Math 209
Assignment 1 — Solutions

1. Identify and sketch the level surface $f(x, y, z) = 1$ for the function $f(x, y, z) = z^2 - 36x^2 - 9y^2$.

   Solution
   The surface $f(x, y, z) = 1$ is a hyperboloid of two sheets.

   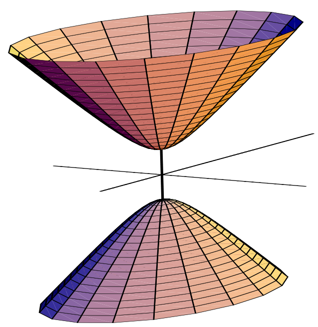

   Figure 1: Level surface $z^2 - 36x^2 - 9y^2 = 1$

2. Find the limit if it exists, or show that the limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$.

   Solution
   Check the one–dimensional limit along the path $y = 0$: $\displaystyle\lim_{(x,0)\to(0,0)} f(x, 0) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$.
   Check the one–dimensional limit along the path $y = x$: $\displaystyle\lim_{(x,x)\to(0,0)} f(x, x) = \lim_{x\to 0} \frac{4x^2}{2x^2} = 2$.
   Since these one-dimensional limits are not the same, the two-dimensional limit
   $\boxed{\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} \text{ does not exist}}$.

3. Find the limit if it exists, or show that the limit does not exist: $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2+y^2-2x-2y}{x^2+y^2-2x+2y+2}$.

   Solution
   Check the one–dimensional limit along the path $x = 1$: $\displaystyle\lim_{(1,y)\to(1,-1)} f(1, y) = \lim_{y\to -1} \frac{y^2-2y-1}{y^2+2y+1}$, which doesn't exist since the denominator goes to zero while the numerator goes to 2. Therefore the two–dimensional limit $\boxed{\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2+y^2-2x-2y}{x^2+y^2-2x+2y+2} \text{ does not exist}}$.

4. Find the limit if it exists, or show that the limit does not exist: $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{\sqrt{x^2+y^2}}$.

   <u>*Solution*</u>

   Notice the following inequality: $0 \leqslant \dfrac{x^2}{\sqrt{x^2+y^2}} \leqslant \dfrac{x^2}{\sqrt{x^2}} = |x|$. Taking limits yields:

$$0 \leqslant \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} \leqslant \lim_{(x,y)\to(0,0)} |x| = \lim_{x\to 0} |x| = 0.$$

   Hence, by the queeze law, $\lim_{(x,y)\to(0,0)} \dfrac{x^2}{\sqrt{x^2+y^2}} = 0$. By an analogous argument we obtain the result $\lim_{(x,y)\to(0,0)} \dfrac{y^2}{\sqrt{x^2+y^2}} = 0$. Therefore $\boxed{\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{\sqrt{x^2+y^2}} = 0}$.

5. Find the limit if it exists, or show that the limit does not exist: $\lim_{(x,y)\to(0,0)} \dfrac{x^4 + y^4}{(x^2+y^2)^{3/2}}$.

   <u>*Solution*</u>

   Notice the following inequality: $0 \leqslant \dfrac{x^4}{(x^2+y^2)^{3/2}} \leqslant \dfrac{x^4}{(x^2)^{3/2}} = |x|$. Taking limits and applying the squeeze law, we get $\lim_{(x,y)\to(0,0)} \dfrac{x^4}{(x^2+y^2)^{3/2}} = 0$. Similarly $\lim_{(x,y)\to(0,0)} \dfrac{y^4}{(x^2+y^2)^{3/2}} = 0$ and hence $\boxed{\lim_{(x,y)\to(0,0)} \dfrac{x^4+y^4}{(x^2+y^2)^{3/2}} = 0}$.

6. Find the limit if it exists, or show that the limit does not exist: $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2}$.

   <u>Hint</u>. Consider using spherical coordinates: $x = \rho \sin\phi \cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.

