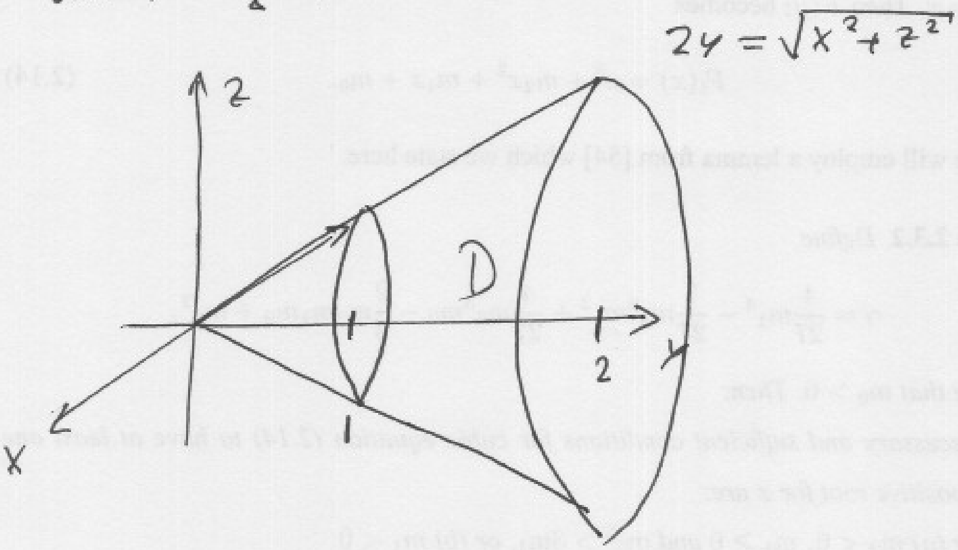
6. D enclosed by $2y = \sqrt{x^2+z^2}$, $y=1$, $y=2$,

$S = \partial D$, find flux of $\vec{F}$ through S:

$$\iint_S \vec{F}\, d\vec{S} = \iiint_D \operatorname{div} \vec{F}\, dV$$

$$\operatorname{div} \vec{F} = 2x + 2y + 2z .$$



Use cylindrical coordinates in (x, z):

$$x = r\cos\theta,$$
$$y = y$$
$$z = r\sin\theta$$

$$D = \{ (r, y, \theta) : 0 \le \theta \le 2\pi, \quad 1 \le y \le 2, \quad 0 \le r \le 2y \}$$

given y, then $2y = \sqrt{x^2+z^2} = r$

hence $0 \le r \le 2y$

$$\iint_S \vec{F}\,d\vec{S} = \iiint_D \operatorname{div}\vec{F}\,dV$$

$$= \int_0^{2\pi}\int_1^2\int_0^{2y} \left(2r\cos\theta \mp 2y + 2r\sin\theta\right) r\,dr\,dy\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \frac{2}{3}r^3\cos\theta \mp y r^2 + \frac{2}{3}r^3\sin\theta \Big|_{r=0}^{r=2y} dy\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \frac{16}{3}y^3\cos\theta \mp 4y^3 + \frac{16}{3}y^3\sin\theta \; dy\,d\theta$$

$$= \int_0^{2\pi} \frac{44}{3}\,\frac{y^4}{4}\Big|_1^2 (\cos\theta$$

$$= \int_0^{2\pi} \frac{y^4}{4}\Big|_1^2 \left(\frac{16}{3}\cos\theta \mp 4 + \frac{16}{3}\sin\theta\right) d\theta$$

$$= \left(4 - \frac{1}{4}\right)\left(\frac{16}{3}\sin\theta \mp 4\theta - \frac{16}{3}\cos\theta\Big|_0^{2\pi}\right)$$

$$= \frac{15}{4}\left(-8\pi - \left(\frac{16}{3} - \frac{16}{3}\right)\right)$$

$$= -30\pi$$

R1 2

9. (a) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \qquad 0 \le t \le 2\pi$

$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\|\vec{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$\int_C y \sin z \, ds = \int_0^{2\pi} \sin t \sin t \sqrt{2} \, dt$$

$$= \sqrt{2} \int_0^{2\pi} \sin^2 t \, dt$$

$$= \frac{\sqrt{2}}{2} \int_0^{2\pi} (1 - \cos 2t) \, dt$$

$$= \frac{\sqrt{2}}{2} (2\pi - 0) = \sqrt{2}\pi$$

(b) Unit sphere:

$\vec{r}(\theta, \phi) = \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$

$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$

$\vec{r}_\theta = \langle -\sin\theta \sin\phi, \cos\theta \sin\phi, 0 \rangle$

$\vec{r}_\phi = \langle \cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi \rangle$

Better idea:

