac{88}{15};$$

$$I_y = \iint_D x^2\lambda(x,y)\,dA = \int_0^2 \int_0^2 x^2(1+\frac{x}{10})\,dy\,dx = \frac{92}{15}.$$

We find that

$$I_x = \frac{88}{15} < \frac{92}{15} = I_y,$$

so it is more difficult to rotate the blade about the y-axis.

7. Find the surface area of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0,0)$, $(0,1)$ and $(2,1)$.

Solution

To simplify the calculation, consider the order of integration.

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \int_0^1 \int_0^{2y} \sqrt{10+16y^2}\,dx\,dy$$

$$= \int_0^1 2y\sqrt{10+16y^2}\,dy = \frac{1}{24}(10+16y^2)^{3/2}\Big|_0^1 = \frac{1}{24}[(26)^{3/2} - (10)^{3/2}].$$

8. Find the surface area of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane.

<u>*Solution*</u>
For this problem polar coordinates are useful.

$$S = \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint\limits_D \sqrt{1 + 4x^2 + 4y^2}\, dA$$
$$= \int_0^{2\pi} \int_0^2 r\sqrt{1 + 4r^2}\, dr\, d\theta = \int_0^{2\pi} \frac{1}{12}(1 + 4r^2)^{3/2}\Big|_0^2\, d\theta = \frac{\pi}{6}[(17)^{3/2} - 1].$$

9. Find the surface area of the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ for $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$.

<u>*Solution*</u>

$$S = \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_0^1 \int_0^1 \sqrt{1 + x + y}\, dy\, dx$$
$$= \int_0^1 \frac{2}{3}(1 + x + y)^{3/2}\Big|_{y=0}^1\, dx = \frac{2}{3}\int_0^1 [(2 + x)^{3/2} - (1 + x)^{3/2}]\, dx$$
$$= \frac{4}{15}[(2 + x)^{5/2} - (1 + x)^{5/2}]\Big|_0^1 = \frac{4}{15}\{[(3)^{5/2} - (2)^{5/2}] - [(2)^{5/2} - (1)^{5/2}]\}$$
$$= \frac{4}{15}[(3)^{5/2} - (2)^{7/2} + 1].$$

10. Find the surface area of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

<u>*Solution*</u>
It is convenient to use cylindrical coordinates. The equations of the sphere and paraboloid in cylindrical coordinates are $r^2 + z^2 = 4z$ and $z = r^2$ respectively. First calculate the curve of intersection of the two surfaces.

$$z + z^2 = 4z \qquad \Longrightarrow \qquad z = 0, 3 \qquad \Longrightarrow \qquad r = 0, \sqrt{3}$$

Thus the points of intersection are $(r, z) = (0, 0)$ and $(\sqrt{3}, 3)$. Calculating partial derivatives, we obtain

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{4 - x^2 - y^2}}, \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{4 - x^2 - y^2}}.$$

Calculating the surface area, we obtain

$$S = \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint\limits_D \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}}\, dr\, d\theta = \int_0^{2\pi} (-2\sqrt{4 - r^2})\Big|_0^{\sqrt{3}}\, d\theta = 4\pi.$$