Math 209

Solutions to assignment 3

Due: 12:00 Noon on Thursday, October 6, 2005.

---

1. Find the minimum of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the condition $x + 2y + 3z = 4$.

   Solution. Let's define $g(x, y, z) = x + 2y + 3z$, so the problem is to find the minimum of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 4$. We have

$$\nabla f = \lambda \nabla g \quad \Leftrightarrow \quad (2x, 2y, 2z) = \lambda(1, 2, 3);$$

   and reading this component by component we obtain $x = \frac{\lambda}{2}$, $y = \lambda$, $z = \frac{3\lambda}{2}$. Plugging this into the constraint we have

$$\frac{\lambda}{2} + 2\lambda + 3\left(\frac{3\lambda}{2}\right) = 4 \quad \Rightarrow \quad \lambda = \frac{4}{7}.$$

   Thus $x = \frac{2}{7}$, $y = \frac{4}{7}$, $z = \frac{6}{7}$, and $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$ is the only critical point. Now we could use the Hessian matrix of f and see that it is positive definite to justify that this critical point gives the minimum. Alternatively, we can note that the function f is unbounded above (even subject to the restriction) and therefore has no maximum, but it has a minimum since it is bounded below by 0. Therefore the minimum subject to the given restriction is

$$\boxed{f\left(\tfrac{2}{7}, \tfrac{4}{7}, \tfrac{6}{7}\right) = \tfrac{56}{49}}.$$

2. Find the maximum value of the function $F(x, y, z) = (x+y+z)^2$, subject to the constraint given by $x^2 + 2y^2 + 3z^2 = 1$.

   Solution.

   Let's define $g(x, y, z) = x^2+2y^2+3z^2$, so the problem is to find the maximum of $F(x, y, z)$ subject to the constraint $g(x, y, z) = 1$. We have

$$\nabla F = \lambda \nabla g \quad \Leftrightarrow \quad (2(x + y + z), 2(x + y + z), 2(x + y + z)) = \lambda(2x, 4y, 6z).$$

   Reading this component by component and including the restriction we get the system of equations

$$x + y + z = \lambda x \qquad \text{(A)}$$

$$x + y + z = 2\lambda y \qquad \text{(B)}$$

$$x + y + z = 3\lambda z \qquad \text{(C)}$$

$$x^2 + 2y^2 + 3z^2 = 1. \qquad \text{(D)}$$

Subtracting (A)–(B) we get $\lambda(x-2y)=0$, so either $\lambda=0$ or $x=2y$. But $\lambda=0$ would give $x=y=z=0$, and $f(0,0,0)=0$ is obviously not the maximum. Therefore we work with $\boxed{x=2y}$.

Subtracting (B)–(C) we get $\lambda(2y-3z)=0$, and since we already discarded the case $\lambda=0$ we are left with $\boxed{z=\frac{2}{3}y}$.

Using the results in the two frames into (D) we get

$$(2y)^2+2y^2+3\left(\frac{2}{3}y\right)^2=1 \quad \Rightarrow \quad y=\pm\sqrt{\frac{3}{22}} \quad \Rightarrow \quad x=\pm 2\sqrt{\frac{3}{22}},\ z=\pm\frac{2}{3}\sqrt{\frac{3}{22}}.$$

It is clear that the maximum of F occurs when x,y,z are all positive, or when they are all negative. Therefore the maximum value is

$$F\left(2\sqrt{\frac{3}{22}},\sqrt{\frac{3}{22}},\frac{2}{3}\sqrt{\frac{3}{22}}\right)=F\left(-2\sqrt{\frac{3}{22}},-\sqrt{\frac{3}{22}},-\frac{2}{3}\sqrt{\frac{3}{22}}\right)=\left(\pm\frac{11}{3}\sqrt{\frac{3}{22}}\right)^2=\frac{11}{6}.$$

3. Find the maximum and minimum values of the function

$$f(x,y,z)=3x-y-3z,$$

subject to the constraints

$$x+y-z=0, \qquad x^2+2z^2=1.$$

Solution. Let's define $g(x,y,z)=x+y-z$ and $h(x,y,z)=x^2+2z^2$, so the problem is to find the maximum of $f(x,y,z)$ subject to the constraints $g(x,y,z)=0$ and $h(x,y,z)=1$. We have

$$\nabla f=\lambda\nabla g+\mu\nabla h \quad \Leftrightarrow \quad (3,-1,-3)=\lambda(1,1,-1)+\mu(2x,0,4z).$$