RB

Use divergence theorem:

$$\operatorname{div} \vec{F} = 0 + 1 + 0 = 1$$

$$\iint_S \vec{F}\, d\vec{S} = \iiint_E 1\, dV = \frac{4}{3}\pi$$

10. $$W = \int_C \vec{F}\, d\vec{r}$$

Check if $\vec{F}$ is conservative

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^4 + z^2 & y^4 + x^2 & z^4 + y^2 \end{vmatrix}$$

$$= \vec{i}\,(2y) + \vec{j}\,(2z) + \vec{k}\,(2x)$$

Not conservative, but still able to use Stokes theorem:

$$\int_C \vec{F}\, d\vec{r} = \iint_S \operatorname{curl} \vec{F}\, d\vec{S}$$

Need parametrization of the sphere:

$$\vec{r}(\theta, \phi) = \langle 2\cos\theta \sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi \rangle$$

$$\vec{r}_\theta = \langle -2\sin\theta\sin\phi,\ 2\cos\theta\sin\phi,\ 0 \rangle$$

$$\vec{r}_\phi = \langle 2\cos\theta\cos\phi,\ 2\sin\theta\cos\phi,\ -2\sin\phi \rangle$$

R14

normal vector

$$\vec{r}_\theta \times \vec{r}_\phi = 4\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta\,\sin\phi & \cos\theta\,\sin\phi & 0 \\ \cos\theta\,\cos\phi & \sin\theta\,\cos\phi & -\sin\phi \end{vmatrix}$$

$$= \vec{i}\,4\left(-\cos\theta\,\sin^2\phi\right) + \vec{j}\,4\left(-\sin\theta\,\sin^2\phi\right)$$

$$+ \vec{k}\,4\left(-\sin^2\theta\,\sin\phi\cos\phi - \cos^2\theta\,\sin\phi\,\cos\phi\right)$$

$$= -\sin\phi\,4\left(\cos\theta\sin\phi\,\vec{i} + \sin\theta\,\sin\phi\,\vec{j} + \cos\phi\,\vec{k}\right)$$

$$= -\sin\phi\,4\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$ <u>inward normal!</u>

$$\cancel{\vec{F}\cdot(\vec{r}_\theta\times\vec{r}_\phi) = -\sin\phi\,(2x\vec{i} + y^2\vec{j} + 2x\vec{k})}$$

$$\cancel{= -\sin\phi\,(2\cos\theta\,\sin\phi\cos\phi + \sin^2\theta\sin^2\phi)}$$

Use outward normal $4\sin\phi\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$

~~etc.~~

Use $\vec{F}\cdot(\vec{r}_\phi \times \vec{r}_\theta) = 4\sin\phi\,(2xy + 2yz + 2xz)$

$$= 4\sin\phi\,\Big(8\cos\theta\,\sin\theta\,\sin^2\phi + 8\sin\theta\,\sin\phi\cos\phi$$

$$+ 8\cos\theta\,\sin\phi\,\cos\phi\Big)$$

$$W=\int_C \vec{F}\,d\vec{r}=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 4\sin\phi\Big(8\cos\theta\sin\theta\sin^2\phi + 8\sin\theta\sin\phi\cos\phi + 8\cos\theta\sin\phi\cos\phi\Big)\,d\phi\,d\theta$$

$$=32\int_0^{\frac{\pi}{2}} \tfrac{1}{2}\sin^2\theta\Big|_0^{\frac{\pi}{2}}\cdot\sin^3\phi - \cos\theta\Big|_0^{\frac{\pi}{2}}\sin^2\phi\cos\phi + \sin\theta\Big|_0^{\frac{\pi}{2}}\sin^2\phi\cos\phi\;d\phi$$

$$=32\int_0^{\frac{\pi}{2}} \tfrac{1}{2}\sin^3\phi + \sin^2\phi\cos\phi + \sin^2\phi\cos\phi\;d\phi$$

$$=32\int_0^{\frac{\pi}{2}} \tfrac{1}{4}\left(1-\cos^2\phi\right)\sin\phi + 2\sin^2\phi\cos\phi\;d\phi$$

$$=32\left[-\frac{\cos\phi}{4}\Big|_0^{\frac{\pi}{2}} + \frac{1}{3\cdot 4}\cos^3\phi\Big|_0^{\frac{\pi}{2}} + \frac{2}{3}\sin^3\phi\Big|_0^{\frac{\pi}{2}}\right]$$

$$=32\left(\frac{1}{4}-\frac{1}{12}+\frac{2}{3}\right)=32\left(\frac{3-1+8}{12}\right)=32\,\frac{10}{12}$$

$$=\frac{80}{3}$$

$$= 32 \int_0^{\frac{\pi}{2}} \frac{1}{4}\left(1 - \cos 2\phi\right)\sin\phi + 2\sin^2\phi\cos\phi \, d\phi$$

$$= 32 \int_0^{\frac{\pi}{2}} \frac{1}{4}\left(\sin\phi - \frac{1}{2}\sin(2\phi+\phi) + \frac{1}{2}\sin(2\phi-\phi)\right)$$
$$+ 2\sin^2\phi\cos\phi \, d\phi$$

