3 Fourier Series

(3.1) Recall: Linear Algebra

Vector space $\mathbb{R}^n$

basis vectors $e_1 = (1, 0, \ldots, 0)$

$e_2 = (0, 1, 0, \ldots, 0)$

Orthogonal basis of $\mathbb{R}^n$ $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$

Orthogonal: $\langle e_i, e_j \rangle = e_i^T e_j = \delta_{ij}$

basis representation: $a \in \mathbb{R}^n$

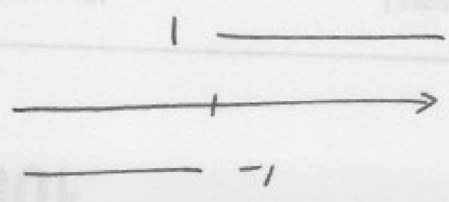
$$a = \sum_{i=1}^{n} a_i e_i \qquad a_j = \langle a \cdot e_j \rangle$$

(3.2) Piecewise smooth functions

Def 1: A function $f(x)$ has a jump discontinuity at x_0, if the right and left sided limits exist and

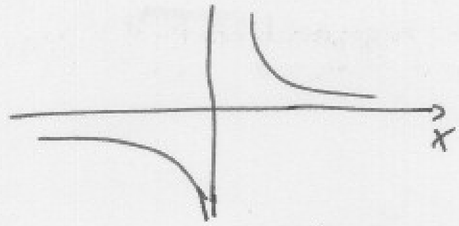
$$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$$

Example 1 Heaviside function $H(x) = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0. \end{cases}$



Example 2 Not a jump discontinuity at $x_0 = 0$:

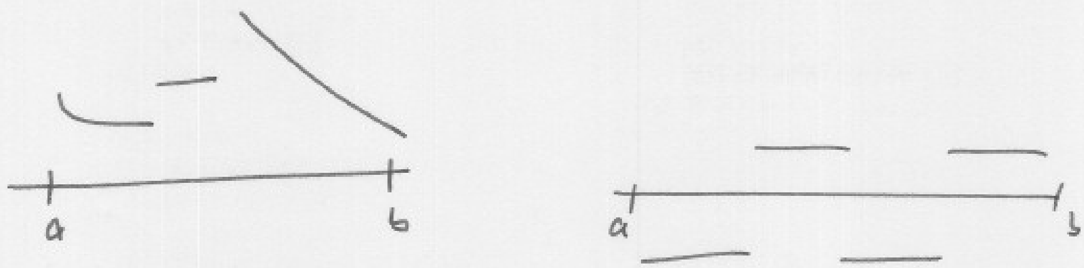
$f(x) = \frac{1}{x}$



$\lim_{x \to 0^-} f(x) = -\infty$ $\lim_{x \to 0+} f(x) = +\infty$.

Def 2: A function $f(x)$ on $[a, b]$ is called piecewise smooth, if f has at most a finite number of jump discontinuities.

e.g.



The set of all piecewise smooth functions on $[a, b]$ is called $PWS[a, b]$.

$PWS[a, b]$ is a vectorspace !

<u>inner product in PWS:</u>

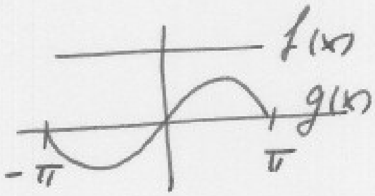
$$f, g \in PWS[a, b]$$

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

<u>Orthogonality in PWS:</u> $\langle f, g \rangle = 0$

<u>examples:</u> $f(x) = 3$, $g(x) = \sin x$ are orthogonal in $PWS[-\pi, \pi]$:

$$\langle 3, \sin x \rangle = \int_{-\pi}^{\pi} 3 \sin x\, dx = -3\left[\cos \pi - \cos(-\pi)\right] = 0$$



$\cos\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal in $PWS[-L, L]$ and in $PWS[0, L]$ for $n \neq m$.