Reading this component by component and including the restrictions we get the system of equations

$$3=\lambda+2\mu x \tag{A}$$
$$-1=\lambda \tag{B}$$
$$-3=-\lambda+4\mu z \tag{C}$$
$$x+y-z=0 \tag{D}$$
$$x^2+2z^2=1. \tag{E}$$

Note that (B) already gives $\boxed{\lambda = -1}$. Using this in (A) and (C) we obtain $x = \frac{2}{\mu}$ and $z = -\frac{1}{\mu}$ respectively. Plugging these expressions for x and z into (E) we get

$$\left(\frac{2}{\mu}\right)^2 + 2\left(-\frac{1}{\mu}\right)^2 = 1 \quad \Rightarrow \quad \boxed{\mu = \pm\sqrt{6}}.$$

Now, from (D) we have $y = z - x$, so we get

$$\mu = \sqrt{6} \quad \Rightarrow \quad x = \frac{2}{\sqrt{6}},\ z = -\frac{1}{\sqrt{6}},\ y = -\frac{3}{\sqrt{6}}.$$

$$\mu = -\sqrt{6} \quad \Rightarrow \quad x = -\frac{2}{\sqrt{6}},\ z = \frac{1}{\sqrt{6}},\ y = \frac{3}{\sqrt{6}}.$$

Since the intersection of $x + y - z = 0$ and $x^2 + 2z^2 = 1$ is closed and bounded, all we need to do now is evaluate f at the critical points we have found.

$$f\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = 2\sqrt{6} \qquad \text{is the maximum value,}$$

$$f\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -2\sqrt{6} \qquad \text{is the minimum value.}$$

4. Find the extreme values of the function $f(x, y, x) = xy + z^2$ on the region described by the inequality $x^2 + y^2 + z^2 \leq 1$. Use Lagrange multipliers to treat the boundary case.

Solution. First we work in the interior: $x^2 + y^2 + z^2 < 1$. to find the critical points we set $\nabla f = 0$. This yields $x = y = 0$, so the only critical point in the interior is $(0, 0, 0)$. But clearly $f(0, 0, 0) = 0$ is neither a maximum nor a minimum. It is also clear that there are no singular points.

Now we work on the boundary: $x^2+y^2+z^2 = 1$. Here we can define $g(x, y, z) = x^2+y^2+z^2$, so the problem is to find the extreme values of $f(x, y, z)$ subject to $g(x, y, z) = 1$. We have

$$\nabla f = \lambda \nabla g \quad \Leftrightarrow \quad (y, x, 2z) = \lambda(2x, 2y, 2z).$$

Reading this component by component and including the restriction we get

$$y = 2\lambda x \tag{A}$$

$$x = 2\lambda y \tag{B}$$

$$2z = 2\lambda z \tag{C}$$

$$x^2 + y^2 + z^2 = 1. \tag{E}$$

Note that (C) implies $2z(1 - \lambda) = 0$, so either $z = 0$ or $\lambda = 1$.

<u>Case 1: $z = 0$</u>. Note that (A) and (B) imply $x^2 = y^2$, and then from (D) we get $x^2 = y^2 = \frac{1}{2}$. this way we get four points: $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0)$.

Case 2: $\lambda = 1$. Now (A) and (B) imply $x = y = 0$, and then from (D) we get $z = \pm 1$. This way we get the two points $(0, 0, \pm 1)$.

Since $x^2 + y^2 + z^2 = 1$ is closed and bounded, all we need to do now is evaluate the function at the points we have found:

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2}$$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = -\frac{1}{2} \quad \text{(this is the global minimum)}$$

$$f(0, 0, \pm 1) = 1 \quad \text{(this is the global maximum).}$$

5. Use Lagrange multipliers to prove that a rectangle with maximum area, that has a given perimeter p, is a square.

   Solution. Let the sides of the rectangle be x and y, so the area is $A(x, y) = xy$. The problem is to maximize the function $A(x, y)$ subject to the constraint $g(x, y) = 2x + 2y = p$ ($p > 0$ is a fixed number). We have

$$\nabla A = \lambda \nabla g \quad \Leftrightarrow \quad (y, x) = \lambda(2, 2).$$

   Reading this component by component we get

$$\begin{cases} y &= 2\lambda \\ x &= 2\lambda \end{cases} \quad \Rightarrow \quad x = y$$

   so the rectangle with maximum area is a square with side length $\frac{p}{4}$.