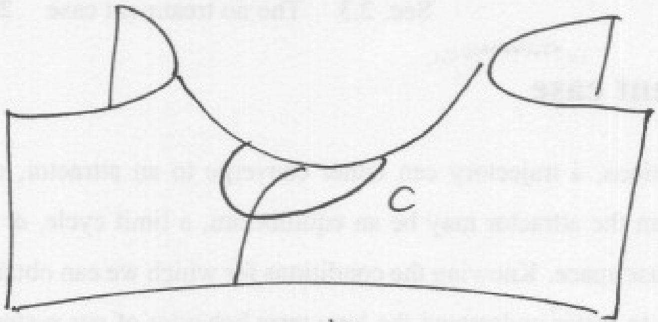
$$= \int_0^{\frac{\pi}{2}} 8\left(\frac{3}{2}\sin\phi - \frac{1}{2}\sin(3\phi)\right) + \overset{64}{\cancel{2}}\sin^2\phi\cos\phi \, d\phi$$

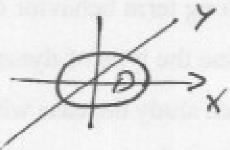
$$= -12\cos\phi \Big|_0^{\frac{\pi}{2}} + \frac{1}{6}\cos(3\phi)\Big|_0^{\frac{\pi}{2}} + \frac{64}{3}\sin^3\phi \Big|_0^{\frac{\pi}{2}}$$

$$= 12 - \frac{1}{6} + \frac{64}{3} = \frac{72 - 1 + 128}{6} = \frac{199}{6}$$

R16a

11.





$$\vec{r}(t) = \langle \sin t, \cos t, -\cos(2t) \rangle$$

$$0 \le t \le 2\pi$$

$\vec{r}(t)$ sits on the surface $z = x^2 - y^2$

$$-\cos(2t) = \sin^2 t - \cos^2 t \quad \checkmark$$

$z = x^2 - y^2$ over the domain $D = \{0 \le x^2 + y^2 \le 1\}$

$$\vec{r}(r, t) = \langle r \sin t, r \cos t, -r^2 \cos(2t) \rangle$$

because $\quad z = r^2 \sin^2 t - r^2 \cos^2 t = -r^2 \cos(2t)$.

$$\vec{r}_r = \langle \sin t, \cos t, -2r \cos(2t) \rangle$$

$$\vec{r}_t = \langle r \cos t, -r \sin t, +2r^2 \sin(2t) \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin t & \cos t & -2r\cos(2t) \\ r\cos t & -r\sin t & 2r^2 \sin(2t) \end{vmatrix}$$

$$= \vec{i}\left(2r^2 \cos t \sin(2t) - 2r^2 \sin t \cos 2t\right)$$

$$+ \vec{j}\left(-2r^2 \sin t \sin(2t) - 2r^2 \cos t \cos 2t\right)$$

$$+ \vec{k}\left(-r \sin^2 t - r \cos^2 t\right)$$

R17

$$= \vec{i}\left(2r^2 \sin(2t-t)\right) + \vec{j}\left(-2r^2 \cos(2t-t)\right)$$

$$- \cancel{\phi}\vec{k}\, r$$

$$= \langle 2r^2 \sin t, -2r^2 \cos t, -r \rangle$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+\sin x & x^2+\cos y & x^3 \end{vmatrix}$$

$$= \vec{i}(0) + \vec{j}(-3x^2) + \vec{k}(2x-1)$$

$$\text{curl } \vec{F} \cdot (\vec{r}_r \times \vec{r}_t) = \langle 0, -3r^2 \sin^2 t, 2r \sin t - 1 \rangle \cdot \langle 2r^2 \sin t, -2r^2 \cos t, -r \rangle$$

$$= -6r^4 \sin^2 t \cos t + 2r^2 \sin t + r$$

$$\int_C \vec{F}\, d\vec{r} = \int_0^{2\pi}\int_0^1 \left(-6r^4 \sin^2 t \cos t + 2r^2 \sin t + r\right) dr\, dt$$

$$= \int_0^{2\pi} \left(-\frac{6}{5} \sin^2 t \cos t + \frac{2}{3} \sin t + \frac{1}{2}\right) dt$$

$$= -\frac{6}{5}\frac{1}{3} \sin^3 t \Big|_0^{2\pi} + \frac{2}{3}(-\cos t)\Big|_0^{2\pi} + \frac{2\pi}{2}$$

$$= +\pi$$

R18