   <u>*Solution*</u>

   Using spherical coordinates $x = \rho \sin\phi \cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$ we have

$$\begin{aligned} x^2+y^2+z^2 &= \rho^2 \sin^2\phi \cos^2\theta + \rho^2\sin^2\phi \sin^2\theta + \rho^2\cos^2\phi \\ &= \rho^2\sin^2\phi\,(\cos^2\theta + \sin^2\theta) + \rho^2\cos^2\phi \\ &= \rho^2\sin^2\phi + \rho^2\cos^2\phi = \rho^2. \end{aligned}$$

   Using the fact that $|\sin\theta| \leqslant 1$, $|\cos\theta| \leqslant 1$, $|\sin\phi| \leqslant 1$, $|\cos\phi| \leqslant 1$ we get

$$0 \leqslant \frac{|xyz|}{x^2+y^2+z^2} = \frac{|\rho\sin\phi\,\cos\theta\,\rho\sin\phi\,\sin\theta\,\rho\cos\phi|}{\rho^2} = \frac{\rho^3|\sin^2\phi|\,|\cos\theta|\,|\sin\theta|\,|\cos\phi|}{\rho^2} \leqslant \rho.$$

   But $\lim_{(x,y,z)\to(0,0,0)} \rho = \lim_{(x,y,z)\to(0,0,0)} (x^2+y^2+z^2) = 0$, so by the squeeze law $\boxed{\lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2} = 0}$.

7. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called *Laplace's Equation*, named after the eminent French mathematician Pierre Simon de Laplace (1749 — 1827). Solutions of this equation are called *harmonic functions* and play a role in problems of heat conduction, fluid flow, and electric potential. Which of the following functions are solutions of Laplace's equation?

(a) $u(x, y) = x^2 + y^2$;

(b) $u(x, y) = \ln(x^2 + y^2)^{3/2}$;

(c) $u(x, y) = \sin x \cosh y + \cos x \sinh y$.

<u>*Solution*</u>

(a) If $u(x, y) = x^2 + y^2$, then $u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$. Therefore $u(x, y) = x^2 + y^2$ is not a solution.

(b) If $u(x, y) = \ln(x^2 + y^2)^{3/2} = \dfrac{3}{2} \ln(x^2 + y^2)$, then we get

$$u_x = \frac{3x}{x^2 + y^2}, \qquad u_y = \frac{3y}{x^2 + y^2}, \qquad u_{xx} = 3\frac{-x^2 + y^2}{(x^2 + y^2)^2}, \qquad u_{yy} = 3\frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Hence $u_{xx} + u_{yy} = 3\dfrac{-x^2 + y^2 + x^2 - y^2}{(x^2 + y^2)^2} = 0$. Therefore $u(x, y) = \ln(x^2 + y^2)^{3/2}$ is a solution.

(c) If $u(x, y) = \sin x \cosh y + \cos x \sinh y$, then

$$u_{xx} = -\sin x \cosh y - \cos x \sinh y, \qquad u_{yy} = \sin x \cosh y + \cos x \sinh y.$$

Hence $u_{xx} + u_{yy} = -\sin x \cosh y - \cos x \sinh y + \sin x \cosh y + \cos x \sinh y = 0$.
Therefore $u(x, y) = \sin x \cosh y + \cos x \sinh y$ is a solution.

8. Find an equation of the tangent plane to $z = e^x \ln y$ at $(3, 1, 0)$.

<u>*Solution*</u>
If $z = f(x, y) = e^x \ln y$ then
$f_x(x, y) = e^x \ln y$, $f_y(x, y) = \dfrac{e^x}{y}$; $\qquad f_x(3, 1) = 0$, $f_y(3, 1) = e^3$.
The equation of the tangent plane is

$$z = f(3, 1) + f_x(3, 1)(x - 3) + f_y(3, 1)(y - 1) = 0 + 0 \cdot (x - 3) + e^3 \cdot (y - 1) \quad \text{or} \quad \boxed{e^3 y - z = e^3}.$$

9. Find the differential of the following function: $w = \dfrac{x + y}{y + z}$.

<u>*Solution*</u>

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz = \frac{(y + z)\, dx + (z - x)\, dy - (x + y)\, dz}{(y + z)^2}.$$

10. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 centimeters high and 4 centimeters in diameter if the metal in the wall is 0.05 centimeters thick and the metal in the top and bottom is 0.1 centimeters thick.

    Solution
    Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = 2\pi r h\, dr + \pi r^2\, dh$ is an estimate of the amount of metal that makes up the container. With $dr = 0.05$ and $dh = 0.20$ we get
    $dV = 2\pi(2)(10)(0.05) + \pi(2)^2(0.20) = 2.80\pi \approx 8.8$ cm^3.