Let $\mathcal{C} := \left\{ \cos\left(\frac{n\pi x}{L}\right),\ n = 0, 1, 2, \dots \right\}$

$\mathcal{S} := \left\{ \sin\left(\frac{m\pi x}{L}\right),\ m = 1, 2, 3, \dots \right\}$

<u>Theorem:</u> $\mathcal{S} \cup \mathcal{C}$ forms an orthogonal basis of $PWS[-L, L]$.

$$\langle 1, \sin(\tfrac{n\pi x}{L}) \rangle = 0$$

$$\langle \sin(\tfrac{n\pi x}{L}), \sin(\tfrac{m\pi x}{L}) \rangle = 0 \qquad n \neq m$$

$$\langle \sin(\tfrac{n\pi x}{L}), \cos(\tfrac{m\pi x}{L}) \rangle = 0$$

etc.

So orthogonal basis $\Rightarrow$ each function $f \in PWS[-L, L]$ must have a basis representation:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

Fourier-series.

We find a_0, a_n and b_n from orthogonality:

Multiply $f(x)$ by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate, $m > 0$:

$$\langle \cos(\tfrac{m\pi x}{L}), f(x) \rangle$$

$$= \langle \cos(\tfrac{m\pi x}{L}), a_0 \rangle + \sum_{n=1}^{\infty} a_n \langle \cos(\tfrac{m\pi x}{L}), \cos(\tfrac{n\pi x}{L}) \rangle$$

$$+ b_n \langle \cos(\tfrac{m\pi x}{L}), \sin(\tfrac{n\pi x}{L}) \rangle$$

$$= a_m \left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle$$

$$\left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = \int_{-L}^{L} \cos^2\left(\frac{m\pi x}{L}\right) dx$$

$$= \int_{-L}^{L} \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2m\pi x}{L}\right) dx$$

$$= L$$

Hence for $m > 0$: $\quad a_m = \frac{1}{L} \left\langle \cos\left(\frac{m\pi x}{L}\right), f(x) \right\rangle$

or

$$\boxed{a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx} \quad m > 0.$$

Similarly:

$$\boxed{b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx} \quad m > 0$$

a_0: Integrate $f(x)$:

$$\int_{-L}^{L} f(x)\, dx = \int_{-L}^{L} a_0\, dx + \sum_{n=1}^{\infty} a_m \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx$$

$$+ b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= 2L\, a_0$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.$$

<u>Theorem</u>: Let $f(x)$ be piecewise smooth on $[-L,L]$.

The Fourierseries

$$FS(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

has the following properties:

(i) Whenever $f(x)$ is continuous we have

$$f(x) = FS(x)$$

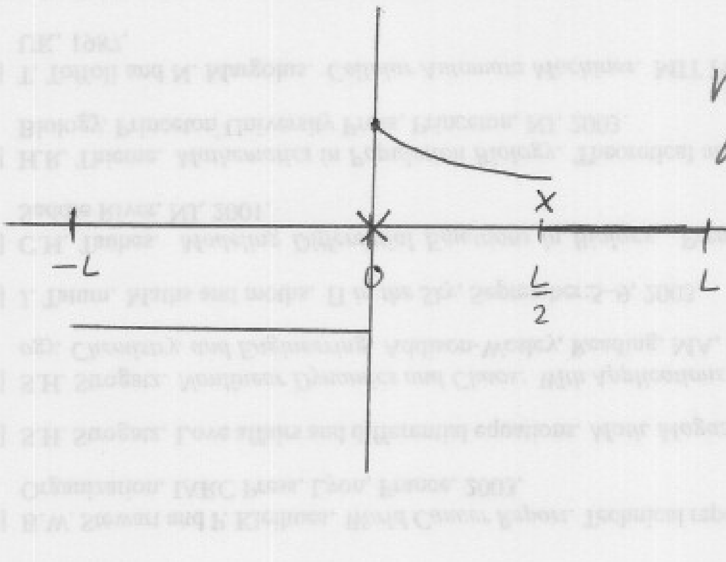
(ii) A jump-discontinuities x_0 we have

$$\frac{f(x^+) + f(x^-)}{2} = FS(x)$$

mean value of the jump.

Sketch the Fourier series of

$$f(x) = \begin{cases} -1 & -L \le x \le 0 \\ e^{-x} & 0 \le x \le \frac{L}{2} \\ 0 & \frac{L}{2} \le x \le L \end{cases}$$



Mark the mean value at a jump by an X.