6. Evaluate

$$\int_0^2 \frac{x}{y^2 + 1} dy.$$

   Solution. Since we are integrating with respect to y, the letter x in the integrand is treated as a constant. We have

$$\int_0^2 \frac{x}{y^2 + 1} dy = x \int_0^2 \frac{1}{y^2 + 1} dy = x \arctan(y)|_{y=0}^{y=2}$$

$$= x(\arctan(2) - \arctan(0)) = \boxed{x \arctan(2)}.$$

7. Calculate the iterated integral

$$\int_1^2 \int_0^1 (x+y)^{-2} dx dy.$$

Solution. We have

$$\begin{aligned}
\int_1^2 \int_0^1 (x+y)^{-2} dx dy &= \int_1^2 \left(-(x+y)^{-1}\Big|_{x=0}^{x=1} \right) dy \\
&= \int_1^2 \left[-(1+y)^{-1} + y^{-1} \right] dy = -\ln(1+y)\Big|_{y=1}^{y=2} + \ln(y)\Big|_{y=1}^{y=2} \\
&= -(\ln(3) - \ln(2)) + \ln(2) - \ln(1) \\
&= -\ln(3) + 2\ln(2) = \boxed{\ln\left(\tfrac{4}{3}\right)}.
\end{aligned}$$

8. Calculate the double integral

$$\iint_R x \sin(x+y)\, dA, \qquad \text{where} \qquad R = [0, \pi/6] \times [0, \pi/3].$$

Solution. In this case it is convenient to integrate first with respect to the variable y. We have

$$\begin{aligned}
\iint_R x \sin(x+y)\, dA &= \int_0^{\pi/6} \int_0^{\pi/3} x \sin(x+y)\, dy\, dx \\
&= \int_0^{\pi/6} \left(-x \cos(x+y)\Big|_{y=0}^{y=\pi/3} \right) dx \\
&= \int_0^{\pi/6} \left(-x \cos(x + \frac{\pi}{3}) + x \cos(x) \right) dx \\
&= \int_0^{\pi/6} x \cos(x)\, dx - \int_0^{\pi/6} x \cos(x + \frac{\pi}{3})\, dx.
\end{aligned}$$

These two single integrals can be computed easily using integration by parts, and this way we get

$$\boxed{\iint_R x \sin(x+y)\, dA = \tfrac{\sqrt{3}}{2} - \tfrac{1}{2} - \tfrac{\pi}{12}}.$$

9. Calculate the double integral

$$\iint_R \frac{x}{x^2+y^2}\, dA, \qquad \text{where} \qquad R = [1,2] \times [0,1].$$

Solution. We will need to use the identity

$$\int \ln(a^2 + x^2)\,dx = x\ln(a^2 + x^2) - 2x + 2a\arctan\left(\frac{x}{a}\right). \qquad (*)$$

which can be obtained using integration by parts.

Integrating first with respect to the variable x, we have

$$\begin{aligned}
\iint_R \frac{x}{x^2+y^2}\,dA &= \int_0^1 \int_1^2 \frac{x}{x^2+y^2}\,dx\,dy = \int_0^1 \frac{1}{2}\int_1^2 \frac{2x}{x^2+y^2}\,dx\,dy \\
&= \frac{1}{2}\int_0^1 \left(\ln(x^2+y^2)\Big|_{x=1}^{x=2}\right)\,dy \\
&= \frac{1}{2}\left(\int_0^1 \ln(4+y^2)\,dy - \int_0^1 \ln(1+y^2)\,dy\right) \\
&= \frac{1}{2}\left(y\ln(4+y^2) - 2y + 4\arctan\left(\frac{y}{2}\right)\Big|_{y=0}^{y=1}\right) \\
&\qquad - \frac{1}{2}\left(y\ln(1+y^2) - 2y + 2\arctan(y)\Big|_{y=0}^{y=1}\right) \\
&\qquad \text{(here we have used the identity (*))} \\
&= \cdots = \boxed{\tfrac{1}{2}\ln\left(\tfrac{5}{2}\right) + 2\arctan\left(\tfrac{1}{2}\right) - \arctan(1)}.
\end{aligned}$$

10. Find the volume of the solid that lies under the hyperbolic paraboloid $z = y^2 - x^2$, and above the square $R = [-1, 1] \times [1, 3]$.

Solution. We can see that the function $f(x, y) = y^2 - x^2$ is nonnegative over the given rectangle. Therefore, calling S the solid we have

$$\begin{aligned}
\mathrm{Vol}(S) &= \iint_R (y^2 - x^2)\,dA = \int_1^3 \int_{-1}^1 (y^2 - x^2)\,dx\,dy \\
&= \int_1^3 \left(y^2x - \frac{x^3}{3}\Big|_{x=-1}^{x=1}\right)\,dy \\
&= \int_1^3 2\left(y^2 - \frac{1}{3}\right)\,dy = \cdots = 16.
\end{aligned}$$

Thus $\boxed{\mathrm{Vol}(S) = 16}$ cubic units.